INDUCTIVE APPROACH TO FIND FRACTIONAL DERIVATIVES AND THEIR HEURISTICS

Abstract. This research article redefines the fractional operators. Fractional operators have been deduced from nature of the operator by Riemann, Wilhelm Leibnitz, Neils Henrik Abel, Liouville etc. these fractional operator have some deviation from reality. This article is an attempt to find expression of fractional operator by inductive method which exactly reflects the reality, expression for explicit fractionl derivative for $0 < \alpha < 1$, given by

$$D^{\alpha}x^{m} = \exp\left(\lim_{r\to 1} {}^{m}_{m-\alpha} \frac{i}{e^{-2\pi it}-1}\left({}^{2\pi}_{0} \frac{e^{it\theta}}{1-re^{-i\theta}}d\theta\right)dt\right)x^{m-\alpha}.$$

and corresponding derivatives and integrals of other special functions, and it's representation in terms of classical classes of elementary functions, is possible iff non elementary integral, ${}^{2\pi}_{0} \frac{e^{it\theta}}{1-re^{-i\theta}}d\theta$ could be evaluated. An odd function changes to even function by operating integer order derivative operator, **what are intermediary states?** here intermediary state means

$$\frac{d^{\alpha}\sin x}{dx^{\alpha}} = a_0x^{1-\alpha} - a_1x^{3-\alpha} + a_2x^{5-\alpha} - \ldots;\ 0 < \alpha < 1, \qquad (0.1)$$

where a_n is function of α and n *only*. A periodic odd or even functions could be represented in a countable orthogonal basis, do the intermediary functions have countable orthogonal basis for their representation? only if $0 < \alpha < 1$, is a rational number.

Keywords: Fractional calculus, Rational change, Chaos, Dimensions of change, classical elementary functions

2020 Mathematics Subject Classification. [2020] primary 26AXX; 26A30, 26A33, 24A36.

1. Introduction

In applied Mathematics and Mathematical analysis, a fractional derivative of arbitrary order, real or complex, first appeared in a letter to Guillaume de I'Hŏpital by Gottfried Wilhelm Leibnitz in 1695 [1]. Later on contributed by Neils Henrik Abel [2], Liouville [3], Oliver Heaviside [4], and foundation of the subject was laid by Liouville [5].

Body Math Some definitions of fractional derivatives and integrals

Let us assume that $f(x)$is a monomial of the form $f(x) = x^k$. The first derivative is as usual

$$f'(x) = \frac{df(x)}{dx} = kx^{k-1}.$$

More general results gives

$$f^{(a)}(x) = \frac{d^a f(x)}{dx^a} = \frac{k!}{(k-a)!} x^{k-a}; a \in \mathbb{N} \tag{1.1}$$

after replacing the factorials with the gamma function,

$$\frac{d^a x^k}{dx^a} = \frac{\Gamma(k+1)}{\Gamma(k-a+1)} x^{k-a}; k > 1$$

For $k = 1$ and $a = \frac{1}{2}$, of course a is replaced by any real or complex number for fractional order derivative.

$$\frac{d^{\frac{1}{2}} x}{dx^{\frac{1}{2}}} = \frac{\Gamma(1+1)}{\Gamma(1-\frac{1}{2}+1)} x^{\frac{1}{2}} = \frac{2}{\sqrt[2]{\pi}} x^{\frac{1}{2}}.$$

For a general function $f(x)$ and $0 < \alpha < 1$, the complete fractional derivative is

$$D^{\alpha} f(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_0^x \frac{f(t)}{(x-t)^{\alpha}} dt.$$

Riemann–Liouville fractional derivative

Computing nth order derivative over the integral of order $(n-\alpha)$, the α order derivative is obtained. It is important to remark that n is the smallest integer greater than α (that is, $n = [\alpha]$).

$${}_aD_t^{\alpha} f(t) = \frac{d^n}{dt^n} {}_aD_t^{\alpha-n} f(t) = \frac{d^n}{dt^n} {}_aI_t^{n-\alpha} f(t).$$

Caputo fractional derivative: Caputo's definition is illustrated as follows, where again $n = [\alpha]$

$$D_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha+1-n}} d\tau.$$

Caputo's derivative does not require fractional order initial condition while solving differential equations.

Caputo–Fabrizio fractional derivative: Without singular kernel is given as

$${}_a^{CF}D_t^{\alpha} f(t) = \frac{1}{1-\alpha} \int_a^t f'(\tau) \exp\left(-\alpha \frac{t-\tau}{1-\alpha}\right) d\tau.$$

frametitleNature of fractional derivatives

The α^{th} derivative of a function f(x) at a point x is a local property only when α is an integer; this is not the true for non-integer power derivatives. Thus, a non-integer fractional derivative of a function $f(x)$ at $x = a$ depends on all values of f, even those far away from α. Therefore, it is expected that the fractional derivative operation involves some sort of boundary conditions.

1.1. **Fractional integral.** Riemann–Liouville fractional integral:

$$ {}_aD_t^{-\alpha}f(t) = {}_aI_t^{\alpha}f(t) = \frac{1}{\Gamma(\alpha)}\int_a^t (t-\tau)^{\alpha-1}f(\tau)d\tau. $$

Hadamard fractional integral:

$$ {}_aD_t^{-\alpha}f(t) = {}_aI_t^{\alpha}f(t) = \frac{1}{\Gamma(\alpha)}\int_a^t \left(\log\frac{t}{\tau}\right)^{\alpha-1} f(\tau)\frac{d\tau}{\tau}. $$

Now consider Navier -Stokes equations in convective form as

$$ \rho\frac{Du}{Dt} = \rho\left(\frac{\partial u}{\partial t} + (u.\nabla)u\right) $$

$$ = -\nabla P + \nabla.\left\{\mu\left[\nabla u + (\nabla u)^T - \frac{2}{3}(\nabla.u)I\right] + \zeta(\nabla.u)I\right\} + \rho g. $$

For planar Couette flow this equation reduces to

$$ \frac{\partial^2 u}{\partial y^2} = 0, \tag{1.2} $$

where y is the spatial coordinate normal to the plates and $u(y)$ is the velocity field. Thus in this case flow is unidirectional, If the lower plate is taken at $y = 0$, with boundary conditions are$u(0) = 0$, $u(h) = U$,then it has exact solution $u(y) = U\frac{y}{h}$. Nonetheless this steady state is not reached instantaneously due to shear stress, time variant of equation (1.2) could be written as

$$ \frac{\partial u}{\partial t} = v\frac{\partial^2 u}{\partial y^2} \tag{1.3} $$

with initial conditions $u(y,0) = 0, 0 < y < h$,along with steady state conditions $u(0,t) = 0, u(h,t) = U;\ t > 0$. Problem in (1.3) could be solved by separation of variable and applying boundary conditions to get final solution as

$$ u(h,t) = U\frac{y}{h} - \frac{2U}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}e^{-n^2\pi^2\frac{vt}{h^2}}\sin\left[n\pi\left(1-\frac{y}{h}\right)\right]. $$

Since the term

$$ \frac{2U}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}e^{-n^2\pi^2\frac{vt}{h^2}}\sin\left[n\pi\left(1-\frac{y}{h}\right)\right] \to 0, as\ t \to \infty $$

, but at $t = 0$,we have

$$ u(h,0) = U\frac{y}{h} - \frac{2U}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\left[n\pi\left(1-\frac{y}{h}\right)\right] $$

i.e. solution at $t = 0$, is a linear combination of of vectors

$$ \left\{\sin\left[n\pi\left(1-\frac{y}{h}\right)\right], n = 1,2,3,...\right\} $$

infinite dimensional vector space with coordinates $\left\{\frac{2U}{\pi}\frac{1}{n}\right\}$, and solutions at any time t, is again some another linear combination of vectors

$$ \left\{\sin\left[n\pi\left(1-\frac{y}{h}\right)\right], n = 1,2,3,...\right\} $$

with coordinates $\left\{\frac{2U}{\pi}\frac{1}{n}e^{-n^2\pi^2\frac{vt}{h^2}}\right\}$.

Question 1. Do we encounter the infinite dimensional representation of solution of the equations of type (1.3) due to boundary conditions only? Or Solution have their natural course i.e. boundary conditions are not the reason for such infinite dimensional representation rather natural changes of a flow problem have infinite dimensions (could be considered as eigen functions)out of which only some are admissible, depending on problems. Then what are **admissibility conditions?** What is the set of all possible dimensions (i.e. all possible eigen functions)?.

Question 2. If we apply integral order derivative e.g. take derivative $\frac{d\sin x}{dx} = \cos x$,notice the change in parameter x and its degrees, for Taylor's series of

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - ...$$

Or

$$\frac{d^\alpha \sin x}{dx^\alpha} = a_0 x - a_1\frac{x^3}{3!} + a_2\frac{x^5}{5!} - ...;;\ 0 < \alpha < 1, \tag{1.4}$$

where a_n is function of α and n *only*. Intermediary state given in (0.1) have countable basis only if $0 < \alpha < 1$, is a rational number.

Question3. Can the expressions in (0.1) and (1.4) are expressible in terms of classical classes of functions as considered by Liouville ([6, pp 668]) i.e. in the form of trigonometric functions, algebraic, rational, inverse trigonometric , logarithmic functions or in terms of functions of these functions etc., or have only non-elementary integral representations([6, pp670])?

Now to answer quetion-1 to question-3, find explicit representation of fractional order derivative operator $D^\alpha; 0 < \alpha < 1$.Since the universal principle of natural changes says that any change tends to minimize energy of the system, one need to find whether only changes which are governed by integral order operations comply with universal principle of change or any arbitrary order could work? if only rational or integral order change are in consonance with universal principle of change then in question-1, eigen functions are not dependent on boundary conditions, rather solution have only these functions as admissible function which are in accordance with universal principle of change.

Nonetheless to find explicit representation of fractional order derivative operator $D^\alpha; 0 < \alpha < 1$, but from formulae and approaches in ([1],[2],[3],[4],[5]), define another form of fractional derivatives and integrals (2.1), as all of the above form of definition involves evaluation of gamma function at different real or complex number. Since evaluation of gamma function other than integers and 1/2, is a tedious task. Involvement of Gamma function also does not gives any insight about evolution of system. To short out this problem, change approach from generalizing outcomes of integer order derivatives to action oriented approach.

2. Main Result

Theorem 1. *Formula for fractional derivatives based on effect of action of action operator D,as*

$$D^{\frac{1}{k}}x^m = \exp\left(\lim_{r\to 1}{}_{m-\frac{1}{k}}^{m}\frac{i}{e^{-2\pi it}-1}\left({}_{0}^{2\pi}\frac{e^{it\theta}}{1-re^{-i\theta}}d\theta\right)dt\right).x^{m-\frac{1}{k}}. \tag{2.1}$$

Proof. Take $f(x) = x^n$ i.e. monomial of degree n, and D be derivative operator, suppose $D^{\frac{1}{k}}$th, order derivative acts on x^n, and gives

$$D^{\frac{1}{k}}x^n = \psi(n)x^{n-\frac{1}{k}}. \tag{2.2}$$

Again apply operator $D^{\frac{1}{k}}$on both side of equation (2.2), and get

$$D^{\frac{1}{k}}D^{\frac{1}{k}}x^n = D^{\frac{2}{k}}x^n = \psi(n)D^{\frac{1}{k}}x^{n-\frac{1}{k}} = \psi(n)\psi\left(n-\frac{1}{k}\right)x^{n-\frac{2}{k}}.$$

where, ψ, is function of degree of x,at the time of operator takes action on it. Now suppose $D^{\frac{1}{k}}$ operated on both side of equation (2.2) then get

$$\underbrace{D^{\frac{1}{k}}D^{\frac{1}{k}}...D^{\frac{1}{k}}x^n}_{k\ times} = Dx^n = \psi(n)\psi\left(n-\frac{1}{k}\right)...\psi\left(n-\frac{k-1}{k}\right)x^{n-1}. \tag{2.3}$$

Since $Dx^n = nx^{n-1}$, compare it with equation (2.3) and get

$$\psi(n)\psi\left(n-\frac{1}{k}\right)...\psi\left(n-\frac{k-1}{k}\right) = n.$$

Take logarithm on both side to get

$${}_{i=0}^{k-1}\log\psi\left(n-\frac{i}{k}\right) = \log n. \tag{2.4}$$

Multiply on both side of equation (2.4) by $\frac{1}{k}$ to get

$${}_{i=0}^{k-1}\log\psi\left(n-\frac{i}{k}\right)\frac{1}{k} = (\log n)\frac{1}{k}. \tag{2.5}$$

Take $k \to \infty$,and put $\frac{i}{k} = \mu; \frac{1}{k} = d\mu$ then summation in equation (2.5) changes to integral as below

$${}_{0}^{1}\log\psi\,(n-\mu)\,d\mu = (\log n)_{0}^{1}\,d\mu = \log n.$$

Put $n - \mu = v$, then integral changes to

$${}_{n-1}^{n}\log\psi\,(v)\,dv = \log n. \tag{2.6}$$

Differentiate equation (2.6) with respect to n, by using Leibnitz rule of differentiation under integral sign, to get

$$\log\psi\,(n) - \log\psi\,(n-1) = \frac{1}{n}. \tag{2.7}$$

Solve equation (2.7) to get

$$\log\psi\left(n-\frac{1}{k}\right) = \left(\lim_{r\to 1}{}_{m-\frac{1}{k}}^{m}\frac{i}{e^{-2\pi it}-1}\left({}_{0}^{2\pi}\frac{e^{it\theta}}{1-re^{-i\theta}}d\theta\right)dt\right).x^{m-\frac{1}{k}}.$$

Thus get

$$D^{\frac{1}{k}}x^m = \exp\left(\lim_{r\to 1} {}^{m}_{m-\frac{1}{k}} \frac{i}{e^{-2\pi it}-1}\left({}^{2\pi}_{0}\frac{e^{it\theta}}{1-re^{-i\theta}}d\theta\right)dt\right).x^{m-\frac{1}{k}}.$$

Now replace $\frac{1}{k}$ byα, $0<\alpha<1$, to get arbitrary order fractional derivative Corresponding fractional order integral is derived to be

$$D^{-\frac{1}{k}}x^m = I^{\frac{1}{k}}x^m$$
$$= \exp\left(-\lim_{r\to 1} {}^{m}_{m-\frac{1}{k}} \frac{i}{e^{-2\pi it}-1}\left({}^{2\pi}_{0}\frac{e^{it\theta}}{1-re^{-i\theta}}d\theta\right)dt\right).x^{m+\frac{1}{k}}.$$

Fractional derivative for special functions

$$D^{\frac{1}{k}}e^x = {}^{\infty}_{m=0} \exp\left(\lim_{r\to 1} {}^{m}_{m-\frac{1}{k}} \frac{i}{e^{-2\pi it}-1}\left({}^{2\pi}_{0}\frac{e^{it\theta}}{1-re^{-i\theta}}d\theta\right)dt\right).\frac{x^{m-\frac{1}{k}}}{m!}.$$

And corresponding fractional integral

$$D^{-\frac{1}{k}}x^m = I^{\frac{1}{k}}x^m$$
$$= {}^{\infty}_{m=0} \exp\left(-\lim_{r\to 1} {}^{m}_{m-\frac{1}{k}} \frac{i}{e^{-2\pi it}-1}\left({}^{2\pi}_{0}\frac{e^{it\theta}}{1-re^{-i\theta}}d\theta\right)dt\right).\frac{x^{m+\frac{1}{k}}}{m!}.$$

Like above define fractional derivative and integral to other special functions. New approach could be easily applied and be replaced in place of usual formulae we use in fluid mechanics. □

Remark 1. *The integral* ${}^{2\pi}_{0}\frac{e^{it\theta}}{1-re^{-i\theta}}d\theta$*, is Cauchy's fractional integral, could not be integrated by Cauchy's residue formula nor is an Cauchy singular integral as* $e^{it\theta}; 0<t<1$*, has branch points. Also* $\frac{e^{it\theta}}{1-re^{-i\theta}}d\theta$*, pertains to class of non-elementary integrals given by Liouville (*[6, pp670]*).*

Corresponding fractional order integral is derived to be

$$D^{-\frac{1}{k}}x^m = I^{\frac{1}{k}}x^m$$
$$= \exp\left(-\lim_{r\to 1} {}^{m}_{m-\frac{1}{k}} \frac{i}{e^{-2\pi it}-1}\left({}^{2\pi}_{0}\frac{e^{it\theta}}{1-re^{-i\theta}}d\theta\right)dt\right).x^{m+\frac{1}{k}}.$$

Fractional derivative for special functions

$$D^{\frac{1}{k}}e^x = {}^{\infty}_{m=0} \exp\left(\lim_{r\to 1} {}^{m}_{m-\frac{1}{k}} \frac{i}{e^{-2\pi it}-1}\left({}^{2\pi}_{0}\frac{e^{it\theta}}{1-re^{-i\theta}}d\theta\right)dt\right).\frac{x^{m-\frac{1}{k}}}{m!}.$$

And corresponding fractional integral

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$$= {}^{\infty}_{m=0} \exp\left(-\lim_{r\to 1} {}^{m}_{m-\frac{1}{k}} \frac{i}{e^{-2\pi it}-1}\left({}^{2\pi}_{0}\frac{e^{it\theta}}{1-re^{-i\theta}}d\theta\right)dt\right).\frac{x^{m+\frac{1}{k}}}{m!}. \qquad (2.8)$$

Limitations: Representation given in (2.8) is an infinite series, it's expression in the form of classical classes of elementary functions is yet to be determined, and expression for fractional derivative required integration of non elementary integral which itself could be evaluated numerically only.

Remark 2. ***Heuristics of fractional changes:*** *Take D^{α}, where α is an irrational number then for any integer n, $n\alpha$ is again an irrational number, thus $D^{n\alpha}$, could never be a total order derivative, hence all the changes derived from irrational fractions will never terminate or stabilize. Therefore the forces shaping the nature surrounding us seems to be driven by rational changes, rest are chaos. Since growth of cells in tissues, petals in flowers are precisely controlled and have exactly same number for the same species, so forces driving biological changes must have some rational representations, rest other changes are cancerous.*

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