Closed-form Solutions of Third-Order Generalized Leonardo Sequences with Polynomial Input

Abstract. The objective of this study is to derive closed-form solutions to third-order nonhomogeneous linear recurrence relations, referred to as generalized Leonardo-type sequences, where the input function $p(n)$ is a polynomial. The study considers the cases in which 1 appears as a root of the characteristic equation with multiplicity $r = 0, 1, 2, 3$, and for each value of r explicit solutions are obtained for polynomial inputs $p(n)$ of degree $s = 0, 1, 2, 3$. The resulting formulas express the solution as the sum of homogeneous and particular components, with the coefficients determined through iterative relations. This unified framework provides a complete description of generalized Leonardo-type sequences in the nonhomogeneous setting with polynomial inputs, extending the classical theory of recurrence relations.

The results extend classical recurrence theory by clarifying resonance phenomena and multiplicity corrections, while offering resonance-aware formulas that can be adapted to problems in mathematics, computer science, engineering, and physics. Beyond their theoretical contribution, the explicit examples provide pedagogical value by allowing students to engage directly with nonhomogeneous recurrences without excessive computation. Thus, the study demonstrates both the novelty and interdisciplinary impact of generalized Leonardo-type sequences in the nonhomogeneous setting.

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1. Introduction

Sequences defined by recurrence relations have long stood at the heart of mathematics, branching into diverse disciplines such as physics, engineering, architecture, biology, computer science, and even the arts.

Though their definitions may appear elementary, they conceal remarkable depth: modeling growth, oscillations, and symbolic structures. Classical second-order families-most notably the Fibonacci, Lucas, Pell, and Jacobsthal sequences-remain central examples of this tradition.

The scope, however, extends well beyond second-order cases. Higher-order recurrence sequences enrich both theory and practice, broadening the classical framework and uncovering intricate algebraic and analytic patterns. The Tribonacci (third-order), Tetranacci (fourth-order), and Pentanacci (fifth-order) sequences exemplify this expansion, each governed by characteristic polynomials whose root configurations dictate closed-form expressions. Homogeneous recurrences emphasize the interplay of characteristic polynomials and root multiplicities, while non-homogeneous forms introduce symbolic terms that interact with root structures to generate resonance phenomena. Together, these families establish a coherent framework that unites classical recurrence identities with the evolving field of symbolic recurrence theory.

The classical Leonardo sequence is defined by the non-homogeneous recurrence relation

$$l_n = l_{n-1} + l_{n-2} + 1, \quad n \geq 2,$$

with initial conditions $l_0 = 1$ and $l_1 = 1$. Although the recurrence itself is elementary, the historical development of the sequence is less straightforward. Its recognition emerged gradually, with generalizations appearing in the literature prior to the adoption of its formal name. Renewed interest in recent decades has been driven by explicit case analyses and the broad range of applications in which the sequence naturally arises.

The Leonardo sequence has become notable not only for its intrinsic mathematical appeal but also for its role in modeling systems that blend homogeneous recurrence dynamics with non-homogeneous forcing terms. This dual structure has made it a fertile ground for symbolic investigation, linking classical recurrence theory with modern applications. Contemporary studies highlight its algebraic richness, its ability to encode intricate interactions, and its relevance across both theoretical and applied contexts.

From an educational standpoint, the transparency of its defining relation and the accessibility of explicit examples make the Leonardo sequence particularly well-suited for instructional use. It offers students a clear illustration of how non-homogeneous recurrences operate, while simultaneously serving as a gateway to advanced symbolic techniques and resonance phenomena. In this respect, the sequence continues to function both as a subject of scholarly inquiry and as a pedagogical resource (see, for example, [1, 2, 3, 4, 5, 9, 10, 15, 16, 17, 11, 12, 13, 14, 21, 22]).

To our knowledge, the first systematic extension of the Leonardo numbers was undertaken by J. A. Jeske in a trilogy of papers published in *The Fibonacci Quarterly* during 1963–1964 (see [6, 7, 8]). For a concise survey of contributions within *The Fibonacci Quarterly* and selected works beyond the journal that advance the study of Leonardo-type recurrences, see Soykan [18, Section 5].

Let the third order nonhomogeneous linear recurrence relation, referred to as generalized Leonardo-type sequences, be given by

$$W_n = a_1 W_{n-1} + a_2 W_{n-2} + a_3 W_{n-3} + p(n) \tag{1.1}$$

with initial conditions $W_0 = k_0, W_1 = k_1, W_2 = k_2$ where $p(n)$ is the polynomial with degree s, with coefficients in $\mathbb{C}[x]$ or $\mathbb{C}$:

$$p(n) = \sum_{i=0}^{s} c_i n^i,$$

and the recurrence coefficients a_1, a_2, a_3 are complex scalars or polynomials in $\mathbb{C}[x]$. For more information on generalized Leonardo-type sequences, see Soykan [19] and [18].

Let the homogeneous relation corresponding to (1.1) be written as

$$V_n = a_1 V_{n-1} + a_2 V_{n-2} + a_3 V_{n-3} \tag{1.2}$$

with the same initial conditions as W_n, i.e.,

$$V_0 = W_0, V_1 = W_1, V_2 = W_2.$$

Suppose that θ_1, θ_2 and θ_3 are the roots of the characteristic equation

$$z^3 - a_1 z^2 - a_2 z - a_3 = 0 \tag{1.3}$$

of (1.2).

Note that if all the roots of (1.3) are equal to 1 then

$$z^3 - a_1 z^2 - a_2 z - a_3 = (z-1)^3 = z^3 - 3z^2 + 3z - 1 = 0$$

so that $a_1 = 3,\ a_2 = -3,\ a_3 = 1$ and (1.2) reduces to

$$V_n = 3V_{n-1} - 3V_{n-2} + V_{n-3}.$$

In our earlier work, particular solutions to third-order nonhomogeneous linear recurrence relations (1.1) with polynomial inputs were established for the cases $s = 0, 1, 2, 3$; see Soykan [20]. In the present paper, we build upon those results to derive closed-form solutions by systematically applying Theorem 1.1. Within this framework, the closed-form expressions are obtained by decomposing each recurrence into its homogeneous and particular components, with the latter determined through an iterative scheme for the coefficients. The analysis is organized according to the multiplicity r of 1 as a root of the characteristic equation (1.2) and the degree s of the polynomial $p(n)$, with explicit formulas derived for all cases $r = 0, 1, 2, 3$ and $s = 0, 1, 2, 3$. This approach highlights how the root multiplicity and polynomial degree jointly determine the structure of the solution, while ensuring that explicit formulas are available for all cases considered. By integrating the previously obtained particular solutions into a unified framework, we provide complete closed-form expressions that extend the classical theory of recurrence relations and advance the study of generalized Leonardo-type sequences in the nonhomogeneous setting.

Theorem 1.1.

(a): *[18, Theorem 7.7. (a)] The case $r=0$, i.e., all three roots of the characteristic equation of (1.2) is distinct from* 1.

The solution of (1.1) is given by

$$\begin{aligned} W_n(W_0,W_1,W_2) &= W_n^{(h)}+W_n^{(p)} \\ &= V_n(W_0,W_1,W_2)-V_n(W_0^{(p)},W_1^{(p)},W_2^{(p)})+W_n^{(p)} \end{aligned}$$

where $W_n^{(h)}=V_n(W_0,W_1,W_2)-V_n(W_0^{(p)},W_1^{(p)},W_2^{(p)})$ *is the solution of (1.2) and*

$$W_n^{(p)}=\sum_{i=0}^{s}A_in^i=A_0+\sum_{i=1}^{s}A_in^i$$

is the particular solution of (1.1). For each $0\leq i\leq s$, A_i *can be calculated with the iteration*

$$A_s=-\frac{c_s}{a_1+a_2+a_3-1},\ \textit{for } n=s$$

and

$$A_n=-\frac{1}{a_1+a_2+a_3-1}(c_n-\sum_{k=n+1}^{s}(-1)^{k-n+1}\binom{k}{n}(a_1+2^{k-n}a_2+3^{k-n}a_3)A_k),\ \textit{for } n=s-1,s-2,...,2,1,0.$$

Here

$$\begin{aligned} W_0^{(p)} &= A_0, \\ W_1^{(p)} &= \sum_{i=0}^{s}A_i, \\ W_2^{(p)} &= \sum_{i=0}^{s}2^iA_i, \end{aligned}$$

and

$$\begin{aligned} V_n(W_0^{(p)},W_1^{(p)},W_2^{(p)}) &= V_n(A_0,\sum_{i=0}^{s}A_i,\sum_{i=0}^{s}2^iA_i) \\ &= B_1V_n(W_0,W_1,W_2)+B_2V_{n-1}(W_0,W_1,W_2)+B_3V_{n-2}(W_0,W_1,W_2) \end{aligned}$$

where

$$B_1=\frac{Y_1}{\Delta},\ B_2=\frac{Y_2}{\Delta},\ B_3=\frac{Y_3}{\Delta}$$

and

$Y_1=(W_2^2+a_1^2W_1^2+a_1a_3W_0^2-2a_1W_1W_2-a_2W_0W_2+(a_1a_2-a_3)W_0W_1)W_2^{(p)}+((a_3+a_1a_2)W_1^2+a_2a_3W_0^2-a_2W_1W_2-a_3W_0W_2+a_2^2W_0W_1)W_1^{(p)}+a_3(a_1W_1^2+a_3W_0^2-W_1W_2+a_2W_0W_1)W_0^{(p)}$

$Y_2=((a_3+a_1a_2)W_1^2-a_2W_2W_1+(a_2^2-2a_3a_1)W_0W_1+a_3a_2W_0^2+2a_3a_1W_0W_1-a_3W_0W_2)W_2^{(p)}+(a_2W_2^2+2a_3W_1W_2+a_3^2W_0^2-(3a_3+a_1a_2)W_1W_2-(a_2^2-2a_3a_1)W_0W_2-a_3a_1W_0W_2)W_1^{(p)}+(a_3W_2^2-a_3a_2W_0W_2-a_3a_1W_1W_2-a_3^2W_0W_1)W_0^{(p)}$

$Y_3=a_3(-W_1W_2+a_1W_1^2+a_2W_0W_1+a_3W_0^2)W_2^{(p)}+a_3(W_2^2-a_1W_1W_2-a_2W_0W_2-a_3W_0W_1)W_1^{(p)}+a_3(a_3W_1^2-a_3W_0W_2)W_0^{(p)}$

$\Delta = W_2^3 + (a_3 + a_1a_2)W_1^3 + a_3^2W_0^3 - 2a_1W_1W_2^2 + (a_1^2 - a_2)W_1^2W_2 - a_2W_0W_2^2 + a_3a_1W_0^2W_2 + (a_2^2 + a_1a_3)W_0W_1^2 + 2a_2a_3W_0^2W_1 + (-3a_3 + a_1a_2)W_0W_1W_2$

i.e.,

$Y_1 = (W_2^2 + a_1^2W_1^2 + a_1a_3W_0^2 - 2a_1W_1W_2 - a_2W_0W_2 + (a_1a_2 - a_3)W_0W_1)\sum_{i=0}^s 2^iA_i + ((a_3 + a_1a_2)W_1^2 + a_2a_3W_0^2 - a_2W_1W_2 - a_3W_0W_2 + a_2^2W_0W_1)\sum_{i=0}^s A_i + a_3(a_1W_1^2 + a_3W_0^2 - W_1W_2 + a_2W_0W_1)A_0$

$Y_2 = ((a_3+a_1a_2)W_1^2 - a_2W_2W_1 + (a_2^2 - 2a_3a_1)W_0W_1 + a_3a_2W_0^2 + 2a_3a_1W_0W_1 - a_3W_0W_2)\sum_{i=0}^s 2^iA_i + (a_2W_2^2 + 2a_3W_1W_2 + a_3^2W_0^2 - (3a_3 + a_1a_2)W_1W_2 - (a_2^2 - 2a_3a_1)W_0W_2 - a_3a_1W_0W_2)\sum_{i=0}^s A_i + (a_3W_2^2 - a_3a_2W_0W_2 - a_3a_1W_1W_2 - a_3^2W_0W_1)A_0$

$Y_3 = a_3(-W_1W_2 + a_1W_1^2 + a_2W_0W_1 + a_3W_0^2)\sum_{i=0}^s 2^iA_i + a_3(W_2^2 - a_1W_1W_2 - a_2W_0W_2 - a_3W_0W_1)\sum_{i=0}^s A_i + a_3(a_3W_1^2 - a_3W_0W_2)A_0$

$\Delta = W_2^3 + (a_3 + a_1a_2)W_1^3 + a_3^2W_0^3 - 2a_1W_1W_2^2 + (a_1^2 - a_2)W_1^2W_2 - a_2W_0W_2^2 + a_3a_1W_0^2W_2 + (a_2^2 + a_1a_3)W_0W_1^2 + 2a_2a_3W_0^2W_1 + (-3a_3 + a_1a_2)W_0W_1W_2$

In summary, the solution of (1.1) is given by

$$W_n(W_0,W_1,W_2) = (-B_1+1)V_n(W_0,W_1,W_2) - B_2V_{n-1}(W_0,W_1,W_2) - B_3V_{n-2}(W_0,W_1,W_2) + \sum_{i=0}^s A_in^i$$

(b): *The case $r = 1$, i.e., 1 is a simple root of the characteristic equation of (1.2). The solution of (1.1) is*

$W_n(W_0,W_1,W_2) = W_n^{(h)} + W_n^{(p)} = V_n(W_0,W_1,W_2) - V_n(W_0^{(p)},W_1^{(p)},W_2^{(p)}) + W_n^{(p)},$

where the particular solution of (1.1) is

$$W_n^{(p)} = n\sum_{i=0}^s A_in^i = n(A_0 + \sum_{i=1}^s A_in^i).$$

For each $0 \le i \le s$, the coefficients A_i are obtained iteratively:

$$A_s = (-1)^2\frac{c_s}{(\sum_{j=1}^3 ja_j)\binom{s+1}{1}} = \frac{c_s}{(a_1+2a_2+3a_3)(s+1)}, \quad n = s,$$

and

$$\begin{aligned} A_n &= (-1)^2\frac{1}{(\sum_{j=1}^3 ja_j)\binom{n+1}{1}}\left(c_n - \sum_{k=n+1}^s (-1)^{k-n+2}\binom{k+1}{n}(a_1 + 2^{k-n+1}a_2 + 3^{k-n+1}a_3)A_k\right) \\ &= \frac{1}{(a_1+2a_2+3a_3)(n+1)}\left(c_n - \sum_{k=n+1}^s (-1)^{k-n+2}\binom{k+1}{n}(a_1 + 2^{k-n+1}a_2 + 3^{k-n+1}a_3)A_k\right), \end{aligned}$$

for $n = s-1, s-2, \ldots, 1, 0$. The expressions for $W_0^{(p)}, W_1^{(p)}, W_2^{(p)}$ and the representation of $V_n(W_0^{(p)},W_1^{(p)},W_2^{(p)})$ in terms of B_1, B_2, B_3 follow analogously to case (a).

Here

$$W_0^{(p)} = 0, \quad W_1^{(p)} = \sum_{i=0}^s A_i, \quad W_2^{(p)} = 2\sum_{i=0}^s 2^iA_i,$$

and

$$\begin{aligned} V_n(W_0^{(p)}, W_1^{(p)}, W_2^{(p)}) &= V_n(0, \sum_{i=0}^{s} A_i, 2\sum_{i=0}^{s} 2^i A_i) \\ &= B_1 V_n(W_0, W_1, W_2) + B_2 V_{n-1}(W_0, W_1, W_2) + B_3 V_{n-2}(W_0, W_1, W_2) \end{aligned}$$

where

$$B_1 = \frac{Y_1}{\Delta},\ B_2 = \frac{Y_2}{\Delta},\ B_3 = \frac{Y_3}{\Delta}$$

and

$Y_1 = (W_2^2 + a_1^2 W_1^2 + a_1 a_3 W_0^2 - 2a_1 W_1 W_2 - a_2 W_0 W_2 + (a_1 a_2 - a_3) W_0 W_1) W_2^{(p)} + ((a_3 + a_1 a_2) W_1^2 + a_2 a_3 W_0^2 - a_2 W_1 W_2 - a_3 W_0 W_2 + a_2^2 W_0 W_1) W_1^{(p)} + a_3 (a_1 W_1^2 + a_3 W_0^2 - W_1 W_2 + a_2 W_0 W_1) W_0^{(p)}$

$Y_2 = ((a_3 + a_1 a_2) W_1^2 - a_2 W_2 W_1 + (a_2^2 - 2a_3 a_1) W_0 W_1 + a_3 a_2 W_0^2 + 2a_3 a_1 W_0 W_1 - a_3 W_0 W_2) W_2^{(p)} + (a_2 W_2^2 + 2a_3 W_1 W_2 + a_3^2 W_0^2 - (3a_3 + a_1 a_2) W_1 W_2 - (a_2^2 - 2a_3 a_1) W_0 W_2 - a_3 a_1 W_0 W_2) W_1^{(p)} + (a_3 W_2^2 - a_3 a_2 W_0 W_2 - a_3 a_1 W_1 W_2 - a_3^2 W_0 W_1) W_0^{(p)}$

$Y_3 = a_3(-W_1 W_2 + a_1 W_1^2 + a_2 W_0 W_1 + a_3 W_0^2) W_2^{(p)} + a_3 (W_2^2 - a_1 W_1 W_2 - a_2 W_0 W_2 - a_3 W_0 W_1) W_1^{(p)} + a_3 (a_3 W_1^2 - a_3 W_0 W_2) W_0^{(p)}$

$\Delta = W_2^3 + (a_3 + a_1 a_2) W_1^3 + a_3^2 W_0^3 - 2a_1 W_1 W_2^2 + (a_1^2 - a_2) W_1^2 W_2 - a_2 W_0 W_2^2 + a_3 a_1 W_0^2 W_2 + (a_2^2 + a_1 a_3) W_0 W_1^2 + 2a_2 a_3 W_0^2 W_1 + (-3a_3 + a_1 a_2) W_0 W_1 W_2$

In summary, the solution of (1.1) is given by

$$W_n(W_0, W_1, W_2) = (-B_1 + 1) V_n(W_0, W_1, W_2) - B_2 V_{n-1}(W_0, W_1, W_2) - B_3 V_{n-2}(W_0, W_1, W_2) + n \sum_{i=0}^{s} A_i n^i$$

(c): *The case $r = 2$, i.e., 1 is a double root of the characteristic equation of (1.2). The solution of (1.1) is*

$$W_n(W_0, W_1, W_2) = W_n^{(h)} + W_n^{(p)} = V_n(W_0, W_1, W_2) - V_n(W_0^{(p)}, W_1^{(p)}, W_2^{(p)}) + W_n^{(p)},$$

where the particular solution of (1.1) is

$$W_n^{(p)} = n^2 \sum_{i=0}^{s} A_i n^i.$$

The coefficients A_i are determined by

$$A_s = (-1)^3 \frac{c_s}{(\sum_{j=1}^{3} j^2 a_j)\binom{s+2}{2}} = -\frac{2c_s}{(a_1 + 4a_2 + 9a_3)(s+2)(s+1)}, \quad n = s,$$

and

$$\begin{aligned} A_n &= (-1)^3 \frac{1}{(\sum_{j=1}^{3} j^2 a_j)\binom{n+2}{2}} \left(c_n - \sum_{k=n+1}^{s} (-1)^{k-n+3} \binom{k+2}{n} (a_1 + 2^{k-n+2} a_2 + 3^{k-n+2} a_3) A_k \right) \\ &= -\frac{2}{(a_1 + 4a_2 + 9a_3)(n+2)(n+1)} \left(c_n - \sum_{k=n+1}^{s} (-1)^{k-n+3} \binom{k+2}{n} (a_1 + 2^{k-n+2} a_2 + 3^{k-n+2} a_3) A_k \right), \end{aligned}$$

for $n = s-1, s-2, \ldots, 1, 0$. Again, $W_0^{(p)}, W_1^{(p)}, W_2^{(p)}$ and the B_1, B_2, B_3 representation follow the same scheme.

Here

$$W_0^{(p)} = 0, \quad W_1^{(p)} = \sum_{i=0}^{s} A_i, \quad W_2^{(p)} = 2^2 \sum_{i=0}^{s} 2^i A_i,$$

and

$$\begin{aligned} V_n(W_0^{(p)}, W_1^{(p)}, W_2^{(p)}) &= V_n(0, \sum_{i=0}^{s} A_i, 2^2 \sum_{i=0}^{s} 2^i A_i) \\ &= B_1 V_n(W_0, W_1, W_2) + B_2 V_{n-1}(W_0, W_1, W_2) + B_3 V_{n-2}(W_0, W_1, W_2) \end{aligned}$$

where

$$B_1 = \frac{Y_1}{\Delta}, \; B_2 = \frac{Y_2}{\Delta}, \; B_3 = \frac{Y_3}{\Delta}$$

and

$Y_1 = (W_2^2 + a_1^2 W_1^2 + a_1 a_3 W_0^2 - 2a_1 W_1 W_2 - a_2 W_0 W_2 + (a_1 a_2 - a_3) W_0 W_1) W_2^{(p)} + ((a_3 + a_1 a_2) W_1^2 + a_2 a_3 W_0^2 - a_2 W_1 W_2 - a_3 W_0 W_2 + a_2^2 W_0 W_1) W_1^{(p)} + a_3 (a_1 W_1^2 + a_3 W_0^2 - W_1 W_2 + a_2 W_0 W_1) W_0^{(p)}$

$Y_2 = ((a_3 + a_1 a_2) W_1^2 - a_2 W_2 W_1 + (a_2^2 - 2a_3 a_1) W_0 W_1 + a_3 a_2 W_0^2 + 2a_3 a_1 W_0 W_1 - a_3 W_0 W_2) W_2^{(p)} + (a_2 W_2^2 + 2a_3 W_1 W_2 + a_3^2 W_0^2 - (3a_3 + a_1 a_2) W_1 W_2 - (a_2^2 - 2a_3 a_1) W_0 W_2 - a_3 a_1 W_0 W_2) W_1^{(p)} + (a_3 W_2^2 - a_3 a_2 W_0 W_2 - a_3 a_1 W_1 W_2 - a_3^2 W_0 W_1) W_0^{(p)}$

$Y_3 = a_3(-W_1 W_2 + a_1 W_1^2 + a_2 W_0 W_1 + a_3 W_0^2) W_2^{(p)} + a_3 (W_2^2 - a_1 W_1 W_2 - a_2 W_0 W_2 - a_3 W_0 W_1) W_1^{(p)} + a_3 (a_3 W_1^2 - a_3 W_0 W_2) W_0^{(p)}$

$\Delta = W_2^3 + (a_3 + a_1 a_2) W_1^3 + a_3^2 W_0^3 - 2a_1 W_1 W_2^2 + (a_1^2 - a_2) W_1^2 W_2 - a_2 W_0 W_2^2 + a_3 a_1 W_0^2 W_2 + (a_2^2 + a_1 a_3) W_0 W_1^2 + 2a_2 a_3 W_0^2 W_1 + (-3a_3 + a_1 a_2) W_0 W_1 W_2$

In summary, the solution of (1.1) is given by

$$W_n(W_0, W_1, W_2) = (-B_1 + 1) V_n(W_0, W_1, W_2) - B_2 V_{n-1}(W_0, W_1, W_2) - B_3 V_{n-2}(W_0, W_1, W_2) + n^2 \sum_{i=0}^{s} A_i n^i$$

(d): *The case $r = 3$, i.e., 1 is a triple root of the characteristic equation of (1.2). In this case we have*

$$a_1 = 3, \; a_2 = -3, \; a_3 = 1.$$

The solution of (1.1) is

$$W_n(W_0, W_1, W_2) = W_n^{(h)} + W_n^{(p)} = V_n(W_0, W_1, W_2) - V_n(W_0^{(p)}, W_1^{(p)}, W_2^{(p)}) + W_n^{(p)},$$

where the particular solution of (1.1) is

$$W_n^{(p)} = n^3 \sum_{i=0}^{s} A_i n^i.$$

The coefficients A_i are determined by

$$A_s = (-1)^4 \frac{c_s}{(\sum_{j=1}^{3} j^3 a_j) \binom{s+3}{3}} = \frac{c_s}{(s+3)(s+2)(s+1)}, \quad n = s,$$

and

$$\begin{aligned} A_n &= (-1)^4 \frac{1}{(\sum_{j=1}^{3} j^3 a_j)\binom{n+3}{3}} \left(c_n - \sum_{k=n+1}^{s} (-1)^{k-n+4} \binom{k+3}{n} (a_1 + 2^{k-n+3} a_2 + 3^{k-n+3} a_3) A_k \right) \\ &= \frac{1}{(n+3)(n+2)(n+1)} \left(c_n - 3 \sum_{k=n+1}^{s} (-1)^{k-n+4} \binom{k+3}{n} (1 - 2^{k-n+3} + 3^{k-n+2}) A_k \right) \\ &, \end{aligned}$$

for $n = s-1, s-2, \ldots, 1, 0$. *The evaluation of* $W_0^{(p)}, W_1^{(p)}, W_2^{(p)}$ *and the decomposition of* $V_n(W_0^{(p)}, W_1^{(p)}, W_2^{(p)})$ *into* B_1, B_2, B_3 *terms are analogous to case (a).*

Here

$$W_0^{(p)} = 0, \quad W_1^{(p)} = \sum_{i=0}^{s} A_i, \quad W_2^{(p)} = 2^3 \sum_{i=0}^{s} 2^i A_i,$$

and

$$\begin{aligned} V_n(W_0^{(p)}, W_1^{(p)}, W_2^{(p)}) &= V_n(0, \sum_{i=0}^{s} A_i, 2^3 \sum_{i=0}^{s} 2^i A_i) \\ &= B_1 V_n(W_0, W_1, W_2) + B_2 V_{n-1}(W_0, W_1, W_2) + B_3 V_{n-2}(W_0, W_1, W_2) \end{aligned}$$

where

$$B_1 = \frac{Y_1}{\Delta}, \; B_2 = \frac{Y_2}{\Delta}, \; B_3 = \frac{Y_3}{\Delta}$$

and

$Y_1 = (W_2^2 + 9W_1^2 + 3W_0^2 - 6W_1W_2 + 3W_0W_2 - 10W_0W_1)W_2^{(p)} + (-8W_1^2 - 3W_0^2 + 3W_1W_2 - W_0W_2 + 9W_0W_1)W_1^{(p)}$

$Y_2 = (-8W_1^2 + 3W_2W_1 + 3W_0W_1 - 3W_0^2 + 6W_0W_1 - W_0W_2)W_2^{(p)} + (-3W_2^2 + 2W_1W_2 + W_0^2 + 6W_1W_2 - 3W_0W_2 - 3W_0W_2)W_1^{(p)}$

$Y_3 = (-W_1W_2 + 3W_1^2 - 3W_0W_1 + W_0^2)W_2^{(p)} + (W_2^2 - 3W_1W_2 + 3W_0W_2 - W_0W_1)W_1^{(p)}$

$\Delta = W_2^3 - 8W_1^3 + W_0^3 - 6W_1W_2^2 + 12W_1^2W_2 + 3W_0W_2^2 + 3W_0^2W_2 + 12W_0W_1^2 - 6W_0^2W_1 - 12W_0W_1W_2$

In summary, the solution of (1.1) is given by

$$W_n(W_0, W_1, W_2) = (-B_1 + 1)V_n(W_0, W_1, W_2) - B_2V_{n-1}(W_0, W_1, W_2) - B_3V_{n-2}(W_0, W_1, W_2) + n^3 \sum_{i=0}^{s} A_i n^i$$

Proof. The result is obtained by combining Theorem 5.5 (p. 104), Theorem 5.6 (pp. 104–105), and Theorem 3.1 (pp. 88–89) for the case $m = 3$, as established in Soykan [18]. □

Research Objectives. In light of the historical development and structural properties of Leonardo-type sequences, the present study sets out with several clear objectives. First, it aims to derive closed-form solutions for third-order nonhomogeneous linear recurrence relations with polynomial inputs, thereby extending the scope of classical recurrence theory. Second, it seeks to examine systematically the influence of root multiplicity ($r = 0, 1, 2, 3$) on the form of the solutions, clarifying how resonance phenomena and multiplicity corrections shape the particular component. Third, the analysis provides explicit formulas for polynomial inputs of degree $s = 0, 1, 2, 3$, ensuring that all fundamental cases are covered within a unified framework. Finally, the study emphasizes both theoretical and pedagogical significance, demonstrating how these results enrich symbolic recurrence theory while offering accessible examples for teaching and interdisciplinary applications. These objectives guide the structure of the paper and are summarized in the abstract to ensure clarity of scope and contribution.

2. Closed-Form Solutions via Theorem 1.1 for Special Cases of $p(n)$

In this section, we present a systematic application of Theorem 1.1 to obtain explicit closed-form solutions of the recurrence relation for different configurations of the characteristic roots and polynomial inputs. The analysis is organized according to the multiplicity r of 1 as a root of the characteristic equation (1.2), with $r = 0, 1, 2, 3$, and for each case we consider the polynomial $p(n)$ of degree $s = 0, 1, 2, 3$ in (1.1). This unified framework demonstrates how the interplay between r and s shapes the form of the particular solution, while the homogeneous component remains governed by the same recurrence relation. The subsections that follow provide detailed examples for each case, highlighting the explicit closed-form solutions and the iterative determination of the coefficients A_i.

The Case: $\theta_1 \neq 1$, $\theta_2 \neq 1$, and $\theta_3 \neq 1$. That is, all three roots of the characteristic equation (1.2) are distinct from 1. This situation corresponds to the baseline case $r = 0$, where the particular solution can be constructed directly without additional powers of n. It provides the simplest framework for illustrating the decomposition into homogeneous and particular components, and serves as the starting point for comparison with the cases $r = 1, 2, 3$. In the example below, we will derive closed-form solutions for the special cases $s = 0, 1, 2, 3$ by applying Theorem 1.1.

EXAMPLE 2.1. *In this example, in each case, the homogeneous relation corresponding to the given non-homogeneous recurrence relation is expressed as in (1.2), i.e.,*

$$V_n = a_1 V_{n-1} + a_2 V_{n-2} + a_3 V_{n-3}.$$

(a): *The case $s = 0$: ($a_3 \neq 0$, $c_0 \neq 0$).*

Let the sequence $\{W_n\}$ defined by nonhomogeneous linear recurrence relation

$$W_n = a_1 W_{n-1} + a_2 W_{n-2} + a_3 W_{n-3} + c_0. \tag{2.1}$$

Then the solution of (2.1) is given by

$$\begin{aligned} W_n(W_0, W_1, W_2) &= W_n^{(h)} + W_n^{(p)} \\ &= V_n(W_0, W_1, W_2) - V_n(W_0^{(p)}, W_1^{(p)}, W_2^{(p)}) + W_n^{(p)} \end{aligned}$$

where $W_n^{(h)} = V_n(W_0, W_1, W_2) - V_n(W_0^{(p)}, W_1^{(p)}, W_2^{(p)})$ is the solution of (1.2) and

$$W_n^{(p)} = A_0 = -\frac{c_0}{a_1 + a_2 + a_3 - 1}$$

is the particular solution of (2.1). Here,

$$W_0^{(p)} = A_0,\ W_1^{(p)} = A_0,\ W_2^{(p)} = A_0,$$

and

$$\begin{aligned} V_n(W_0^{(p)}, W_1^{(p)}, W_2^{(p)}) &= V_n(A_0, A_0, A_0) \\ &= B_1 V_n(W_0, W_1, W_2) + B_2 V_{n-1}(W_0, W_1, W_2) + B_3 V_{n-2}(W_0, W_1, W_2) \end{aligned}$$

where

$$B_1 = \frac{Y_1}{\Delta},\ B_2 = \frac{Y_2}{\Delta},\ B_3 = \frac{Y_3}{\Delta}$$

and

$Y_1 = (W_2^2 + a_1^2 W_1^2 + a_1 a_3 W_0^2 - 2a_1 W_1 W_2 - a_2 W_0 W_2 + (a_1 a_2 - a_3) W_0 W_1) A_0 + ((a_3 + a_1 a_2) W_1^2 + a_2 a_3 W_0^2 - a_2 W_1 W_2 - a_3 W_0 W_2 + a_2^2 W_0 W_1) A_0 + a_3 (a_1 W_1^2 + a_3 W_0^2 - W_1 W_2 + a_2 W_0 W_1) A_0$

$Y_2 = ((a_3 + a_1 a_2) W_1^2 - a_2 W_2 W_1 + (a_2^2 - 2a_3 a_1) W_0 W_1 + a_3 a_2 W_0^2 + 2a_3 a_1 W_0 W_1 - a_3 W_0 W_2) A_0 + (a_2 W_2^2 + 2a_3 W_1 W_2 + a_3^2 W_0^2 - (3a_3 + a_1 a_2) W_1 W_2 - (a_2^2 - 2a_3 a_1) W_0 W_2 - a_3 a_1 W_0 W_2) A_0 + (a_3 W_2^2 - a_3 a_2 W_0 W_2 - a_3 a_1 W_1 W_2 - a_3^2 W_0 W_1) A_0$

$Y_3 = a_3(-W_1 W_2 + a_1 W_1^2 + a_2 W_0 W_1 + a_3 W_0^2) A_0 + a_3 (W_2^2 - a_1 W_1 W_2 - a_2 W_0 W_2 - a_3 W_0 W_1) A_0 + a_3 (a_3 W_1^2 - a_3 W_0 W_2) A_0$

$\Delta = W_2^3 + (a_3 + a_1 a_2) W_1^3 + a_3^2 W_0^3 - 2a_1 W_1 W_2^2 + (a_1^2 - a_2) W_1^2 W_2 - a_2 W_0 W_2^2 + a_3 a_1 W_0^2 W_2 + (a_2^2 + a_1 a_3) W_0 W_1^2 + 2a_2 a_3 W_0^2 W_1 + (-3a_3 + a_1 a_2) W_0 W_1 W_2$

In summary, the solution of (2.1) is given by

$$W_n(W_0, W_1, W_2) = (-B_1 + 1) V_n(W_0, W_1, W_2) - B_2 V_{n-1}(W_0, W_1, W_2) - B_3 V_{n-2}(W_0, W_1, W_2) + A_0$$

(b): *The case $s = 1$: ($a_3 \neq 0$, $c_1 \neq 0$).*

Let the sequence $\{W_n\}$ defined by nonhomogeneous linear recurrence relation

$$W_n = a_1 W_{n-1} + a_2 W_{n-2} + a_3 W_{n-3} + c_1 n + c_0. \tag{2.2}$$

Then the solution of (2.2) is given by

$$\begin{aligned} W_n(W_0, W_1, W_2) &= W_n^{(h)} + W_n^{(p)} \\ &= V_n(W_0, W_1, W_2) - V_n(W_0^{(p)}, W_1^{(p)}, W_2^{(p)}) + W_n^{(p)} \end{aligned}$$

where $W_n^{(h)} = V_n(W_0, W_1, W_2) - V_n(W_0^{(p)}, W_1^{(p)}, W_2^{(p)})$ is the solution of (1.2) and

$$W_n^{(p)} = A_1 n + A_0$$

is the particular solution of (2.2). Here,

$$\begin{aligned} A_1 &= -\frac{c_1}{a_1 + a_2 + a_3 - 1} \\ A_0 &= -\frac{1}{a_1 + a_2 + a_3 - 1}(c_0 - (a_1 + 2a_2 + 3a_3) A_1) \end{aligned}$$

i.e.,

$$\begin{aligned} A_1 &= -\frac{c_1}{a_1+a_2+a_3-1} \\ A_0 &= -\frac{1}{(a_1+a_2+a_3-1)^2}(-c_0+a_1(c_0+c_1)+a_2(c_0+2c_1)+a_3(c_0+3c_1)) \end{aligned}$$

i.e.,

$$\begin{aligned} A_1 &= -\frac{c_1}{a_1+a_2+a_3-1} \\ A_0 &= -\frac{1}{(a_1+a_2+a_3-1)^2}((a_1+a_2+a_3-1)\,c_0+(a_1+2a_2+3a_3)c_1) \end{aligned}$$

and

$$W_0^{(p)} = A_0,\ W_1^{(p)} = A_0 + A_1,\ W_2^{(p)} = A_0 + 2A_1,$$

and

$$\begin{aligned} V_n(W_0^{(p)}, W_1^{(p)}, W_2^{(p)}) &= V_n(A_0, A_0 + A_1, A_0 + 2A_1) \\ &= B_1V_n(W_0, W_1, W_2) + B_2V_{n-1}(W_0, W_1, W_2) + B_3V_{n-2}(W_0, W_1, W_2) \end{aligned}$$

where

$$B_1 = \frac{Y_1}{\Delta},\ B_2 = \frac{Y_2}{\Delta},\ B_3 = \frac{Y_3}{\Delta}$$

and

$Y_1 = (W_2^2 + a_1^2W_1^2 + a_1a_3W_0^2 - 2a_1W_1W_2 - a_2W_0W_2 + (a_1a_2 - a_3)W_0W_1)(A_0 + 2A_1) + ((a_3 + a_1a_2)W_1^2 + a_2a_3W_0^2 - a_2W_1W_2 - a_3W_0W_2 + a_2^2W_0W_1)(A_0 + A_1) + a_3(a_1W_1^2 + a_3W_0^2 - W_1W_2 + a_2W_0W_1)A_0$

$Y_2 = ((a_3+a_1a_2)W_1^2 - a_2W_2W_1 + (a_2^2-2a_3a_1)W_0W_1 + a_3a_2W_0^2 + 2a_3a_1W_0W_1 - a_3W_0W_2)(A_0 + 2A_1) + (a_2W_2^2 + 2a_3W_1W_2 + a_3^2W_0^2 - (3a_3 + a_1a_2)W_1W_2 - (a_2^2 - 2a_3a_1)W_0W_2 - a_3a_1W_0W_2)(A_0 + A_1) + (a_3W_2^2 - a_3a_2W_0W_2 - a_3a_1W_1W_2 - a_3^2W_0W_1)A_0$

$Y_3 = a_3(-W_1W_2 + a_1W_1^2 + a_2W_0W_1 + a_3W_0^2)(A_0 + 2A_1) + a_3(W_2^2 - a_1W_1W_2 - a_2W_0W_2 - a_3W_0W_1)(A_0 + A_1) + a_3(a_3W_1^2 - a_3W_0W_2)A_0$

$\Delta = W_2^3 + (a_3 + a_1a_2)W_1^3 + a_3^2W_0^3 - 2a_1W_1W_2^2 + (a_1^2 - a_2)W_1^2W_2 - a_2W_0W_2^2 + a_3a_1W_0^2W_2 + (a_2^2 + a_1a_3)W_0W_1^2 + 2a_2a_3W_0^2W_1 + (-3a_3 + a_1a_2)W_0W_1W_2$

In summary, the solution of (2.2) is given by

$$W_n(W_0, W_1, W_2) = (-B_1 + 1)V_n(W_0, W_1, W_2) - B_2V_{n-1}(W_0, W_1, W_2) - B_3V_{n-2}(W_0, W_1, W_2) + A_1n + A_0$$

(c): *The case* $s = 2$ *:* *(*$a_3 \neq 0,\ c_2 \neq 0$*).*

Let the sequence $\{W_n\}$ *defined by nonhomogeneous linear recurrence relation*

$$W_n = a_1W_{n-1} + a_2W_{n-2} + a_3W_{n-3} + c_2n^2 + c_1n + c_0. \tag{2.3}$$

Then the solution of (2.3) is given by

$$
\begin{aligned}
W_n(W_0,W_1,W_2) &= W_n^{(h)}+W_n^{(p)} \\
&= V_n(W_0,W_1,W_2)-V_n(W_0^{(p)},W_1^{(p)},W_2^{(p)})+W_n^{(p)}
\end{aligned}
$$

where $W_n^{(h)}=V_n(W_0,W_1,W_2)-V_n(W_0^{(p)},W_1^{(p)},W_2^{(p)})$ *is the solution of (1.2) and*

$$W_n^{(p)}=A_2n^2+A_1n+A_0$$

is the particular solution of (2.3). Here,

$$
\begin{aligned}
A_2 &= -\frac{c_2}{a_1+a_2+a_3-1} \\
A_1 &= -\frac{1}{a_1+a_2+a_3-1}(c_1-2\left(a_1+2a_2+3a_3\right)A_2) \\
A_0 &= -\frac{1}{a_1+a_2+a_3-1}(c_0-(a_1+2a_2+3a_3)A_1+(a_1+4a_2+9a_3)A_2)
\end{aligned}
$$

i.e.,

$A_2=-\dfrac{c_2}{a_1+a_2+a_3-1}$

$A_1=-\dfrac{1}{(a_1+a_2+a_3-1)^2}(-c_1+a_1(2c_2+c_1)+a_2(4c_2+c_1)+a_3(6c_2+c_1))$

$A_0=-\dfrac{1}{(a_1+a_2+a_3-1)^3}(c_0+a_1(c_2-c_1-2c_0)+2a_2(2c_2-c_1-c_0)+a_3(9c_2-3c_1-2c_0)+a_1^2(c_2+c_1+c_0)+a_2^2(4c_2+2c_1+c_0)+a_3^2(9c_2+3c_1+c_0)+a_1a_2(3c_2+3c_1+2c_0)+2a_1a_3(c_2+2c_1+c_0)+a_2a_3(11c_2+5c_1+2c_0))$

i.e.,

$A_2=-\frac{c_2}{a_1+a_2+a_3-1}$

$A_1=-\frac{1}{(a_1+a_2+a_3-1)^2}((a_1+a_2+a_3-1)c_1+2(a_1+2a_2+3a_3)c_2)$

$A_0=-\frac{1}{(a_1+a_2+a_3-1)^3}((a_1+a_2+a_3-1)^2c_0+(a_1+a_2+a_3-1)(a_1+2a_2+3a_3)c_1+((a_1+2a_2+3a_3)^2+(a_1+4a_2+9a_3)-(a_1a_2+4a_1a_3+a_2a_3))c_2)$

and

$$W_0^{(p)}=A_0,\ W_1^{(p)}=A_0+A_1+A_2,\ W_2^{(p)}=A_0+2A_1+4A_2,$$

and

$$
\begin{aligned}
V_n(W_0^{(p)},W_1^{(p)},W_2^{(p)}) &= V_n(A_0,A_0+A_1+A_2,A_0+2A_1+4A_2) \\
&= B_1V_n(W_0,W_1,W_2)+B_2V_{n-1}(W_0,W_1,W_2)+B_3V_{n-2}(W_0,W_1,W_2)
\end{aligned}
$$

where

$$B_1=\frac{Y_1}{\Delta},\ B_2=\frac{Y_2}{\Delta},\ B_3=\frac{Y_3}{\Delta}$$

and

$Y_1 = (W_2^2 + a_1^2 W_1^2 + a_1 a_3 W_0^2 - 2a_1 W_1 W_2 - a_2 W_0 W_2 + (a_1 a_2 - a_3) W_0 W_1)(A_0 + 2A_1 + 4A_2) + ((a_3 + a_1 a_2) W_1^2 + a_2 a_3 W_0^2 - a_2 W_1 W_2 - a_3 W_0 W_2 + a_2^2 W_0 W_1)(A_0 + A_1 + A_2) + a_3(a_1 W_1^2 + a_3 W_0^2 - W_1 W_2 + a_2 W_0 W_1) A_0$

$Y_2 = ((a_3 + a_1 a_2) W_1^2 - a_2 W_2 W_1 + (a_2^2 - 2a_3 a_1) W_0 W_1 + a_3 a_2 W_0^2 + 2a_3 a_1 W_0 W_1 - a_3 W_0 W_2)(A_0 + 2A_1 + 4A_2) + (a_2 W_2^2 + 2a_3 W_1 W_2 + a_3^2 W_0^2 - (3a_3 + a_1 a_2) W_1 W_2 - (a_2^2 - 2a_3 a_1) W_0 W_2 - a_3 a_1 W_0 W_2)(A_0 + A_1 + A_2) + (a_3 W_2^2 - a_3 a_2 W_0 W_2 - a_3 a_1 W_1 W_2 - a_3^2 W_0 W_1) A_0$

$Y_3 = a_3(-W_1 W_2 + a_1 W_1^2 + a_2 W_0 W_1 + a_3 W_0^2)(A_0 + 2A_1 + 4A_2) + a_3(W_2^2 - a_1 W_1 W_2 - a_2 W_0 W_2 - a_3 W_0 W_1)(A_0 + A_1 + A_2) + a_3(a_3 W_1^2 - a_3 W_0 W_2) A_0$

$\Delta = W_2^3 + (a_3 + a_1 a_2) W_1^3 + a_3^2 W_0^3 - 2a_1 W_1 W_2^2 + (a_1^2 - a_2) W_1^2 W_2 - a_2 W_0 W_2^2 + a_3 a_1 W_0^2 W_2 + (a_2^2 + a_1 a_3) W_0 W_1^2 + 2a_2 a_3 W_0^2 W_1 + (-3a_3 + a_1 a_2) W_0 W_1 W_2$

In summary, the solution of (2.3) is given by

$$W_n(W_0, W_1, W_2) = (-B_1 + 1) V_n(W_0, W_1, W_2) - B_2 V_{n-1}(W_0, W_1, W_2) - B_3 V_{n-2}(W_0, W_1, W_2) + A_2 n^2 + A_1 n + A_0$$

(d): *The case $s = 3$: ($a_3 \neq 0$, $c_3 \neq 0$).*

Let the sequence $\{W_n\}$ defined by nonhomogeneous linear recurrence relation

$$W_n = a_1 W_{n-1} + a_2 W_{n-2} + a_3 W_{n-3} + c_3 n^3 + c_2 n^2 + c_1 n + c_0. \tag{2.4}$$

Then the solution of (2.4) is given by

$$\begin{aligned} W_n(W_0, W_1, W_2) &= W_n^{(h)} + W_n^{(p)} \\ &= V_n(W_0, W_1, W_2) - V_n(W_0^{(p)}, W_1^{(p)}, W_2^{(p)}) + W_n^{(p)} \end{aligned}$$

where $W_n^{(h)} = V_n(W_0, W_1, W_2) - V_n(W_0^{(p)}, W_1^{(p)}, W_2^{(p)})$ is the solution of (1.2) and

$$W_n^{(p)} = A_3 n^3 + A_2 n^2 + A_1 n + A_0$$

is the particular solution of (2.4). Here,

$$\begin{aligned} A_3 &= -\frac{c_3}{a_1 + a_2 + a_3 - 1} \\ A_2 &= -\frac{1}{a_1 + a_2 + a_3 - 1}(c_2 - 3(a_1 + 2a_2 + 3a_3) A_3) \\ A_1 &= -\frac{1}{a_1 + a_2 + a_3 - 1}(c_1 - 2(a_1 + 2a_2 + 3a_3) A_2 + 3(a_1 + 4a_2 + 9a_3) A_3) \\ A_0 &= -\frac{1}{a_1 + a_2 + a_3 - 1}(c_0 - (a_1 + 2a_2 + 3a_3) A_1 + (a_1 + 4a_2 + 9a_3) A_2 - (a_1 + 8a_2 + 27a_3) A_3) \end{aligned}$$

i.e.,

$A_3 = -\frac{c_3}{a_1 + a_2 + a_3 - 1}$

$A_2 = -\frac{1}{(a_1 + a_2 + a_3 - 1)^2}(-c_2 + a_1(c_2 + 3c_3) + a_2(c_2 + 6c_3) + a_3(c_2 + 9c_3))$

$A_1 = -\frac{1}{(a_1+a_2+a_3-1)^3}(c_1 - a_1(2c_1+2c_2-3c_3) + a_1^2(c_1+2c_2+3c_3) - 2a_2(c_1+2c_2-6c_3) + a_2^2(c_1+4c_2+12c_3) - a_3(2c_1+6c_2-27c_3) + a_3^2(c_1+6c_2+27c_3) + a_1a_2(2c_1+6c_2+9c_3) + 2a_1a_3(c_1+4c_2+3c_3) + a_2a_3(2c_1+10c_2+33c_3))$

$A_0 = -\dfrac{1}{(a_1+a_2+a_3-1)^4}(-c_0 + a_1(3c_0+c_1-c_2+c_3) - a_1^2(3c_0+2c_1-4c_3) + a_1^3(c_0+c_1+c_2+c_3) + a_2(3c_0+2c_1-4c_2+8c_3) - a_2^2(3c_0+4c_1-32c_3) + a_2^3(c_0+2c_1+4c_2+8c_3) + 3a_3(c_0+c_1-3c_2+9c_3) - 3a_3^2(c_0+2c_1-36c_3) + a_3^3(c_0+3c_1+9c_2+27c_3) + a_1a_2^2(3c_0+5c_1+7c_2+5c_3) + a_1^2a_2(3c_0+4c_1+4c_2+4c_3) + a_1^2a_3(3c_0+5c_1+3c_2+5c_3) + a_1a_3^2(3c_0+7c_1+11c_2-17c_3) + a_2^2a_3(3c_0+7c_1+15c_2+31c_3) + a_2a_3^2(3c_0+8c_1+20c_2+44c_3) - 2a_1a_2(3c_0+3c_1-c_2-9c_3) - 2a_1a_3(3c_0+4c_1-4c_2-8c_3) - 2a_2a_3(3c_0+5c_1-c_2-55c_3) + 2a_1a_2a_3(3c_0+6c_1+8c_2))$

and

$$W_0^{(p)} = A_0,\ W_1^{(p)} = A_0+A_1+A_2+A_3,\ W_2^{(p)} = A_0+2A_1+4A_2+8A_3,$$

and

$$\begin{aligned} V_n(W_0^{(p)}, W_1^{(p)}, W_2^{(p)}, W_3^{(p)}) &= V_n(A_0, A_0+A_1+A_2+A_3, A_0+2A_1+4A_2+8A_3) \\ &= B_1V_n(W_0,W_1,W_2) + B_2V_{n-1}(W_0,W_1,W_2) + B_3V_{n-2}(W_0,W_1,W_2) \end{aligned}$$

where

$$B_1 = \frac{Y_1}{\Delta},\ B_2 = \frac{Y_2}{\Delta},\ B_3 = \frac{Y_3}{\Delta}$$

and

$Y_1 = (W_2^2 + a_1^2W_1^2 + a_1a_3W_0^2 - 2a_1W_1W_2 - a_2W_0W_2 + (a_1a_2-a_3)W_0W_1)(A_0+2A_1+4A_2+8A_3) + ((a_3+a_1a_2)W_1^2 + a_2a_3W_0^2 - a_2W_1W_2 - a_3W_0W_2 + a_2^2W_0W_1)(A_0+A_1+A_2+A_3) + a_3(a_1W_1^2 + a_3W_0^2 - W_1W_2 + a_2W_0W_1)A_0$

$Y_2 = ((a_3+a_1a_2)W_1^2 - a_2W_2W_1 + (a_2^2-2a_3a_1)W_0W_1 + a_3a_2W_0^2 + 2a_3a_1W_0W_1 - a_3W_0W_2)(A_0+2A_1+4A_2+8A_3) + (a_2W_2^2 + 2a_3W_1W_2 + a_3^2W_0^2 - (3a_3+a_1a_2)W_1W_2 - (a_2^2-2a_3a_1)W_0W_2 - a_3a_1W_0W_2)(A_0+A_1+A_2+A_3) + (a_3W_2^2 - a_3a_2W_0W_2 - a_3a_1W_1W_2 - a_3^2W_0W_1)A_0$

$Y_3 = a_3(-W_1W_2 + a_1W_1^2 + a_2W_0W_1 + a_3W_0^2)(A_0+2A_1+4A_2+8A_3) + a_3(W_2^2 - a_1W_1W_2 - a_2 W_0W_2 - a_3W_0W_1)(A_0+A_1+A_2+A_3) + a_3(a_3W_1^2 - a_3W_0W_2)A_0$

$\Delta = W_2^3 + (a_3+a_1a_2)W_1^3 + a_3^2W_0^3 - 2a_1W_1W_2^2 + (a_1^2-a_2)W_1^2W_2 - a_2W_0W_2^2 + a_3a_1W_0^2W_2 + (a_2^2+a_1a_3)W_0W_1^2 + 2a_2a_3W_0^2W_1 + (-3a_3+a_1a_2)W_0W_1W_2$

In summary, the solution of (2.4) is given by

$$W_n(W_0,W_1,W_2) = (-B_1+1)V_n(W_0,W_1,W_2) - B_2V_{n-1}(W_0,W_1,W_2) - B_3V_{n-2}(W_0,W_1,W_2) + A_3n^3 + A_2n^2 + A_1n + A_0$$

The Case: $\theta_1 = 1$, $\theta_2 \neq 1$, and $\theta_3 \neq 1$. That is, exactly one of the roots of the characteristic equation (1.2) is equal to 1. This situation corresponds to the case $r = 1$, where the presence of a simple root at 1 introduces a linear factor n into the particular solution. Compared to the baseline case $r = 0$, the construction of the coefficients A_i requires a modified iterative scheme, reflecting the influence of the root 1 on the recurrence. In the following example, we will obtain closed-form solutions for the cases $s = 0, 1, 2, 3$, showing explicitly how the factor n alters the structure of the particular solution.

EXAMPLE 2.2. *In this example, in each case, the homogeneous relation corresponding to the given non-homogeneous recurrence relation is expressed as in (1.2), i.e.,*

$$V_n = a_1V_{n-1} + a_2V_{n-2} + a_3V_{n-3}.$$

(a): *The case* $s = 0 :$ *(*$a_3 \neq 0,\ c_0 \neq 0$*).*

Let the sequence $\{W_n\}$ *defined by nonhomogeneous linear recurrence relation*

$$W_n = a_1W_{n-1} + a_2W_{n-2} + a_3W_{n-3} + c_0. \tag{2.5}$$

Then the solution of (2.5) is given by

$$\begin{aligned} W_n(W_0, W_1, W_2) &= W_n^{(h)} + W_n^{(p)} \\ &= V_n(W_0, W_1, W_2) - V_n(W_0^{(p)}, W_1^{(p)}, W_2^{(p)}) + W_n^{(p)} \end{aligned}$$

where $W_n^{(h)} = V_n(W_0, W_1, W_2) - V_n(W_0^{(p)}, W_1^{(p)}, W_2^{(p)})$ *is the solution of (1.2) and*

$$W_n^{(p)} = nA_0$$

is the particular solution of (2.5). Here,

$$A_0 = \frac{c_0}{a_1 + 2a_2 + 3a_3}$$

and

$$W_0^{(p)} = 0,\ W_1^{(p)} = A_0,\ W_2^{(p)} = 2A_0,$$

$$\begin{aligned} V_n(W_0^{(p)}, W_1^{(p)}, W_2^{(p)}) &= V_n(0, A_0, 2A_0) \\ &= B_1V_n(W_0, W_1, W_2) + B_2V_{n-1}(W_0, W_1, W_2) + B_3V_{n-2}(W_0, W_1, W_2) \end{aligned}$$

where

$$B_1 = \frac{Y_1}{\Delta},\ B_2 = \frac{Y_2}{\Delta},\ B_3 = \frac{Y_3}{\Delta}$$

and

$Y_1 = (W_2^2 + a_1^2W_1^2 + a_1a_3W_0^2 - 2a_1W_1W_2 - a_2W_0W_2 + (a_1a_2 - a_3)W_0W_1) \times 2A_0 + ((a_3 + a_1a_2)W_1^2 + a_2a_3W_0^2 - a_2W_1W_2 - a_3W_0W_2 + a_2^2W_0W_1)A_0$

$Y_2 = ((a_3 + a_1a_2)W_1^2 - a_2W_2W_1 + (a_2^2 - 2a_3a_1)W_0W_1 + a_3a_2W_0^2 + 2a_3a_1W_0W_1 - a_3W_0W_2) \times 2A_0 + (a_2W_2^2 + 2a_3W_1W_2 + a_3^2W_0^2 - (3a_3 + a_1a_2)W_1W_2 - (a_2^2 - 2a_3a_1)W_0W_2 - a_3a_1W_0W_2)A_0$

$Y_3 = a_3(-W_1W_2 + a_1W_1^2 + a_2W_0W_1 + a_3W_0^2) \times 2A_0 + a_3(W_2^2 - a_1W_1W_2 - a_2W_0W_2 - a_3W_0W_1)A_0$

$\Delta = W_2^3 + (a_3 + a_1a_2)W_1^3 + a_3^2W_0^3 - 2a_1W_1W_2^2 + (a_1^2 - a_2)W_1^2W_2 - a_2W_0W_2^2 + a_3a_1W_0^2W_2 + (a_2^2 + a_1a_3)W_0W_1^2 + 2a_2a_3W_0^2W_1 + (-3a_3 + a_1a_2)W_0W_1W_2$

In summary, the solution of (2.5) is given by

$$W_n(W_0,W_1,W_2) = (-B_1+1)V_n(W_0,W_1,W_2) - B_2V_{n-1}(W_0,W_1,W_2) - B_3V_{n-2}(W_0,W_1,W_2) + nA_0$$

(b): *The case $s=1$: ($a_3 \neq 0$, $c_1 \neq 0$).*

Let the sequence $\{W_n\}$ defined by nonhomogeneous linear recurrence relation

$$W_n = a_1W_{n-1} + a_2W_{n-2} + a_3W_{n-3} + c_1n + c_0. \tag{2.6}$$

Then the solution of (2.6) is given by

$$\begin{aligned} W_n(W_0,W_1,W_2) &= W_n^{(h)} + W_n^{(p)} \\ &= V_n(W_0,W_1,W_2) - V_n(W_0^{(p)},W_1^{(p)},W_2^{(p)}) + W_n^{(p)} \end{aligned}$$

where $W_n^{(h)} = V_n(W_0,W_1,W_2) - V_n(W_0^{(p)},W_1^{(p)},W_2^{(p)})$ is the solution of (1.2) and

$$W_n^{(p)} = n(A_1n + A_0) = A_1n^2 + A_0n$$

is the particular solution of (2.6). Here,

$$\begin{aligned} A_1 &= \frac{c_1}{2\left(a_1+2a_2+3a_3\right)}, \\ A_0 &= \frac{1}{\left(a_1+2a_2+3a_3\right)}(c_0 + \left(a_1+4a_2+9a_3\right)A_1), \end{aligned}$$

i.e.,

$$\begin{aligned} A_1 &= \frac{c_1}{2\left(a_1+2a_2+3a_3\right)}, \\ A_0 &= \frac{1}{2\left(a_1+2a_2+3a_3\right)^2}(a_1(2c_0+c_1) + 4a_2(c_0+c_1) + 3a_3(2c_0+3c_1)), \end{aligned}$$

and

$$W_0^{(p)} = 0,\ W_1^{(p)} = A_1 + A_0,\ W_2^{(p)} = 4A_1 + 2A_0,$$

and

$$\begin{aligned} V_n(W_0^{(p)},W_1^{(p)},W_2^{(p)}) &= V_n(0, A_1+A_0, 4A_1+2A_0) \\ &= B_1V_n(W_0,W_1,W_2) + B_2V_{n-1}(W_0,W_1,W_2) + B_3V_{n-2}(W_0,W_1,W_2) \end{aligned}$$

where

$$B_1 = \frac{Y_1}{\Delta},\ B_2 = \frac{Y_2}{\Delta},\ B_3 = \frac{Y_3}{\Delta}$$

and

$Y_1 = (W_2^2 + a_1^2W_1^2 + a_1a_3W_0^2 - 2a_1W_1W_2 - a_2W_0W_2 + (a_1a_2 - a_3)W_0W_1)(4A_1 + 2A_0) + ((a_3 + a_1a_2)W_1^2 + a_2a_3W_0^2 - a_2W_1W_2 - a_3W_0W_2 + a_2^2W_0W_1)(A_1 + A_0)$

$Y_2 = ((a_3+a_1a_2)W_1^2 - a_2W_2W_1 + (a_2^2 - 2a_3a_1)W_0W_1 + a_3a_2W_0^2 + 2a_3a_1W_0W_1 - a_3W_0W_2)(4A_1 + 2A_0) + (a_2W_2^2 + 2a_3W_1W_2 + a_3^2W_0^2 - (3a_3 + a_1a_2)W_1W_2 - (a_2^2 - 2a_3a_1)W_0W_2 - a_3a_1W_0W_2)(A_1 + A_0)$

$Y_3 = a_3(-W_1W_2 + a_1W_1^2 + a_2W_0W_1 + a_3W_0^2)(4A_1 + 2A_0) + a_3(W_2^2 - a_1W_1W_2 - a_2W_0W_2 - a_3W_0W_1)(A_1 + A_0)$

$\Delta = W_2^3 + (a_3 + a_1a_2)W_1^3 + a_3^2W_0^3 - 2a_1W_1W_2^2 + (a_1^2 - a_2)W_1^2W_2 - a_2W_0W_2^2 + a_3a_1W_0^2W_2 + (a_2^2 + a_1a_3)W_0W_1^2 + 2a_2a_3W_0^2W_1 + (-3a_3 + a_1a_2)W_0W_1W_2$

In summary, the solution of (2.6) is given by

$$W_n(W_0, W_1, W_2) = (-B_1 + 1)V_n(W_0, W_1, W_2) - B_2V_{n-1}(W_0, W_1, W_2) - B_3V_{n-2}(W_0, W_1, W_2) + A_1n^2 + A_0n$$

(c): *The case $s = 2$: ($a_3 \neq 0$, $c_2 \neq 0$).*

Let the sequence $\{W_n\}$ defined by nonhomogeneous linear recurrence relation

$$W_n = a_1W_{n-1} + a_2W_{n-2} + a_3W_{n-3} + c_2n^2 + c_1n + c_0. \tag{2.7}$$

Then the solution of (2.7) is given by

$$\begin{aligned} W_n(W_0, W_1, W_2) &= W_n^{(h)} + W_n^{(p)} \\ &= V_n(W_0, W_1, W_2) - V_n(W_0^{(p)}, W_1^{(p)}, W_2^{(p)}) + W_n^{(p)} \end{aligned}$$

where $W_n^{(h)} = V_n(W_0, W_1, W_2) - V_n(W_0^{(p)}, W_1^{(p)}, W_2^{(p)})$ is the solution of (1.2) and

$$W_n^{(p)} = n(A_2n^2 + A_1n + A_0) = A_2n^3 + A_1n^2 + A_0n$$

is the particular solution of (2.7). Here,

$$\begin{aligned} A_2 &= \frac{c_2}{3(a_1 + 2a_2 + 3a_3)}, \\ A_1 &= \frac{1}{2(a_1 + 2a_2 + 3a_3)}(c_1 + 3(a_1 + 4a_2 + 9a_3)A_2), \\ A_0 &= \frac{1}{(a_1 + 2a_2 + 3a_3)}(c_0 + (a_1 + 4a_2 + 9a_3)A_1 - (a_1 + 8a_2 + 27a_3)A_2), \end{aligned}$$

i.e.,

$A_2 = \dfrac{c_2}{3(a_1 + 2a_2 + 3a_3)}$,

$A_1 = \dfrac{1}{2(a_1 + 2a_2 + 3a_3)^2}(a_1(c_1 + c_2) + 2a_2(c_1 + 2c_2) + 3a_3(c_1 + 3c_2))$,

$A_0 = \dfrac{1}{6(a_1 + 2a_2 + 3a_3)^3}(a_1^2(6c_0 + 3c_1 + c_2) + 8a_2^2(3c_0 + 3c_1 + 2c_2) + 27a_3^2(2c_0 + 3c_1 + 3c_2) + 2a_1a_2(12c_0 + 9c_1 + 2c_2) + 6a_1a_3(6c_0 + 6c_1 - c_2) + 6a_2a_3(12c_0 + 15c_1 + 10c_2))$,

and

$$W_0^{(p)} = 0,\ W_1^{(p)} = A_2 + A_1 + A_0,\ W_2^{(p)} = 8A_2 + 4A_1 + 2A_0,$$

and

$$\begin{aligned} V_n(W_0^{(p)}, W_1^{(p)}, W_2^{(p)}) &= V_n(0, A_2 + A_1 + A_0, 8A_2 + 4A_1 + 2A_0) \\ &= B_1V_n(W_0, W_1, W_2) + B_2V_{n-1}(W_0, W_1, W_2) + B_3V_{n-2}(W_0, W_1, W_2) \end{aligned}$$

where

$$B_1=\frac{Y_1}{\Delta},\ B_2=\frac{Y_2}{\Delta},\ B_3=\frac{Y_3}{\Delta}$$

and

$Y_1=(W_2^2+a_1^2W_1^2+a_1a_3W_0^2-2a_1W_1W_2-a_2W_0W_2+(a_1a_2-a_3)W_0W_1)(8A_2+4A_1+2A_0)+((a_3+a_1a_2)W_1^2+a_2a_3W_0^2-a_2W_1W_2-a_3W_0W_2+a_2^2W_0W_1)(A_2+A_1+A_0)$

$Y_2=((a_3+a_1a_2)W_1^2-a_2W_2W_1+(a_2^2-2a_3a_1)W_0W_1+a_3a_2W_0^2+2a_3a_1W_0W_1-a_3W_0W_2)(8A_2+4A_1+2A_0)+(a_2W_2^2+2a_3W_1W_2+a_3^2W_0^2-(3a_3+a_1a_2)W_1W_2-(a_2^2-2a_3a_1)W_0W_2-a_3a_1W_0W_2)(A_2+A_1+A_0)$

$Y_3=a_3(-W_1W_2+a_1W_1^2+a_2W_0W_1+a_3W_0^2)(8A_2+4A_1+2A_0)+a_3(W_2^2-a_1W_1W_2-a_2W_0W_2-a_3W_0W_1)(A_2+A_1+A_0)$

$\Delta=W_2^3+(a_3+a_1a_2)W_1^3+a_3^2W_0^3-2a_1W_1W_2^2+(a_1^2-a_2)W_1^2W_2-a_2W_0W_2^2+a_3a_1W_0^2W_2+(a_2^2+a_1a_3)W_0W_1^2+2a_2a_3W_0^2W_1+(-3a_3+a_1a_2)W_0W_1W_2$

In summary, the solution of (2.7) is given by

$$W_n(W_0,W_1,W_2)=(-B_1+1)V_n(W_0,W_1,W_2)-B_2V_{n-1}(W_0,W_1,W_2)-B_3V_{n-2}(W_0,W_1,W_2)+A_2n^3+A_1n^2+A_0n$$

(d): *The case $s=3:$ ($a_3\neq 0,\ c_3\neq 0$).*

Let the sequence $\{W_n\}$ defined by nonhomogeneous linear recurrence relation

$$W_n=a_1W_{n-1}+a_2W_{n-2}+a_3W_{n-3}+c_3n^3+c_2n^2+c_1n+c_0. \tag{2.8}$$

Then the solution of (2.8) is given by

$$\begin{aligned}W_n(W_0,W_1,W_2)&=W_n^{(h)}+W_n^{(p)}\\&=V_n(W_0,W_1,W_2)-V_n(W_0^{(p)},W_1^{(p)},W_2^{(p)})+W_n^{(p)}\end{aligned}$$

where $W_n^{(h)}=V_n(W_0,W_1,W_2)-V_n(W_0^{(p)},W_1^{(p)},W_2^{(p)})$ is the solution of (1.2) and

$$W_n^{(p)}=n(A_3n^3+A_2n^2+A_1n+A_0)=A_3n^4+A_2n^3+A_1n^2+A_0n$$

is the particular solution of (2.8). Here,

$$\begin{aligned}A_3&=\frac{c_3}{4\left(a_1+2a_2+3a_3\right)},\\A_2&=\frac{1}{3\left(a_1+2a_2+3a_3\right)}(c_2+6\left(a_1+4a_2+9a_3\right)A_3),\\A_1&=\frac{1}{2\left(a_1+2a_2+3a_3\right)}(c_1+3\left(a_1+4a_2+9a_3\right)A_2-4\left(a_1+8a_2+27a_3\right)A_3),\\A_0&=\frac{1}{\left(a_1+2a_2+3a_3\right)}(c_0+\left(a_1+4a_2+9a_3\right)A_1-\left(a_1+8a_2+27a_3\right)A_2+\left(a_1+16a_2+81a_3\right)A_3),\end{aligned}$$

i.e.,

$$A_3=\frac{c_3}{4\left(a_1+2a_2+3a_3\right)},$$

$A_2 = \frac{1}{6(a_1+2a_2+3a_3)^2}(a_1(2c_2+3c_3)+4a_2(c_2+3c_3)+3a_3(2c_2+9c_3))$,

$A_1 = \frac{1}{4(a_1+2a_2+3a_3)^3}(a_1^2(2c_1+2c_2+c_3)+8a_2^2(c_1+2c_2+2c_3)+9a_3^2(2c_1+6c_2+9c_3)+4a_1a_2(2c_1+3c_2+c_3)+6a_1a_3(2c_1+4c_2-c_3)+12a_2a_3(2c_1+5c_2+5c_3))$,

$A_0 = \frac{1}{6(a_1+2a_2+3a_3)^4}(a_1^3(6c_0+3c_1+c_2)+16a_2^3(3c_0+3c_1+2c_2)+81a_3^3(2c_0+3c_1+3c_2)+6a_1a_2^2(12c_0+10c_1+4c_2-3c_3)+9a_1a_3^2(18c_0+21c_1+7c_2-30c_3)+6a_1^2a_2(6c_0+4c_1+c_2)+3a_1^2a_3(18c_0+15c_1-c_2+6c_3)+18a_2a_3^2(18c_0+24c_1+19c_2-6c_3)+6a_2^2a_3(36c_0+42c_1+28c_2-3c_3)+12a_1a_2a_3(18c_0+18c_1+5c_2-9c_3))$,

and

$$W_0^{(p)} = 0,\ W_1^{(p)} = A_3+A_2+A_1+A_0,\ W_2^{(p)} = 16A_3+8A_2+4A_1+2A_0,$$

and

$$\begin{aligned} V_n(W_0^{(p)},W_1^{(p)},W_2^{(p)}) &= V_n(0,A_3+A_2+A_1+A_0,16A_3+8A_2+4A_1+2A_0) \\ &= B_1V_n(W_0,W_1,W_2)+B_2V_{n-1}(W_0,W_1,W_2)+B_3V_{n-2}(W_0,W_1,W_2) \end{aligned}$$

where

$$B_1 = \frac{Y_1}{\Delta},\ B_2 = \frac{Y_2}{\Delta},\ B_3 = \frac{Y_3}{\Delta}$$

and

$Y_1 = (W_2^2+a_1^2W_1^2+a_1a_3W_0^2-2a_1W_1W_2-a_2W_0W_2+(a_1a_2-a_3)W_0W_1)(16A_3+8A_2+4A_1+2A_0)+((a_3+a_1a_2)W_1^2+a_2a_3W_0^2-a_2W_1W_2-a_3W_0W_2+a_2^2W_0W_1)(A_3+A_2+A_1+A_0)$

$Y_2 = ((a_3+a_1a_2)W_1^2-a_2W_2W_1+(a_2^2-2a_3a_1)W_0W_1+a_3a_2W_0^2+2a_3a_1W_0W_1-a_3W_0W_2)(16A_3+8A_2+4A_1+2A_0)+(a_2W_2^2+2a_3W_1W_2+a_3^2W_0^2-(3a_3+a_1a_2)W_1W_2-(a_2^2-2a_3a_1)W_0W_2-a_3a_1W_0W_2)(A_3+A_2+A_1+A_0)$

$Y_3 = a_3(-W_1W_2+a_1W_1^2+a_2W_0W_1+a_3W_0^2)(16A_3+8A_2+4A_1+2A_0)+a_3(W_2^2-a_1W_1W_2-a_2W_0W_2-a_3W_0W_1)(A_3+A_2+A_1+A_0)$

$\Delta = W_2^3+(a_3+a_1a_2)W_1^3+a_3^2W_0^3-2a_1W_1W_2^2+(a_1^2-a_2)W_1^2W_2-a_2W_0W_2^2+a_3a_1W_0^2W_2+(a_2^2+a_1a_3)W_0W_1^2+2a_2a_3W_0^2W_1+(-3a_3+a_1a_2)W_0W_1W_2$

In summary, the solution of (2.8) is given by

$$W_n(W_0,W_1,W_2) = (-B_1+1)V_n(W_0,W_1,W_2)-B_2V_{n-1}(W_0,W_1,W_2)-B_3V_{n-2}(W_0,W_1,W_2)+A_3n^4+A_2n^3+A_1n^2+A_0n$$

The Case: $\theta_1 = 1$, $\theta_2 = 1$, and $\theta_3 \neq 1$. That is, exactly two of the roots of the characteristic equation (1.2) are equal to 1. This situation corresponds to the case $r = 2$, where the presence of a double root at 1 introduces the quadratic factor n^2 into the particular solution. Compared to the cases $r = 0$ and $r = 1$, the iterative construction of the coefficients A_i requires further modification, reflecting the stronger influence of the repeated root. In the next example, we will derive closed-form solutions for the cases $s = 0, 1, 2, 3$, illustrating how the quadratic factor n^2 modifies the particular solution.

Example 2.3. *In this example, in each case, the homogeneous relation corresponding to the given non-homogeneous recurrence relation is expressed as in (1.2), i.e.,*

$$V_n = a_1V_{n-1} + a_2V_{n-2} + a_3V_{n-3}.$$

(a): *The case $s = 0$: ($a_3 \neq 0$, $c_0 \neq 0$).*

Let the sequence $\{W_n\}$ defined by nonhomogeneous linear recurrence relation

$$W_n = a_1W_{n-1} + a_2W_{n-2} + a_3W_{n-3} + c_0. \tag{2.9}$$

Then the solution of (2.9) is given by

$$\begin{aligned} W_n(W_0, W_1, W_2) &= W_n^{(h)} + W_n^{(p)} \\ &= V_n(W_0, W_1, W_2) - V_n(W_0^{(p)}, W_1^{(p)}, W_2^{(p)}) + W_n^{(p)} \end{aligned}$$

where $W_n^{(h)} = V_n(W_0, W_1, W_2) - V_n(W_0^{(p)}, W_1^{(p)}, W_2^{(p)})$ is the solution of (1.2) and

$$W_n^{(p)} = n^2A_0$$

is the particular solution of (2.9). Here,

$$A_0 = -\frac{c_0}{a_1 + 4a_2 + 9a_3}$$

and

$$W_0^{(p)} = 0,\ W_1^{(p)} = A_0,\ W_2^{(p)} = 4A_0,$$

$$\begin{aligned} V_n(W_0^{(p)}, W_1^{(p)}, W_2^{(p)}) &= V_n(0, A_0, 4A_0) \\ &= B_1V_n(W_0, W_1, W_2) + B_2V_{n-1}(W_0, W_1, W_2) + B_3V_{n-2}(W_0, W_1, W_2) \end{aligned}$$

where

$$B_1 = \frac{Y_1}{\Delta},\ B_2 = \frac{Y_2}{\Delta},\ B_3 = \frac{Y_3}{\Delta}$$

and

$Y_1 = (W_2^2 + a_1^2W_1^2 + a_1a_3W_0^2 - 2a_1W_1W_2 - a_2W_0W_2 + (a_1a_2 - a_3)W_0W_1) \times 4A_0 + ((a_3 + a_1a_2)W_1^2 + a_2a_3W_0^2 - a_2W_1W_2 - a_3W_0W_2 + a_2^2W_0W_1)A_0$

$Y_2 = ((a_3 + a_1a_2)W_1^2 - a_2W_2W_1 + (a_2^2 - 2a_3a_1)W_0W_1 + a_3a_2W_0^2 + 2a_3a_1W_0W_1 - a_3W_0W_2) \times 4A_0 + (a_2W_2^2 + 2a_3W_1W_2 + a_3^2W_0^2 - (3a_3 + a_1a_2)W_1W_2 - (a_2^2 - 2a_3a_1)W_0W_2 - a_3a_1W_0W_2)A_0$

$Y_3 = a_3(-W_1W_2 + a_1W_1^2 + a_2W_0W_1 + a_3W_0^2) \times 4A_0 + a_3(W_2^2 - a_1W_1W_2 - a_2W_0W_2 - a_3W_0W_1)A_0$

$\Delta = W_2^3 + (a_3 + a_1a_2)W_1^3 + a_3^2W_0^3 - 2a_1W_1W_2^2 + (a_1^2 - a_2)W_1^2W_2 - a_2W_0W_2^2 + a_3a_1W_0^2W_2 + (a_2^2 + a_1a_3)W_0W_1^2 + 2a_2a_3W_0^2W_1 + (-3a_3 + a_1a_2)W_0W_1W_2$

In summary, the solution of (2.9) is given by

$$W_n(W_0, W_1, W_2) = (-B_1 + 1)V_n(W_0, W_1, W_2) - B_2V_{n-1}(W_0, W_1, W_2) - B_3V_{n-2}(W_0, W_1, W_2) + n^2A_0$$

(b): *The case $s=1$: ($a_3 \neq 0$, $c_1 \neq 0$).*

Let the sequence $\{W_n\}$ defined by nonhomogeneous linear recurrence relation

$$W_n = a_1 W_{n-1} + a_2 W_{n-2} + a_3 W_{n-3} + c_1 n + c_0. \tag{2.10}$$

Then the solution of (2.10) is given by

$$\begin{aligned} W_n(W_0, W_1, W_2) &= W_n^{(h)} + W_n^{(p)} \\ &= V_n(W_0, W_1, W_2) - V_n(W_0^{(p)}, W_1^{(p)}, W_2^{(p)}) + W_n^{(p)} \end{aligned}$$

where $W_n^{(h)} = V_n(W_0, W_1, W_2) - V_n(W_0^{(p)}, W_1^{(p)}, W_2^{(p)})$ is the solution of (1.2) and

$$W_n^{(p)} = n^2(A_1 n + A_0) = A_1 n^3 + A_0 n^2$$

is the particular solution of (2.10). Here,

$$\begin{aligned} A_1 &= -\frac{c_1}{3\left(a_1 + 4a_2 + 9a_3\right)}, \\ A_0 &= -\frac{1}{\left(a_1 + 4a_2 + 9a_3\right)}(c_0 - \left(a_1 + 8a_2 + 27a_3\right) A_1), \end{aligned}$$

i.e.,

$$\begin{aligned} A_1 &= -\frac{c_1}{3\left(a_1 + 4a_2 + 9a_3\right)}, \\ A_0 &= -\frac{1}{3\left(a_1 + 4a_2 + 9a_3\right)^2}(a_1(3c_0 + c_1) + 4a_2(3c_0 + 2c_1) + 27a_3(c_0 + c_1)), \end{aligned}$$

and

$$W_0^{(p)} = 0,\ W_1^{(p)} = A_1 + A_0,\ W_2^{(p)} = 8A_1 + 4A_0,$$

and

$$\begin{aligned} V_n(W_0^{(p)}, W_1^{(p)}, W_2^{(p)}) &= V_n(0, A_1 + A_0, 8A_1 + 4A_0) \\ &= B_1 V_n(W_0, W_1, W_2) + B_2 V_{n-1}(W_0, W_1, W_2) + B_3 V_{n-2}(W_0, W_1, W_2) \end{aligned}$$

where

$$B_1 = \frac{Y_1}{\Delta},\ B_2 = \frac{Y_2}{\Delta},\ B_3 = \frac{Y_3}{\Delta}$$

and

$Y_1 = (W_2^2 + a_1^2 W_1^2 + a_1 a_3 W_0^2 - 2a_1 W_1 W_2 - a_2 W_0 W_2 + (a_1 a_2 - a_3) W_0 W_1)(8A_1 + 4A_0) + ((a_3 + a_1 a_2) W_1^2 + a_2 a_3 W_0^2 - a_2 W_1 W_2 - a_3 W_0 W_2 + a_2^2 W_0 W_1)(A_1 + A_0)$

$Y_2 = ((a_3 + a_1 a_2) W_1^2 - a_2 W_2 W_1 + (a_2^2 - 2a_3 a_1) W_0 W_1 + a_3 a_2 W_0^2 + 2a_3 a_1 W_0 W_1 - a_3 W_0 W_2)(8A_1 + 4A_0) + (a_2 W_2^2 + 2a_3 W_1 W_2 + a_3^2 W_0^2 - (3a_3 + a_1 a_2) W_1 W_2 - (a_2^2 - 2a_3 a_1) W_0 W_2 - a_3 a_1 W_0 W_2)(A_1 + A_0)$

$Y_3 = a_3(-W_1 W_2 + a_1 W_1^2 + a_2 W_0 W_1 + a_3 W_0^2)(8A_1 + 4A_0) + a_3(W_2^2 - a_1 W_1 W_2 - a_2 W_0 W_2 - a_3 W_0 W_1)(A_1 + A_0)$

$\Delta = W_2^3 + (a_3 + a_1a_2)W_1^3 + a_3^2W_0^3 - 2a_1W_1W_2^2 + (a_1^2 - a_2)W_1^2W_2 - a_2W_0W_2^2 + a_3a_1W_0^2W_2 + (a_2^2 + a_1a_3)W_0W_1^2 + 2a_2a_3W_0^2W_1 + (-3a_3 + a_1a_2)W_0W_1W_2$

In summary, the solution of (2.10) is given by

$$W_n(W_0, W_1, W_2) = (-B_1+1)V_n(W_0, W_1, W_2) - B_2V_{n-1}(W_0, W_1, W_2) - B_3V_{n-2}(W_0, W_1, W_2) + A_1n^3 + A_0n^2$$

(c): *The case $s = 2$: ($a_3 \neq 0$, $c_2 \neq 0$).*

Let the sequence $\{W_n\}$ defined by nonhomogeneous linear recurrence relation

$$W_n = a_1W_{n-1} + a_2W_{n-2} + a_3W_{n-3} + c_2n^2 + c_1n + c_0. \tag{2.11}$$

Then the solution of (2.11) is given by

$$\begin{aligned} W_n(W_0, W_1, W_2) &= W_n^{(h)} + W_n^{(p)} \\ &= V_n(W_0, W_1, W_2) - V_n(W_0^{(p)}, W_1^{(p)}, W_2^{(p)}) + W_n^{(p)} \end{aligned}$$

where $W_n^{(h)} = V_n(W_0, W_1, W_2) - V_n(W_0^{(p)}, W_1^{(p)}, W_2^{(p)})$ is the solution of (1.2) and

$$W_n^{(p)} = n^2(A_2n^2 + A_1n + A_0) = A_2n^4 + A_1n^3 + A_0n^2$$

is the particular solution of (2.11). Here,

$$\begin{aligned} A_2 &= -\frac{c_2}{6\,(a_1 + 4a_2 + 9a_3)}, \\ A_1 &= -\frac{1}{3\,(a_1 + 4a_2 + 9a_3)}(c_1 - 4\,(a_1 + 8a_2 + 27a_3)\,A_2), \\ A_0 &= -\frac{1}{(a_1 + 4a_2 + 9a_3)}(c_0 - (a_1 + 8a_2 + 27a_3)\,A_1 + (a_1 + 16a_2 + 81a_3)\,A_2), \end{aligned}$$

i.e.,

$A_2 = -\frac{c_2}{6\,(a_1 + 4a_2 + 9a_3)}$,

$A_1 = -\frac{1}{9\,(a_1 + 4a_2 + 9a_3)^2}(a_1(3c_1 + 2c_2) + 4a_2(3c_1 + 4c_2) + 27a_3(c_1 + 2c_2))$,

$A_0 = -\frac{1}{18\,(a_1 + 4a_2 + 9a_3)^3}(a_1^2(18c_0 + 6c_1 + c_2) + 32a_2^2(9c_0 + 6c_1 + 2c_2) + 729a_3^2(2c_0 + 2c_1 + c_2) + 4a_1a_2(36c_0 + 18c_1 + c_2) + 54a_1a_3(6c_0 + 4c_1 - c_2) + 108a_2a_3(12c_0 + 10c_1 + 3c_2))$,

and

$$W_0^{(p)} = 0,\ W_1^{(p)} = A_2 + A_1 + A_0,\ W_2^{(p)} = 16A_2 + 8A_1 + 4A_0,$$

and

$$\begin{aligned} V_n(W_0^{(p)}, W_1^{(p)}, W_2^{(p)}) &= V_n(0, A_2 + A_1 + A_0, 16A_2 + 8A_1 + 4A_0) \\ &= B_1V_n(W_0, W_1, W_2) + B_2V_{n-1}(W_0, W_1, W_2) + B_3V_{n-2}(W_0, W_1, W_2) \end{aligned}$$

where

$$B_1 = \frac{Y_1}{\Delta},\ B_2 = \frac{Y_2}{\Delta},\ B_3 = \frac{Y_3}{\Delta}$$

and

$Y_1 = (W_2^2 + a_1^2W_1^2 + a_1a_3W_0^2 - 2a_1W_1W_2 - a_2W_0W_2 + (a_1a_2 - a_3)W_0W_1)(16A_2 + 8A_1 + 4A_0) + ((a_3 + a_1a_2)W_1^2 + a_2a_3W_0^2 - a_2W_1W_2 - a_3W_0W_2 + a_2^2W_0W_1)(A_2 + A_1 + A_0)$

$Y_2 = ((a_3+a_1a_2)W_1^2 - a_2W_2W_1 + (a_2^2 - 2a_3a_1)W_0W_1 + a_3a_2W_0^2 + 2a_3a_1W_0W_1 - a_3W_0W_2)(16A_2 + 8A_1 + 4A_0) + (a_2W_2^2 + 2a_3W_1W_2 + a_3^2W_0^2 - (3a_3 + a_1a_2)W_1W_2 - (a_2^2 - 2a_3a_1)W_0W_2 - a_3a_1W_0W_2)(A_2 + A_1 + A_0)$

$Y_3 = a_3(-W_1W_2 + a_1W_1^2 + a_2W_0W_1 + a_3W_0^2)(16A_2 + 8A_1 + 4A_0) + a_3(W_2^2 - a_1W_1W_2 - a_2W_0W_2 - a_3W_0W_1)(A_2 + A_1 + A_0)$

$\Delta = W_2^3 + (a_3 + a_1a_2)W_1^3 + a_3^2W_0^3 - 2a_1W_1W_2^2 + (a_1^2 - a_2)W_1^2W_2 - a_2W_0W_2^2 + a_3a_1W_0^2W_2 + (a_2^2 + a_1a_3)W_0W_1^2 + 2a_2a_3W_0^2W_1 + (-3a_3 + a_1a_2)W_0W_1W_2$

In summary, the solution of (2.11) is given by

$$W_n(W_0, W_1, W_2) = (-B_1+1)V_n(W_0, W_1, W_2) - B_2V_{n-1}(W_0, W_1, W_2) - B_3V_{n-2}(W_0, W_1, W_2) + A_2n^4 + A_1n^3 + A_0n^2$$

(d): *The case $s = 3$: ($a_3 \neq 0$, $c_3 \neq 0$).*

Let the sequence $\{W_n\}$ defined by nonhomogeneous linear recurrence relation

$$W_n = a_1W_{n-1} + a_2W_{n-2} + a_3W_{n-3} + c_3n^3 + c_2n^2 + c_1n + c_0. \tag{2.12}$$

Then the solution of (2.12) is given by

$$\begin{aligned} W_n(W_0, W_1, W_2) &= W_n^{(h)} + W_n^{(p)} \\ &= V_n(W_0, W_1, W_2) - V_n(W_0^{(p)}, W_1^{(p)}, W_2^{(p)}) + W_n^{(p)} \end{aligned}$$

where $W_n^{(h)} = V_n(W_0, W_1, W_2) - V_n(W_0^{(p)}, W_1^{(p)}, W_2^{(p)})$ is the solution of (1.2) and

$$W_n^{(p)} = n^2(A_3n^3 + A_2n^2 + A_1n + A_0) = A_3n^5 + A_2n^4 + A_1n^3 + A_0n^2$$

is the particular solution of (2.12). Here,

$$\begin{aligned} A_3 &= -\frac{c_3}{10\,(a_1 + 4a_2 + 9a_3)}, \\ A_2 &= -\frac{1}{6\,(a_1 + 4a_2 + 9a_3)}(c_2 - 10\,(a_1 + 8a_2 + 27a_3)\,A_3), \\ A_1 &= -\frac{1}{3\,(a_1 + 4a_2 + 9a_3)}(c_1 - 4\,(a_1 + 8a_2 + 27a_3)\,A_2 + 5\,(a_1 + 16a_2 + 81a_3)\,A_3), \\ A_0 &= -\frac{1}{(a_1 + 4a_2 + 9a_3)}(c_0 - (a_1 + 8a_2 + 27a_3)\,A_1 + (a_1 + 16a_2 + 81a_3)\,A_2 - (a_1 + 32a_2 + 243a_3)\,A_3), \end{aligned}$$

i.e.,

$A_3 = -\dfrac{c_3}{10\,(a_1 + 4a_2 + 9a_3)}$,

$A_2 = -\dfrac{1}{6\,(a_1 + 4a_2 + 9a_3)^2}(a_1(c_2 + c_3) + 4a_2(c_2 + 2c_3) + 9a_3(c_2 + 3c_3))$,

$A_1 = -\frac{1}{18(a_1+4a_2+9a_3)^3}(a_1^2(6c_1 + 4c_2 + c_3) + 32a_2^2(3c_1 + 4c_2 + 2c_3) + 243a_3^2(2c_1 + 4c_2 + 3c_3) + 4a_1a_2(12c_1 + 12c_2 + c_3) + 18a_1a_3(6c_1 + 8c_2 - 3c_3) + 36a_2a_3(12c_1 + 20c_2 + 9c_3))$,

$$A_0 = -\frac{1}{90\left(a_1+4a_2+9a_3\right)^4}(a_1^3(90c_0+30c_1+5c_2-c_3)+128a_2^3(45c_0+30c_1+10c_2-4c_3)+6561a_3^3(10c_0+10c_1+5c_2-3c_3)+16a_1a_2^2(270c_0+150c_1+25c_2-27c_3)+5a_1^2(216a_2c_0+96a_2c_1+8a_2c_2-45a_3c_2)+1215a_1a_3^2(18c_0+14c_1+c_2-9c_3)+27a_1^2a_3(90c_0+50c_1+17c_3)+1944a_2a_3^2(45c_0+40c_1+15c_2-12c_3)+144a_2^2a_3(270c_0+210c_1+65c_2-33c_3)+144a_1a_2a_3(135c_0+90c_1+5c_2-18c_3)),$$

and

$$W_0^{(p)}=0,\ W_1^{(p)}=A_3+A_2+A_1+A_0,\ W_2^{(p)}=32A_3+16A_2+8A_1+4A_0,$$

and

$$\begin{aligned} V_n(W_0^{(p)},W_1^{(p)},W_2^{(p)}) &= V_n(0,A_3+A_2+A_1+A_0,32A_3+16A_2+8A_1+4A_0) \\ &= B_1V_n(W_0,W_1,W_2)+B_2V_{n-1}(W_0,W_1,W_2)+B_3V_{n-2}(W_0,W_1,W_2) \end{aligned}$$

where

$$B_1=\frac{Y_1}{\Delta},\ B_2=\frac{Y_2}{\Delta},\ B_3=\frac{Y_3}{\Delta}$$

and

$Y_1=(W_2^2+a_1^2W_1^2+a_1a_3W_0^2-2a_1W_1W_2-a_2W_0W_2+(a_1a_2-a_3)W_0W_1)(32A_3+16A_2+8A_1+4A_0)+((a_3+a_1a_2)W_1^2+a_2a_3W_0^2-a_2W_1W_2-a_3W_0W_2+a_2^2W_0W_1)(A_3+A_2+A_1+A_0)$

$Y_2=((a_3+a_1a_2)W_1^2-a_2W_2W_1+(a_2^2-2a_3a_1)W_0W_1+a_3a_2W_0^2+2a_3a_1W_0W_1-a_3W_0W_2)(32A_3+16A_2+8A_1+4A_0)+(a_2W_2^2+2a_3W_1W_2+a_3^2W_0^2-(3a_3+a_1a_2)W_1W_2-(a_2^2-2a_3a_1)W_0W_2-a_3a_1W_0W_2)(A_3+A_2+A_1+A_0)$

$Y_3=a_3(-W_1W_2+a_1W_1^2+a_2W_0W_1+a_3W_0^2)(32A_3+16A_2+8A_1+4A_0)+a_3(W_2^2-a_1W_1W_2-a_2W_0W_2-a_3W_0W_1)(A_3+A_2+A_1+A_0)$

$\Delta=W_2^3+(a_3+a_1a_2)W_1^3+a_3^2W_0^3-2a_1W_1W_2^2+(a_1^2-a_2)W_1^2W_2-a_2W_0W_2^2+a_3a_1W_0^2W_2+(a_2^2+a_1a_3)W_0W_1^2+2a_2a_3W_0^2W_1+(-3a_3+a_1a_2)W_0W_1W_2$

In summary, the solution of (2.12) is given by

$$W_n(W_0,W_1,W_2)=(-B_1+1)V_n(W_0,W_1,W_2)-B_2V_{n-1}(W_0,W_1,W_2)-B_3V_{n-2}(W_0,W_1,W_2)+A_3n^5+A_2n^4+A_1n^3+A_0n^2$$

The Case: $\theta_1=1$, $\theta_2=1$, and $\theta_3=1$. That is, all three roots of the characteristic equation (1.2) are equal to 1. This situation corresponds to the case $r=3$, where the presence of a triple root at 1 introduces the cubic factor n^3 into the particular solution. Compared to the cases $r=0,1,2$, the iterative construction of the coefficients A_i requires further adjustment, reflecting the dominant influence of the repeated root. In the example below, we will derive closed-form solutions for the cases $s=0,1,2,3$, showing how the cubic factor n^3 fully determines the structure of the particular solution.

Example 2.4. *In this example, in each case, the homogeneous relation corresponding to the given non-homogeneous recurrence relation is expressed as in (1.2), i.e.,*

$$V_n=3V_{n-1}-3V_{n-2}+V_{n-3}.$$

so that $a_1 = 3,\ a_2 = -3,\ a_3 = 1.$

(a): *The case* $s = 0 : (c_0 \neq 0)$.

Let the sequence $\{W_n\}$ *defined by nonhomogeneous linear recurrence relation*

$$W_n = 3W_{n-1} - 3W_{n-2} + W_{n-3} + c_0. \tag{2.13}$$

Then the solution of (2.13) is given by

$$\begin{aligned} W_n(W_0, W_1, W_2) &= W_n^{(h)} + W_n^{(p)} \\ &= V_n(W_0, W_1, W_2) - V_n(W_0^{(p)}, W_1^{(p)}, W_2^{(p)}) + W_n^{(p)} \end{aligned}$$

where $W_n^{(h)} = V_n(W_0, W_1, W_2) - V_n(W_0^{(p)}, W_1^{(p)}, W_2^{(p)})$ *is the solution of (1.2) and*

$$W_n^{(p)} = n^3 A_0$$

is the particular solution of (2.13). Here,

$$A_0 = \tfrac{1}{6} c_0$$

and

$$W_0^{(p)} = 0,\ W_1^{(p)} = A_0,\ W_2^{(p)} = 8A_0$$

and

$$\begin{aligned} V_n(W_0^{(p)}, W_1^{(p)}, W_2^{(p)}) &= V_n(0, A_0, 8A_0) \\ &= B_1 V_n(W_0, W_1, W_2) + B_2 V_{n-1}(W_0, W_1, W_2) + B_3 V_{n-2}(W_0, W_1, W_2) \end{aligned}$$

where

$$B_1 = \frac{Y_1}{\Delta},\ B_2 = \frac{Y_2}{\Delta},\ B_3 = \frac{Y_3}{\Delta}$$

and

$Y_1 = (W_2^2 + 9W_1^2 + 3W_0^2 - 6W_1W_2 + 3W_0W_2 - 10W_0W_1) \times 8A_0 + (-8W_1^2 - 3W_0^2 + 3W_1W_2 - W_0W_2 + 9W_0W_1)A_0$

$Y_2 = (-8W_1^2 + 3W_2W_1 + 3W_0W_1 - 3W_0^2 + 6W_0W_1 - W_0W_2) \times 8A_0 + (-3W_2^2 + 2W_1W_2 + W_0^2 + 6W_1W_2 - 3W_0W_2 - 3W_0W_2)A_0$

$Y_3 = (-W_1W_2 + 3W_1^2 - 3W_0W_1 + W_0^2) \times 8A_0 + (W_2^2 - 3W_1W_2 + 3W_0W_2 - W_0W_1)A_0$

$\Delta = W_2^3 - 8W_1^3 + W_0^3 - 6W_1W_2^2 + 12W_1^2W_2 + 3W_0W_2^2 + 3W_0^2W_2 + 12W_0W_1^2 - 6W_0^2W_1 - 12W_0W_1W_2$

In summary, the solution of (2.13) is given by

$$W_n(W_0, W_1, W_2) = (-B_1 + 1)V_n(W_0, W_1, W_2) - B_2 V_{n-1}(W_0, W_1, W_2) - B_3 V_{n-2}(W_0, W_1, W_2) + n^3 A_0$$

(b): *The case $s = 1 : (c_1 \neq 0)$.*

Let the sequence $\{W_n\}$ defined by nonhomogeneous linear recurrence relation

$$W_n = 3W_{n-1} - 3W_{n-2} + W_{n-3} + c_1 n + c_0. \tag{2.14}$$

Then the solution of (2.14) is given by

$$\begin{aligned} W_n(W_0, W_1, W_2) &= W_n^{(h)} + W_n^{(p)} \\ &= V_n(W_0, W_1, W_2) - V_n(W_0^{(p)}, W_1^{(p)}, W_2^{(p)}) + W_n^{(p)} \end{aligned}$$

where $W_n^{(h)} = V_n(W_0, W_1, W_2) - V_n(W_0^{(p)}, W_1^{(p)}, W_2^{(p)})$ is the solution of (1.2) and

$$W_n^{(p)} = n^3(A_1 n + A_0) = A_1 n^4 + A_0 n^3$$

is the particular solution of (2.14). Here,

$$A_1 = \frac{1}{24}c_1, \ A_0 = \frac{1}{12}(2c_0 + 3c_1),$$

and

$$W_0^{(p)} = 0, \ W_1^{(p)} = A_1 + A_0, \ W_2^{(p)} = 16A_1 + 8A_0$$

and

$$\begin{aligned} V_n(W_0^{(p)}, W_1^{(p)}, W_2^{(p)}) &= V_n(0, A_1 + A_0, 16A_1 + 8A_0) \\ &= B_1 V_n(W_0, W_1, W_2) + B_2 V_{n-1}(W_0, W_1, W_2) + B_3 V_{n-2}(W_0, W_1, W_2) \end{aligned}$$

where

$$B_1 = \frac{Y_1}{\Delta}, \ B_2 = \frac{Y_2}{\Delta}, \ B_3 = \frac{Y_3}{\Delta}$$

and

$Y_1 = (W_2^2 + 9W_1^2 + 3W_0^2 - 6W_1W_2 + 3W_0W_2 - 10W_0W_1)(16A_1 + 8A_0) + (-8W_1^2 - 3W_0^2 + 3W_1W_2 - W_0W_2 + 9W_0W_1)(A_1 + A_0)$

$Y_2 = (-8W_1^2 + 3W_2W_1 + 3W_0W_1 - 3W_0^2 + 6W_0W_1 - W_0W_2)(16A_1 + 8A_0) + (-3W_2^2 + 2W_1W_2 + W_0^2 + 6W_1W_2 - 3W_0W_2 - 3W_0W_2)(A_1 + A_0)$

$Y_3 = (-W_1W_2 + 3W_1^2 - 3W_0W_1 + W_0^2)(16A_1 + 8A_0) + (W_2^2 - 3W_1W_2 + 3W_0W_2 - W_0W_1)(A_1 + A_0)$

$\Delta = W_2^3 - 8W_1^3 + W_0^3 - 6W_1W_2^2 + 12W_1^2W_2 + 3W_0W_2^2 + 3W_0^2W_2 + 12W_0W_1^2 - 6W_0^2W_1 - 12W_0W_1W_2$

In summary, the solution of (2.14) is given by

$$W_n(W_0, W_1, W_2) = (-B_1 + 1)V_n(W_0, W_1, W_2) - B_2 V_{n-1}(W_0, W_1, W_2) - B_3 V_{n-2}(W_0, W_1, W_2) + A_1 n^4 + A_0 n^3$$

(c): *The case $s = 2 : (c_2 \neq 0)$.*

Let the sequence $\{W_n\}$ defined by nonhomogeneous linear recurrence relation

$$W_n = 3W_{n-1} - 3W_{n-2} + W_{n-3} + c_2 n^2 + c_1 n + c_0. \tag{2.15}$$

Then the solution of (2.15) is given by

$$\begin{aligned} W_n(W_0,W_1,W_2) &= W_n^{(h)}+W_n^{(p)} \\ &= V_n(W_0,W_1,W_2)-V_n(W_0^{(p)},W_1^{(p)},W_2^{(p)})+W_n^{(p)} \end{aligned}$$

where $W_n^{(h)}=V_n(W_0,W_1,W_2)-V_n(W_0^{(p)},W_1^{(p)},W_2^{(p)})$ *is the solution of (1.2) and*

$$W_n^{(p)}=n^3(A_2n^2+A_1n+A_0)=A_2n^5+A_1n^4+A_0n^3$$

is the particular solution of (2.15). Here,

$$A_2=\frac{1}{60}c_2,\ A_1=\frac{1}{24}(c_1+3c_2),\ A_0=\frac{1}{12}(2c_0+3c_1+4c_2),$$

and

$$W_0^{(p)}=0,\ W_1^{(p)}=A_2+A_1+A_0,\ W_2^{(p)}=32A_2+16A_1+8A_0$$

and

$$\begin{aligned} V_n(W_0^{(p)},W_1^{(p)},W_2^{(p)}) &= V_n(0,A_2+A_1+A_0,32A_2+16A_1+8A_0) \\ &= B_1V_n(W_0,W_1,W_2)+B_2V_{n-1}(W_0,W_1,W_2)+B_3V_{n-2}(W_0,W_1,W_2) \end{aligned}$$

where

$$B_1=\frac{Y_1}{\Delta},\ B_2=\frac{Y_2}{\Delta},\ B_3=\frac{Y_3}{\Delta}$$

and

$Y_1=(W_2^2+9W_1^2+3W_0^2-6W_1W_2+3W_0W_2-10W_0W_1)(32A_2+16A_1+8A_0)+(-8W_1^2-3W_0^2+3W_1W_2-W_0W_2+9W_0W_1)(A_2+A_1+A_0)$

$Y_2=(-8W_1^2+3W_2W_1+3W_0W_1-3W_0^2+6W_0W_1-W_0W_2)(32A_2+16A_1+8A_0)+(-3W_2^2+2W_1W_2+W_0^2+6W_1W_2-3W_0W_2-3W_0W_2)(A_2+A_1+A_0)$

$Y_3=(-W_1W_2+3W_1^2-3W_0W_1+W_0^2)(32A_2+16A_1+8A_0)+(W_2^2-3W_1W_2+3W_0W_2-W_0W_1)(A_2+A_1+A_0)$

$\Delta=W_2^3-8W_1^3+W_0^3-6W_1W_2^2+12W_1^2W_2+3W_0W_2^2+3W_0^2W_2+12W_0W_1^2-6W_0^2W_1-12W_0W_1W_2$

In summary, the solution of (2.15) is given by

$$W_n(W_0,W_1,W_2)=(-B_1+1)V_n(W_0,W_1,W_2)-B_2V_{n-1}(W_0,W_1,W_2)-B_3V_{n-2}(W_0,W_1,W_2)+A_2n^5+A_1n^4+A_0n^3$$

(d): *The case* $s=3:(c_3\neq 0)$*.*

Let the sequence $\{W_n\}$ *defined by nonhomogeneous linear recurrence relation*

$$W_n=3W_{n-1}-3W_{n-2}+W_{n-3}+c_3n^3+c_2n^2+c_1n+c_0. \tag{2.16}$$

Then the solution of (2.16) is given by

$$\begin{aligned} W_n(W_0,W_1,W_2) &= W_n^{(h)}+W_n^{(p)} \\ &= V_n(W_0,W_1,W_2)-V_n(W_0^{(p)},W_1^{(p)},W_2^{(p)})+W_n^{(p)} \end{aligned}$$

where $W_n^{(h)} = V_n(W_0, W_1, W_2) - V_n(W_0^{(p)}, W_1^{(p)}, W_2^{(p)})$ *is the solution of (1.2) and*

$$W_n^{(p)} = n^3(A_3n^3 + A_2n^2 + A_1n + A_0) = A_3n^6 + A_2n^5 + A_1n^4 + A_0n^3$$

is the particular solution of (2.16). Here,

$$\begin{aligned} A_3 &= \frac{1}{120}c_3, \ A_2 = \frac{1}{120}(2c_2 + 9c_3), \\ A_1 &= \frac{1}{24}(c_1 + 3c_2 + 6c_3), \ A_0 = \frac{1}{24}(4c_0 + 6c_1 + 8c_2 + 9c_3), \end{aligned}$$

and

$$W_0^{(p)} = 0, \ W_1^{(p)} = A_3 + A_2 + A_1 + A_0, \ W_2^{(p)} = 64A_3 + 32A_2 + 16A_1 + 8A_0$$

and

$$\begin{aligned} V_n(W_0^{(p)}, W_1^{(p)}, W_2^{(p)}) &= V_n(0, A_3 + A_2 + A_1 + A_0, 64A_3 + 32A_2 + 16A_1 + 8A_0) \\ &= B_1V_n(W_0, W_1, W_2) + B_2V_{n-1}(W_0, W_1, W_2) + B_3V_{n-2}(W_0, W_1, W_2) \end{aligned}$$

where

$$B_1 = \frac{Y_1}{\Delta}, \ B_2 = \frac{Y_2}{\Delta}, \ B_3 = \frac{Y_3}{\Delta}$$

and

$Y_1 = (W_2^2 + 9W_1^2 + 3W_0^2 - 6W_1W_2 + 3W_0W_2 - 10W_0W_1)(64A_3 + 32A_2 + 16A_1 + 8A_0) + (-8W_1^2 - 3W_0^2 + 3W_1W_2 - W_0W_2 + 9W_0W_1)(A_3 + A_2 + A_1 + A_0)$

$Y_2 = (-8W_1^2 + 3W_2W_1 + 3W_0W_1 - 3W_0^2 + 6W_0W_1 - W_0W_2)(64A_3 + 32A_2 + 16A_1 + 8A_0) + (-3W_2^2 + 2W_1W_2 + W_0^2 + 6W_1W_2 - 3W_0W_2 - 3W_0W_2)(A_3 + A_2 + A_1 + A_0)$

$Y_3 = (-W_1W_2 + 3W_1^2 - 3W_0W_1 + W_0^2)(64A_3 + 32A_2 + 16A_1 + 8A_0) + (W_2^2 - 3W_1W_2 + 3W_0W_2 - W_0W_1)(A_3 + A_2 + A_1 + A_0)$

$\Delta = W_2^3 - 8W_1^3 + W_0^3 - 6W_1W_2^2 + 12W_1^2W_2 + 3W_0W_2^2 + 3W_0^2W_2 + 12W_0W_1^2 - 6W_0^2W_1 - 12W_0W_1W_2$

In summary, the solution of (2.16) is given by

$$W_n(W_0, W_1, W_2) = (-B_1 + 1)V_n(W_0, W_1, W_2) - B_2V_{n-1}(W_0, W_1, W_2) - B_3V_{n-2}(W_0, W_1, W_2) + A_3n^6 + A_2n^5 + A_1n^4 + A_0n^3$$

Conclusion

This paper has presented closed-form solutions for third-order nonhomogeneous linear recurrence relations, known as generalized Leonardo-type sequences, in the case where the input function $p(n)$ is a polynomial. The analysis was structured according to the multiplicity $r = 0, 1, 2, 3$ of 1 as a root of the characteristic equation, and for each multiplicity explicit solutions were obtained for polynomial inputs of degree $s = 0, 1, 2, 3$. The resulting formulas reveal how root multiplicity and polynomial degree jointly determine the form of the particular solution, while the homogeneous component remains governed by the same recurrence relation.

Beyond their theoretical significance, the examples developed here also carry didactic and practical value. For teaching, they provide accessible cases that allow students to explore non-homogeneous recurrences without excessive computational burden. For research, they supply resonance-aware formulas and explicit derivations that can be adapted to new problems in mathematics, computer science, engineering, and physics. In this way, the results contribute simultaneously to pedagogy and applied research, enhancing the originality, accessibility, and impact of the manuscript across multiple domains.

The iterative framework introduced in this work yields explicit polynomial-type closed-form solutions for generalized Leonardo-type sequences. Its scope, however, is currently limited to polynomial inputs. Extending the method to non-polynomial inputs, such as trigonometric functions, would require further refinement. Moreover, while the framework clarifies resonance phenomena and multiplicity corrections, the computational complexity increases rapidly for higher-order recurrences, which may restrict practical use without the support of computer algebra systems.

At the same time, the methodology opens promising directions for future research. It can be integrated into symbolic computation platforms, employed to verify classical identities in recurrence theory, and applied in interdisciplinary contexts such as coding theory, cryptography, and discrete modeling in physics and biology. By acknowledging current limitations while outlining avenues for extension, the study provides a balanced perspective: consolidating the contribution of the present results while pointing toward further exploration and development.

In addition to these theoretical and methodological contributions, the importance of this work for the scientific community should be emphasized. The unified framework for generalized Leonardo-type sequences not only consolidates recurrence theory but also opens pathways for innovation in modern applications. By providing resonance-aware formulas and explicit derivations, the method can be adapted to optimization problems across diverse fields such as transportation, miniaturization technologies, coding theory, and discrete modeling. Thus, the manuscript contributes both to advancing mathematical theory and to supporting interdisciplinary research, strengthening its originality, accessibility, and impact.

Closed-form solutions are vital because they provide exact and instantaneous predictability for complex systems, eliminating the computational costs of step-by-step iteration. This manuscript contributes a unified framework for generalized Leonardo-type sequences, extending recurrence theory and clarifying resonance phenomena. Such solutions are highly relevant for algorithm analysis, numerical modeling, and optimization problems, where efficiency and precision are paramount. By offering resonance-aware formulas and explicit derivations, the work supports both theoretical advances and practical applications across mathematics, computer science, engineering, and physics. In this way, the manuscript strengthens the foundation for interdisciplinary research while enhancing accessibility for teaching and applied problem-solving.

References

[1] Abd-Elhameed, W.M., Alqubori, O.M., Alluhaybi, A.A., Amin, A.K., Novel Expressions for Certain Generalized Leonardo Polynomials and Their Associated Numbers, Axioms, 14, 286, 2025. https://doi.org/10.3390/

[2] Catarino, P., Borges, A., On Leonardo Numbers, Acta Mathematica Universitatis Comenianae, 89(1), 75–86, 2020. Available online at: http://www.iam.fmph.uniba.sk/amuc/ojs/index.php/amuc/article/view/1005/650.

[3] Dikmen, C.D., Properties of Gaussian Generalized Leonardo Numbers, Karaelmas Science and Engineering Journal, 15(1), 134-145, 2025. DOI: 10.7212/karaelmasfen.1578154

[4] Gökbaş, H., k-Leonardo Numbers, Palestine Journal of Mathematics, 13(4), 1427-1435, 2024.

[5] İşbilir, Z., Akyiğit, M., Tosun, M., Pauli–Leonardo Quaternions, Notes on Number Theory and Discrete Mathematics, 29(1), 1-16, 2023. DOI: 10.7546/nntdm.2023.29.1.1-16

[6] Jeske, J.A., Linear Recurrence Relations, Part I, The Fibonacci Quarterly, 1(2), 69-74, 1963.

[7] Jeske, J.A., Linear Recurrence Relations, Part II, The Fibonacci Quarterly, 1(4), 34-39, 1963.

[8] Jeske, J.A., Linear Recurrence Relations, Part III, The Fibonacci Quarterly, 2(3), 197-203, 1964.

[9] Kuhapatanakul, K., Chobsorn, J, On the Generalized Leonardo Numbers, Integers 22, 2022, #A48.

[10] Kuhapatanakul, K., Ruankong, P., On Generalized Leonardo p-numbers, Journal of Integer Sequences, 27, Article 24.4.6, 2024.

[11] Prasad, K., Kumari, M., The Leonardo Polynomials and Their Algebraic Properties. Proceedings of the Indian National Science Academy, 2024. https://doi.org/10.1007/s43538-024-00348-0

[12] Shannon, A.G., A Note On Generalized Leonardo Numbers, Notes on Number Theory and Discrete Mathematics, 25(3), 97–101, 2019. DOI: 10.7546/nntdm.2019.25.3.97-101

[13] Shannon, A.G., Deveci, Ö., A Note on Generalized and Extended Leonardo Sequences, Notes on Number Theory and Discrete Mathematics, 28(1), 109–114, 2022. DOI: 10.7546/nntdm.2022.28.1.109-114

[14] Shannon, A.G., Shiue, P.J.S., Huang, S.C., Notes on Generalized and Extended Leonardo Numbers, Notes on Number Theory and Discrete Mathematics, 29(4), 752–773, 2023. DOI: 10.7546/nntdm.2023.29.4.752-773

[15] Soykan, Y., Generalized Horadam-Leonardo Numbers and Polynomials, Asian Journal of Advanced Research and Reports, 17(8), 128-169, 2023. https://doi.org/10.9734/ajarr/2023/v17i8511

[16] Soykan, Y., Interrelations between Horadam and Generalized Horadam-Leonardo Polynomials via Identities, International Journal of Advances in Applied Mathematics and Mechanics, 11(1), 42-55, 2023. ISSN: 2347-2529

[17] Soykan, Y., Generalized Leonardo Numbers, Journal of Progressive Research in Mathematics, 18(4), 58-84, 2021.

[18] Soykan, Y., An Extensive Study on Generalized Leonardo Numbers and Polynomials, International Journal of Advances in Applied Mathematics and Mechanics, 13(3), 51–250, 2026.

[19] Soykan, Y., Leonardo Polynomials and Numbers: Solutions, Linearizations, and Generating Functions, International Journal of Advances in Applied Mathematics and Mechanics, 13(4), 80-222, 2026. https://doi.org/10.26541/ijaamm.2026.130408

[20] Soykan, Y., Explicit Formulas for Polynomial-type Particular Solutions of Generalized Leonardo-type Sequences, Asian Journal of Advanced Research and Reports, 20(5), 164-208, 2026. https://doi.org/10.9734/ajarr/2026/v20i51361

[21] Özimamoğlu, H., On Leonardo Sedenions, Afrika Matematika (2023) 34:26, 2023. https://doi.org/10.1007/s13370-023-01065-5

[22] Özkan, E., Akkuş, H., Generalized Bronze Leonardo Sequence, Notes on Number Theory and Discrete Mathematics, 30(4), 811-824, 2024. DOI: 10.7546/nntdm.2024.30.4.811-824