A Note on Explicit Particular Solutions for Third and Fourth order Generalized Leonardo-Type Recurrences with Polynomial-Exponential Input

Abstract. Sequences, both classical and modern in scope, can be analyzed through a versatile framework that remains central to mathematics, namely recurrence relations. Previous investigations established explicit iterative procedures for constructing polynomial–exponential particular solutions of generalized Leonardo-type sequences. Building upon that framework, this article develops illustrative examples for the cases

$$m = 3, 4,$$

where the forcing term is given by $C(n) = p(n)d^n$, with $p(n) = \sum_{i=0}^{s} c_i n^i$ a polynomial in n. For such recurrences, we derive particular solutions of the form

$$W_n^{(C)} = n^r \left(\sum_{i=0}^{s} A_i n^i \right) d^n,$$

and demonstrate the computation of the coefficients A_i via the established iterative scheme. These formulas not only provide constructive clarity but also demonstrate how the iterative procedure systematically determines the polynomial part of the solution. The examples reveal how the multiplicity r of the root d in the characteristic polynomial governs the structure of the solution, while resonance phenomena emerge when the forcing term interacts with repeated characteristic roots. Such resonance effects are highlighted in detail, showing their decisive role in shaping the solution's form and complexity.

In addition to the explicit constructions, a brief literature review is included to situate Leonardo-type sequences within their historical development and to highlight recent advances in generalized Leonardo-type recurrences. This contextualization underscores the enduring role of recurrence relations in number theory, discrete mathematics, and symbolic computation. By presenting explicit cases, the paper offers a transparent and accessible illustration of the general theory, reinforcing the connection between abstract recurrence analysis and concrete symbolic computation, while also pointing toward potential applications in computational mathematics and combinatorial modeling.

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1. Introduction

Recurrence sequences, generated through recurrence relations, have long been recognized as fundamental objects in mathematics, with influence extending well beyond the discipline into physics, engineering, biology, computer science, architecture, and even artistic analysis. Despite their elementary formulation, they encapsulate profound structures: modeling growth dynamics, oscillatory behavior, and intricate symbolic identities. Classical second-order families such as the Fibonacci, Lucas, Pell, and Jacobsthal sequences exemplify this richness and continue to serve as paradigmatic cases.

The scope of recurrence theory, however, is not confined to second-order constructions. Higher-order families occupy an equally prominent position, both in abstract theory and in applied modeling. These generalizations broaden the classical framework and uncover deeper algebraic and analytic phenomena. Third-order examples such as the Tribonacci sequence, fourth-order examples such as the Tetranacci sequence, and fifth-order examples such as the Pentanacci sequence extend the paradigm, each governed by characteristic polynomials whose root structures dictate closed-form representations. Homogeneous recurrences highlight the decisive role of characteristic polynomials and root multiplicities, whereas non-homogeneous recurrences introduce external symbolic inputs whose interaction with the root configuration gives rise to resonance phenomena. Taken together, these families establish a coherent framework that unites classical recurrence identities with modern developments in symbolic recurrence theory.

The classical Leonardo numbers are defined by the non-homogeneous recurrence relation

$$l_n = l_{n-1} + l_{n-2} + 1, \quad n \geq 2,$$

with initial conditions $l_0 = 1$ and $l_1 = 1$.

Although the recurrence itself is elementary, the historical development of the Leonardo sequence is not fully transparent. Its emergence appears to have been gradual, with generalizations and extensions studied well before the sequence acquired its formal name. The modern revival of interest has been closely associated with the analysis of explicit cases and their diverse applications.

Over time, the Leonardo sequence has attracted renewed attention, not only for its intrinsic mathematical elegance but also for its versatility in modeling hybrid recurrence systems that combine homogeneous dynamics with non-homogeneous inputs. This dual character has made it a fertile subject for symbolic exploration, bridging classical recurrence theory with contemporary applications. Recent studies emphasize its structural richness, its ability to encode complex interactions, and its relevance to both theoretical investigations and applied modeling.

From a pedagogical perspective, the clarity of its defining recurrence and the accessibility of explicit examples render the Leonardo sequence particularly suitable for textbooks and teaching materials. It provides students with a concrete illustration of how non-homogeneous recurrence relations operate, while simultaneously serving as an entry point to deeper symbolic methods and resonance phenomena. In this way, the Leonardo sequence functions both as a continuing subject of scholarly inquiry and as a valuable educational tool (see, for example, [1, 2, 3, 5, 6, 10, 11, 16, 17, 18, 12, 13, 14, 15, 21, 22]).

To the best of our knowledge, the first systematic extension of the Leonardo numbers was undertaken by J. A. Jeske in a trilogy of papers published in *The Fibonacci Quarterly* during 1963–1964 (see [7, 8, 9]). For a concise literature review, particularly of contributions in *The Fibonacci Quarterly*, as well as an overview of selected works outside the journal that advance the study of Leonardo-type recurrence relations, see Soykan [19, Section 5].

We first recall the definition of m-order homogeneous linear recurrence relations.

Definition 1.1. *A sequence* $\{V_n\}_{n\geq 0}$ *is called a homogeneous (linear) recurrence relation order* $m\in\mathbb{N}$ *if it satisfies*

$$V_n=\sum_{k=1}^{m}a_kV_{n-k}=a_1V_{n-1}+a_2V_{n-2}...+a_mV_{n-m} \tag{1.1}$$

for

$$m\geq 1$$

with the initial conditions $V_0,V_1,...,V_{m-1}$

and

$$V_n=a_0, \tag{1.2}$$

for

$$m=0.$$

The recurrence coefficients $a_1,a_2,\ldots,a_m$ *and the initial conditions* $V_0,V_1,...,V_{m-1}$ *are complex scalars. We allow each coefficient* a_i*, for* $1\leq i\leq m$*, to be identically zero.*

The integer m *is called the order of the linear recurrence.*

The characteristic polynomial of the sequence $(V_n)_{n\geq 0}$ is given by

$$A(z)=z^m-\sum_{k=1}^{m}a_kz^{m-k}=z^m-a_1z^{m-1}-a_2z^{m-2}-...-a_{m-1}z-a_m=(z-\theta_1)^{u_1}(z-\theta_2)^{u_2}...(z-\theta_v)^{u_v}$$

with distinct $\theta_1,\theta_2,...,\theta_v$ and $u_1+u_2+...+u_v=m$. $\theta_1,\theta_2,...,\theta_v$ are called the (characteristic) root of characteristic equation

$$A(z)=z^m-a_1z^{m-1}-a_2z^{m-2}-...-a_{m-1}z-a_m=(z-\theta_1)^{u_1}(z-\theta_2)^{u_2}...(z-\theta_v)^{u_v}=0. \tag{1.3}$$

For $m\geq 1$, consider the sequence (W_n) defined by the recurrence relation (a **generalized Leonardo-type sequence**)

$$W_n=\sum_{k=1}^{m}a_kW_{n-k}+p(n)bd^n=\sum_{k=1}^{m}a_kW_{n-k}+C(n) \tag{1.4}$$

with initial conditions $W_0,W_1,\ldots,W_{m-1}$ and the recurrence coefficients $a_1,a_2,\ldots,a_m$ are complex scalars or polynomials in $\mathbb{C}[x]$ and with the input function

$$C(n)=p(n)bd^n$$

where

$$p(n) := p(n,x) = \sum_{i=0}^{s} c_i n^i$$

denotes a polynomial in n of order s, with coefficients belonging to $\mathbb{C}[x]$ or $\mathbb{C}$ and $b \in \mathbb{C}[x]$ or $\mathbb{C}$, and $d \in \mathbb{C}$ or $\mathbb{R}$. For more information on generalized Leonardo-type sequences, see Soykan [20] and [19].

We consider the homogeneous recurrence relation (1.1) and its characteristic equation (1.3), corresponding to the sequence (W_n) defined by (1.4).

The particular solution $W_n^{(C)}$ of (1.4) is of the form

$$W_n^{(C)} = n^r(\sum_{i=0}^{s} A_i n^i)d^n = n^r(A_0 + \sum_{i=1}^{s} A_i n^i)d^n, \tag{1.5}$$

where the coefficients $A_i \in \mathbb{C}[x]$ or $\mathbb{C}$ and r is the multiplicity of d as a root of the characteristic equation (1.3), (if d is not a root of characteristic equation (1.3) then $r = 0$).

We proceed to formulate a theorem that provides explicit iterative expressions for the coefficients appearing in the particular solution $W_n^{(C)}$ of (1.4). The derivation is governed by the relationship between the parameter d and the characteristic roots of (1.3). When d coincides with a root of multiplicity r, the iterative procedure requires precise adjustments that reflect this multiplicity, ensuring the correct construction of the solution.

Theorem 1.2. *[20, p.100, Theorem 5.1] For each $0 \leq i \leq s$, A_i given in (1.5) can be calculated with the iteration as follows:*

- *If $r = 0$, i.e., none of the roots of the characteristic equation (1.3) equals d, then*

$$A_s = -\frac{c_s b d^m}{a_1 d^{m-1} + a_2 d^{m-2} + a_2 d^{m-3} + \ldots + a_{m-2} d^2 + a_{m-1} d + a_m - d^m} = -\frac{c_s b d^m}{-d^m + \sum_{j=1}^{m} a_j d^{m-j}}, \text{ for } n = s$$

  and

$$A_n = -\frac{1}{-d^m + \sum_{j=1}^{m} a_j d^{m-j}}(c_n b d^m - \sum_{k=n+1}^{s} (-1)^{k-n+1}\binom{k}{n}(\sum_{j=1}^{m} j^{k-n} a_j d^{m-j})A_k)$$

  for $n = s-1, s-2, \ldots, 2, 1, 0$.

- *If $r > 0$ then*

$$A_s = (-1)^{r+1}\frac{c_s b d^m}{(\sum_{j=1}^{m} j^r a_j \times d^{m-j})\binom{s+r}{r}}, \text{ for } n = s$$

  and

$$A_n = (-1)^{r+1}\frac{1}{(\sum_{j=1}^{m} j^r a_j \times d^{m-j})\binom{n+r}{r}}(c_n b d^m - \sum_{k=n+1}^{s} (-1)^{k+r-n+1}\binom{k+r}{n}(\sum_{j=1}^{m} j^{k+r-n} a_j \times d^{m-j})A_k)$$

  for $n = s-1, s-2, \ldots, 2, 1, 0$.

In the following sections, we present explicit particular solutions to (1.4) for $m = 1, 2, 3, 4$, where

$$C(n) = p(n)bd^n, \quad p(n) \text{ is a polynomial in } n.$$

We seek solutions of the form

$$W_n^{(C)} = P(n)d^n,$$

where $P(n)$ is itself a polynomial in n.

2. Special Cases

2.1. The Case $m = 3$. Consider the homogeneous relation

$$V_n = a_1V_{n-1} + a_2V_{n-2} + a_3V_{n-3} \tag{2.1}$$

with the initial conditions V_0, V_1, V_2. Suppose that $\theta_1, \theta_2, \theta_3$ are the roots of characteristic equation

$$z^3 - a_1z^2 - a_2z - a_3 = 0 \tag{2.2}$$

associated with (2.1). Note that if all the roots of (2.2) are equal to d then

$$z^3 - a_1z^2 - a_2z - a_3 = (z-d)^3 = z^3 - 3dz^2 + 3d^2z - d^3 = 0$$

so that $a_1 = 3d$, $a_2 = -3d^2$, $a_3 = d^3$ and (2.1) reduces to

$$V_n = 3dV_{n-1} - 3d^2V_{n-2} + d^3V_{n-3}.$$

We now turn to an example that demonstrates the results derived above.

Example 2.1. *Consider the sequence (W_n) defined by the recurrence relation*

$$W_n = a_1W_{n-1} + a_2W_{n-2} + a_3W_{n-3} + p(n)bd^n$$

where $p(n) := p(n,x)$ is a polynomial in n of order s, with coefficients belonging to $\mathbb{C}[x]$ or $\mathbb{C}$ and $b \in \mathbb{C}[x]$ or $\mathbb{C}$, and $d \in \mathbb{C}$ or $\mathbb{R}$. We seek a particular solution

$$W_n^C = P(n)d^n$$

for the cases $s = 0, 1, 2, 3$ where $P(n)$ is itself a polynomial in n. The order (degree) and coffecients of $P(n)$ depend on the multiplicity r of d as a root of the characteristic equation (2.2) and W_n^C satisfy

$$W_n^C = a_1W_{n-1}^C + a_2W_{n-2}^C + a_3W_{n-3}^C + p(n)bd^n$$

i.e.,

$$P(n)d^n = a_1P(n-1)d^{n-1} + a_2P(n-2)d^{n-2} + a_3P(n-3)d^{n-3} + p(n)bd^n.$$

In each case of s, we consider the homogeneous relation (2.1) and its characteristic equation (2.2), corresponding to the sequence (W_n) with the same initial conditions as W_n, i.e.,

$$V_0 = W_0, V_1 = W_1, V_2 = W_2.$$

We investigate all cases of multiplicity r of of d as a root of the characteristic equation (2.2):

(a): $m = 3$, $s = 0$. *Consider the sequence (W_n) defined by*

$$W_n = a_1W_{n-1} + a_2W_{n-2} + a_3W_{n-3} + c_0bd^n.$$

(i): *Case $r = 0$, i.e., none of the roots of the characteristic equation equals d:*

$$W_n^C = A_0d^n, \quad A_0 = -\frac{c_0bd^3}{a_1d^2 + a_2d + a_3 - d^3}.$$

(ii): *Case $r=1$, i.e., exactly one root of the characteristic equation equals d:*

$$W_n^C = nA_0d^n, \quad A_0 = \frac{c_0bd^3}{(a_1d^2+2a_2d+3a_3)}.$$

(iii): *Case $r=2$, i.e., exactly two roots of the characteristic equation equal d:*

$$W_n^C = n^2A_0d^n, \quad A_0 = -\frac{c_0bd^3}{(a_1d^2+4a_2d+9a_3)}.$$

(iv): *Case $r=3$, i.e., all three roots of the characteristic equation equal d:*

$$W_n^C = n^3A_0d^n, \quad A_0 = \frac{1}{6}bc_0.$$

(b): $m=3$, $s=1$. *Consider the sequence (W_n) defined by*

$$W_n = a_1W_{n-1} + a_2W_{n-2} + a_3W_{n-3} + (c_1n+c_0)bd^n.$$

(i): *Case $r=0$ (no root equal to d):*

$$W_n^C = (A_1n+A_0)d^n$$

where

$$\begin{aligned} A_1 &= -\frac{c_1bd^3}{(a_1d^2+a_2d+a_3-d^3)}, \\ A_0 &= -\frac{1}{(a_1d^2+a_2d+a_3-d^3)}(c_0bd^3-(a_1d^2+2a_2d+3a_3)A_1), \end{aligned}$$

i.e.,

$$\begin{aligned} A_1 &= -\frac{c_1bd^3}{(a_1d^2+a_2d+a_3-d^3)}, \\ A_0 &= -\frac{bd^3}{(a_1d^2+a_2d+a_3-d^3)^2}(-c_0d^3+a_1(c_0+c_1)d^2+a_2(c_0+2c_1)d+a_3(c_0+3c_1)). \end{aligned}$$

(ii): *Case $r=1$ (exactly one root equal to d):*

$$W_n^C = n(A_1n+A_0)d^n$$

where

$$\begin{aligned} A_1 &= \frac{c_1bd^3}{2\left(a_1d^2+2a_2d+3a_3\right)}, \\ A_0 &= \frac{1}{(a_1d^2+2a_2d+3a_3)}(c_0bd^3+\left(a_1d^2+4a_2d+9a_3\right)A_1), \end{aligned}$$

i.e.,

$$\begin{aligned} A_1 &= \frac{c_1bd^3}{2\left(a_1d^2+2a_2d+3a_3\right)}, \\ A_0 &= \frac{bd^3}{2\left(a_1d^2+2a_2d+3a_3\right)^2}(a_1(2c_0+c_1)d^2+4a_2(c_0+c_1)d+3a_3(2c_0+3c_1)). \end{aligned}$$

(iii): *Case* $r = 2$ *(exactly two roots equal to* d*):*

$$W_n^C = n^2(A_1 n + A_0)d^n$$

where

$$\begin{aligned} A_1 &= -\frac{c_1 b d^3}{3\left(a_1 d^2 + 4a_2 d + 9a_3\right)}, \\ A_0 &= -\frac{1}{\left(a_1 d^2 + 4a_2 d + 9a_3\right)}\left(c_0 b d^3 - \left(a_1 d^2 + 8a_2 d + 27a_3\right) A_1\right), \end{aligned}$$

i.e.,

$$\begin{aligned} A_1 &= -\frac{c_1 b d^3}{3\left(a_1 d^2 + 4a_2 d + 9a_3\right)}, \\ A_0 &= -\frac{b d^3}{3\left(a_1 d^2 + 4a_2 d + 9a_3\right)^2}\left(a_1(3c_0 + c_1)d^2 + 4a_2(3c_0 + 2c_1)d + 27a_3(c_0 + c_1)\right). \end{aligned}$$

(iv): *Case* $r = 3$ *(all three roots equal to* d*):*

$$W_n^C = n^3(A_1 n + A_0)d^n$$

where

$$\begin{aligned} A_1 &= \frac{1}{24} b c_1, \\ A_0 &= \frac{1}{12} b(2c_0 + 3c_1). \end{aligned}$$

(c): $m = 3$, $s = 2$. *Consider the sequence* (W_n) *defined by*

$$W_n = a_1 W_{n-1} + a_2 W_{n-2} + a_3 W_{n-3} + (c_2 n^2 + c_1 n + c_0) b d^n.$$

(i): *Case* $r = 0$ *(no root equal to* d*):*

$$W_n^C = (A_2 n^2 + A_1 n + A_0)d^n$$

where

$$\begin{aligned} A_2 &= -\frac{c_2 b d^3}{(a_1 d^2 + a_2 d + a_3 - d^3)}, \\ A_1 &= -\frac{1}{(a_1 d^2 + a_2 d + a_3 - d^3)}(c_1 b d^3 - 2(a_1 d^2 + 2a_2 d + 3a_3)A_2), \\ A_0 &= -\frac{1}{(a_1 d^2 + a_2 d + a_3 - d^3)}(c_0 b d^3 - (a_1 d^2 + 2a_2 d + 3a_3)A_1 + (a_1 d^2 + 4a_2 d + 9a_3)A_2), \end{aligned}$$

i.e.,

$$A_2 = -\frac{c_2 b d^3}{(a_1 d^2 + a_2 d + a_3 - d^3)},$$

$$A_1 = -\frac{b d^3}{(a_1 d^2 + a_2 d + a_3 - d^3)^2}\left(-c_1 d^3 + a_1\left(c_1 + 2c_2\right)d^2 + a_2\left(c_1 + 4c_2\right)d + a_3\left(c_1 + 6c_2\right)\right),$$

$$A_0 = -\frac{b d^3}{(a_1 d^2 + a_2 d + a_3 - d^3)^3}(c_0 d^6 + a_1^2(c_0 + c_1 + c_2)d^4 + a_2^2(c_0 + 2c_1 + 4c_2)d^2 + a_3^2(c_0 + 3c_1 + 9c_2) - a_1(2c_0 + c_1 - c_2)d^5 - 2a_2(c_0 + c_1 - 2c_2)d^4 - a_3(2c_0 + 3c_1 - 9c_2)d^3 + a_1 a_2(2c_0 + 3c_1 + 3c_2)d^3 + 2a_1 a_3(c_0 + 2c_1 + c_2)d^2 + a_2 a_3(2c_0 + 5c_1 + 11c_2)d).$$

(ii): *Case $r=1$ (exactly one root equal to d):*

$$W_n^C = n(A_2n^2 + A_1n + A_0)d^n$$

where

$$\begin{aligned}
A_2 &= \frac{c_2bd^3}{3\left(a_1d^2+2a_2d+3a_3\right)},\\
A_1 &= \frac{1}{2\left(a_1d^2+2a_2d+3a_3\right)}(c_1bd^3 + 3\left(a_1d^2+4a_2d+9a_3\right)A_2),\\
A_0 &= \frac{1}{\left(a_1d^2+2a_2d+3a_3\right)}(c_0bd^3 + \left(a_1d^2+4a_2d+9a_3\right)A_1 - \left(a_1d^2+8a_2d+27a_3\right)A_2),
\end{aligned}$$

i.e.,

$$A_2 = \frac{c_2bd^3}{3\left(a_1d^2+2a_2d+3a_3\right)},$$

$$A_1 = \frac{bd^3}{2\left(a_1d^2+2a_2d+3a_3\right)^2}(a_1(c_1+c_2)d^2 + 2a_2(c_1+2c_2)d + 3a_3(c_1+3c_2)),$$

$$A_0 = \frac{bd^3}{6\left(a_1d^2+2a_2d+3a_3\right)^3}(a_1^2(6c_0+3c_1+c_2)d^4 + 8a_2^2(3c_0+3c_1+2c_2)d^2 + 27a_3^2(2c_0+3c_1+3c_2) + 2a_1a_2(12c_0+9c_1+2c_2)d^3 + 6a_1a_3(6c_0+6c_1-c_2)d^2 + 6a_2a_3(12c_0+15c_1+10c_2)d).$$

(iii): *Case $r=2$ (exactly two roots equal to d):*

$$W_n^C = n^2(A_2n^2 + A_1n + A_0)d^n$$

where

$$\begin{aligned}
A_2 &= -\frac{c_2bd^3}{6\left(a_1d^2+4a_2d+9a_3\right)},\\
A_1 &= -\frac{1}{3\left(a_1d^2+4a_2d+9a_3\right)}(c_1bd^3 - 4\left(a_1d^2+8a_2d+27a_3\right)A_2),\\
A_0 &= -\frac{1}{\left(a_1d^2+4a_2d+9a_3\right)}(c_0bd^3 - \left(a_1d^2+8a_2d+27a_3\right)A_1 + \left(a_1d^2+16a_2d+81a_3\right)A_2),
\end{aligned}$$

i.e.,

$$A_2 = -\frac{c_2bd^3}{6\left(a_1d^2+4a_2d+9a_3\right)},$$

$$A_1 = -\frac{bd^3}{9\left(a_1d^2+4a_2d+9a_3\right)^2}(a_1(3c_1+2c_2)d^2 + 4a_2(3c_1+4c_2)d + 27a_3(c_1+2c_2)),$$

$$A_0 = -\frac{bd^3}{18\left(a_1d^2+4a_2d+9a_3\right)^3}(a_1^2(18c_0+6c_1+c_2)d^4 + 32a_2^2(9c_0+6c_1+2c_2)d^2 + 729a_3^2(2c_0+2c_1+c_2) + 4a_1a_2(36c_0+18c_1+c_2)d^3 + 54a_1a_3(6c_0+4c_1-c_2)d^2 + 108a_2a_3(12c_0+10c_1+3c_2)d).$$

(iv): *Case $r=3$ (all three roots equal to d):*

$$W_n^C = n^3(A_2n^2 + A_1n + A_0)d^n$$

where

$$\begin{aligned}
A_2 &= \frac{1}{60}bc_2,\\
A_1 &= \frac{1}{24}b(c_1+3c_2),\\
A_0 &= \frac{1}{12}b(2c_0+3c_1+4c_2).
\end{aligned}$$

(d): $m = 3$, $s = 3$. *Consider the sequence* (W_n) *defined by*

$$W_n = a_1W_{n-1} + a_2W_{n-2} + a_3W_{n-3} + (c_3n^3 + c_2n^2 + c_1n + c_0)bd^n.$$

(i): *Case* $r = 0$ *(no root equal to* d*):*

$$W_n^C = (A_3n^3 + A_2n^2 + A_1n + A_0)d^n$$

where

$A_3 = -\dfrac{c_3bd^3}{(a_1d^2 + a_2d + a_3 - d^3)}$,

$A_2 = -\dfrac{1}{(a_1d^2 + a_2d + a_3 - d^3)}(c_2bd^3 - 3(a_1d^2 + 2a_2d + 3a_3)A_3)$,

$A_1 = -\dfrac{1}{(a_1d^2 + a_2d + a_3 - d^3)}(c_1bd^3 - 2(a_1d^2 + 2a_2d + 3a_3)A_2 + 3(a_1d^2 + 4a_2d + 9a_3)A_3)$,

$A_0 = -\dfrac{1}{(a_1d^2 + a_2d + a_3 - d^3)}(c_0bd^3 - (a_1d^2 + 2a_2d + 3a_3)A_1 + (a_1d^2 + 4a_2d + 9a_3)A_2 - (a_1d^2 +$
$8a_2d + 27a_3)A_3)$,

i.e.,

$A_3 = -\dfrac{c_3bd^3}{(a_1d^2 + a_2d + a_3 - d^3)}$,

$A_2 = -\dfrac{bd^3}{(a_1d^2 + a_2d + a_3 - d^3)^2}(-c_2d^3 + a_1(c_2 + 3c_3)d^2 + a_2(c_2 + 6c_3)d + a_3(c_2 + 9c_3))$,

$A_1 = -\dfrac{bd^3}{(a_1d^2 + a_2d + a_3 - d^3)^3}(c_1d^6 + a_1^2(c_1 + 2c_2 + 3c_3)d^4 + a_2^2(c_1 + 4c_2 + 12c_3)d^2 + a_3^2(c_1 + 6c_2 + 27c_3)$
$- a_1(2c_1 + 2c_2 - 3c_3)d^5 - 2a_2(c_1 + 2c_2 - 6c_3)d^4 - a_3(2c_1 + 6c_2 - 27c_3)d^3 + a_1a_2(2c_1 + 6c_2 + 9c_3)d^3 +$
$2a_1a_3(c_1 + 4c_2 + 3c_3)d^2 + a_2a_3(2c_1 + 10c_2 + 33c_3)d)$,

$A_0 = -\dfrac{bd^3}{(a_1d^2 + a_2d + a_3 - d^3)^4}(-c_0d^9 + a_1^3(c_0 + c_1 + c_2 + c_3)d^6 + a_2^3(c_0 + 2c_1 + 4c_2 + 8c_3)d^3 +$
$a_3^3(c_0 + 3c_1 + 9c_2 + 27c_3) - a_1^2(3c_0 + 2c_1 - 4c_3)d^7 - a_2^2(3c_0 + 4c_1 - 32c_3)d^5 - 3a_3^2(c_0 + 2c_1 - 36c_3)d^3 +$
$a_1(3c_0 + c_1 - c_2 + c_3)d^8 + a_2(3c_0 + 2c_1 - 4c_2 + 8c_3)d^7 + 3a_3(c_0 + c_1 - 3c_2 + 9c_3)d^6 - 2a_1a_2(3c_0 + 3c_1 - c_2 -$
$9c_3)d^6 - 2a_1a_3(3c_0 + 4c_1 - 4c_2 - 8c_3)d^5 - 2a_2a_3(3c_0 + 5c_1 - c_2 - 55c_3)d^4 + a_1a_2^2(3c_0 + 5c_1 + 7c_2 + 5c_3)$
$d^4 + a_1^2a_2(3c_0 + 4c_1 + 4c_2 + 4c_3)d^5 + a_1a_3^2(3c_0 + 7c_1 + 11c_2 - 17c_3)d^2 + a_1^2a_3(3c_0 + 5c_1 + 3c_2 + 5c_3)$
$d^4 + a_2a_3^2(3c_0 + 8c_1 + 20c_2 + 44c_3)d + a_2^2a_3(3c_0 + 7c_1 + 15c_2 + 31c_3)d^2 + 2a_1a_2a_3(3c_0 + 6c_1 + 8c_2)d^3)$.

(ii): *Case* $r = 1$ *(exactly one root equal to* d*):*

$$W_n^C = n(A_3n^3 + A_2n^2 + A_1n + A_0)d^n$$

where

$A_3 = \dfrac{c_3bd^3}{4\,(a_1d^2 + 2a_2d + 3a_3)}$,

$A_2 = \dfrac{1}{3\,(a_1d^2 + 2a_2d + 3a_3)}(c_2bd^3 + 6(a_1d^2 + 4a_2d + 9a_3)A_3)$,

$A_1 = \dfrac{1}{2\,(a_1d^2 + 2a_2d + 3a_3)}(c_1bd^3 + 3(a_1d^2 + 4a_2d + 9a_3)A_2 - 4(a_1d^2 + 8a_2d + 27a_3)A_3)$,

$A_0 = \dfrac{1}{(a_1d^2 + 2a_2d + 3a_3)}(c_0bd^3 + (a_1d^2 + 4a_2d + 9a_3)A_1 - (a_1d^2 + 8a_2d + 27a_3)A_2 + (a_1d^2 + 16a_2d +$
$81a_3)A_3)$,

i.e.,

$A_3 = \dfrac{c_3bd^3}{4\,(a_1d^2 + 2a_2d + 3a_3)}$,

$$A_2 = \frac{bd^3}{6\left(a_1 d^2 + 2a_2 d + 3a_3\right)^2}(a_1(2c_2 + 3c_3)d^2 + 4a_2(c_2 + 3c_3)d + 3a_3(2c_2 + 9c_3)),$$

$$A_1 = \frac{bd^3}{4\left(a_1 d^2 + 2a_2 d + 3a_3\right)^3}(a_1^2(2c_1 + 2c_2 + c_3)d^4 + 8a_2^2(c_1 + 2c_2 + 2c_3)d^2 + 9a_3^2(2c_1 + 6c_2 + 9c_3) + 4a_1a_2(2c_1 + 3c_2 + c_3)d^3 + 6a_1a_3(2c_1 + 4c_2 - c_3)d^2 + 12a_2a_3(2c_1 + 5c_2 + 5c_3)d),$$

$$A_0 = \frac{bd^3}{6\left(a_1 d^2 + 2a_2 d + 3a_3\right)^4}(a_1^3(6c_0 + 3c_1 + c_2)d^6 + 16a_2^3(3c_0 + 3c_1 + 2c_2)d^3 + 81a_3^3(2c_0 + 3c_1 + 3c_2) + 6a_1a_2^2(12c_0 + 10c_1 + 4c_2 - 3c_3)d^4 + 6a_1^2a_2(6c_0 + 4c_1 + c_2)d^5 + 9a_1a_3^2(18c_0 + 21c_1 + 7c_2 - 30c_3)d^2 + 3a_1^2a_3(18c_0 + 15c_1 - c_2 + 6c_3)d^4 + 18a_2a_3^2(18c_0 + 24c_1 + 19c_2 - 6c_3)d + 6a_2^2a_3(36c_0 + 42c_1 + 28c_2 - 3c_3)d^2 + 12a_1a_2a_3(18c_0 + 18c_1 + 5c_2 - 9c_3)d^3).$$

(iii): *Case $r = 2$ (exactly two roots equal to d):*

$$W_n^C = n^2(A_3n^3 + A_2n^2 + A_1n + A_0)d^n$$

where

$$A_3 = -\frac{c_3bd^3}{10\left(a_1 d^2 + 4a_2 d + 9a_3\right)},$$

$$A_2 = -\frac{1}{6\left(a_1 d^2 + 4a_2 d + 9a_3\right)}(c_2bd^3 - 10(a_1d^2 + 8a_2d + 27a_3)A_3),$$

$$A_1 = -\frac{1}{3\left(a_1 d^2 + 4a_2 d + 9a_3\right)}(c_1bd^3 - 4(a_1d^2 + 8a_2d + 27a_3)A_2 + 5(a_1d^2 + 16a_2d + 81a_3)A_3),$$

$$A_0 = -\frac{1}{\left(a_1 d^2 + 4a_2 d + 9a_3\right)}(c_0bd^3 - (a_1d^2 + 8a_2d + 27a_3)A_1 + (a_1d^2 + 16a_2d + 81a_3)A_2 - (a_1d^2 + 32a_2d + 243a_3)A_3),$$

i.e.,

$$A_3 = -\frac{c_3bd^3}{10\left(a_1 d^2 + 4a_2 d + 9a_3\right)},$$

$$A_2 = -\frac{bd^3}{6\left(a_1 d^2 + 4a_2 d + 9a_3\right)^2}(a_1(c_2 + c_3)d^2 + 4a_2(c_2 + 2c_3)d + 9a_3(c_2 + 3c_3)),$$

$$A_1 = -\frac{bd^3}{18\left(a_1 d^2 + 4a_2 d + 9a_3\right)^3}(a_1^2(6c_1 + 4c_2 + c_3)d^4 + 32a_2^2(3c_1 + 4c_2 + 2c_3)d^2 + 243a_3^2(2c_1 + 4c_2 + 3c_3) + 4a_1a_2(12c_1 + 12c_2 + c_3)d^3 + 18a_1a_3(6c_1 + 8c_2 - 3c_3)d^2 + 36a_2a_3(12c_1 + 20c_2 + 9c_3)d),$$

$$A_0 = -\frac{bd^3}{90\left(a_1 d^2 + 4a_2 d + 9a_3\right)^4}(a_1^3(90c_0 + 30c_1 + 5c_2 - c_3)d^6 + 128a_2^3(45c_0 + 30c_1 + 10c_2 - 4c_3)d^3 + 6561a_3^3(10c_0 + 10c_1 + 5c_2 - 3c_3) + 16a_1a_2^2(270c_0 + 150c_1 + 25c_2 - 27c_3)d^4 + 40a_1^2a_2(27c_0 + 12c_1 + c_2)d^5 + 1215a_1a_3^2(18c_0 + 14c_1 + c_2 - 9c_3)d^2 + 9a_1^2a_3(270c_0 + 150c_1 - 25c_2 + 51c_3)d^4 + 1944a_2a_3^2(45c_0 + 40c_1 + 15c_2 - 12c_3)d + 144a_2^2a_3(270c_0 + 210c_1 + 65c_2 - 33c_3)d^2 + 144a_1a_2a_3(135c_0 + 90c_1 + 5c_2 - 18c_3)d^3).$$

(iv): *Case $r = 3$ (all three roots equal to d):*

$$W_n^C = n^3(A_3n^3 + A_2n^2 + A_1n + A_0)d^n$$

where

$$\begin{aligned} A_3 &= \frac{1}{120}bc_3, \\ A_2 &= \frac{1}{120}b(2c_2 + 9c_3), \\ A_1 &= \frac{1}{24}b(c_1 + 3c_2 + 6c_3), \\ A_0 &= \frac{1}{24}b(4c_0 + 6c_1 + 8c_2 + 9c_3). \end{aligned}$$

2.2. The Case $m = 4$. Consider the homogeneous relation

$$V_n = a_1V_{n-1} + a_2V_{n-2} + a_3V_{n-3} + a_4V_{n-4} \tag{2.3}$$

with the initial conditions V_0, V_1, V_2, V_3. Suppose that $\theta_1, \theta_2, \theta_3, \theta_4$ are the roots of characteristic equation

$$z^4 - a_1z^3 - a_2z^2 - a_3z - a_4 = 0 \tag{2.4}$$

associated with (2.3). Note that if all the roots of (2.4) are equal to d then

$$z^4 - a_1z^3 - a_2z^2 - a_3z - a_4 = (z-d)^4 = z^4 - 4dz^3 + 6d^2z^2 - 4d^3z + d^4 = 0$$

so that $a_1 = 4d$, $a_2 = -6d^2$, $a_3 = 4d^3$, $a_4 = -d^4$ and (2.3) reduces to

$$V_n = 4dV_{n-1} - 6d^2V_{n-2} + 4d^3V_{n-3} - d^4V_{n-4}.$$

Example 2.2. *Consider the sequence (W_n) defined by the recurrence relation*

$$W_n = a_1W_{n-1} + a_2W_{n-2} + a_3W_{n-3} + a_4W_{n-4} + p(n)bd^n$$

where $p(n) := p(n,x)$ is a polynomial in n of order s, with coefficients belonging to $\mathbb{C}[x]$ or $\mathbb{C}$ and $b \in \mathbb{C}[x]$ or $\mathbb{C}$, and $d \in \mathbb{C}$ or $\mathbb{R}$. We seek a particular solution

$$W_n^C = P(n)d^n$$

for the cases $s = 0, 1, 2, 3$ where $P(n)$ is itself a polynomial in n. The order (degree) and coffecients of $P(n)$ depend on the multiplicity r of d as a root of the characteristic equation (2.4) and W_n^C satisfy

$$W_n^C = a_1W_{n-1}^C + a_2W_{n-2}^C + a_3W_{n-3}^C + a_4W_{n-4}^C + p(n)bd^n$$

i.e.,

$$P(n)d^n = a_1P(n-1)d^{n-1} + a_2P(n-2)d^{n-2} + a_3P(n-3)d^{n-3} + a_4P(n-4)d^{n-4} + p(n)bd^n.$$

In each case of s, we consider the homogeneous relation (2.3) and its characteristic equation (2.4), corresponding to the sequence (W_n) with the same initial conditions as W_n, i.e.,

$$V_0 = W_0, V_1 = W_1, V_2 = W_2, V_3 = W_3.$$

We investigate all cases of multiplicity r of of d as a root of the characteristic equation (2.4):

- **(a):** $m = 4$, $s = 0$. *Consider the sequence (W_n) defined by*

$$W_n = a_1W_{n-1} + a_2W_{n-2} + a_3W_{n-3} + a_4W_{n-4} + c_0bd^n.$$

  - **(i):** *Case $r = 0$, i.e., none of the roots of the characteristic equation equals d:*

$$W_n^C = A_0d^n, \quad A_0 = -\frac{c_0bd^4}{a_1d^3 + a_2d^2 + a_3d + a_4 - d^4}.$$

  - **(ii):** *Case $r = 1$, i.e., exactly one root of the characteristic equation equals d:*

$$W_n^C = nA_0d^n, \quad A_0 = \frac{c_0bd^4}{(a_1d^3 + 2a_2d^2 + 3a_3d + 4a_4)}.$$

(iii): *Case* $r=2$*, i.e., exactly two roots of the characteristic equation equal* d*:*

$$W_n^C = n^2 A_0 d^n, \quad A_0 = -\frac{bc_0 d^4}{(a_1 d^3 + 4a_2 d^2 + 9a_3 d + 16a_4)}.$$

(iv): *Case* $r=3$*, i.e., exactly three roots of the characteristic equation equal* d*:*

$$W_n^C = n^3 A_0 d^n, \quad A_0 = \frac{c_0 b d^4}{(a_1 d^3 + 8a_2 d^2 + 27a_3 d + 64a_4)}.$$

(v): *Case* $r=4$*, i.e., all four roots of the characteristic equation equal* d*:*

$$W_n^C = n^4 A_0 d^n, \quad A_0 = \frac{1}{24} b c_0.$$

(b): $m=4$, $s=1$*. Consider the sequence* (W_n) *defined by*

$$W_n = a_1 W_{n-1} + a_2 W_{n-2} + a_3 W_{n-3} + a_4 W_{n-4} + (c_1 n + c_0) b d^n.$$

(i): *Case* $r=0$ *(no root equal to* d*):*

$$W_n^C = (A_1 n + A_0) d^n$$

where

$$\begin{aligned} A_1 &= -\frac{c_1 b d^4}{(a_1 d^3 + a_2 d^2 + a_3 d + a_4 - d^4)}, \\ A_0 &= -\frac{1}{(a_1 d^3 + a_2 d^2 + a_3 d + a_4 - d^4)}(c_0 b d^4 - (a_1 d^3 + 2a_2 d^2 + 3a_3 d + 4a_4) A_1), \end{aligned}$$

i.e.,

$$A_1 = -\frac{c_1 b d^4}{(a_1 d^3 + a_2 d^2 + a_3 d + a_4 - d^4)},$$

$$A_0 = -\frac{b d^4}{(a_1 d^3 + a_2 d^2 + a_3 d + a_4 - d^4)^2}(-c_0 d^4 + a_1(c_0 + c_1) d^3 + a_2(c_0 + 2c_1) d^2 + a_3(c_0 + 3c_1) d + a_4(c_0 + 4c_1)).$$

(ii): *Case* $r=1$ *(exactly one root equal to* d*):*

$$W_n^C = n(A_1 n + A_0) d^n$$

where

$$\begin{aligned} A_1 &= \frac{c_1 b d^4}{2\left(d^3 a_1 + 2a_2 d^2 + 3d a_3 + 4a_4\right)}, \\ A_0 &= \frac{1}{(d^3 a_1 + 2a_2 d^2 + 3d a_3 + 4a_4)}(c_0 b d^4 + \left(d^3 a_1 + 4d^2 a_2 + 9d a_3 + 16a_4\right) A_1), \end{aligned}$$

i.e.,

$$A_1 = \frac{c_1 b d^4}{2\left(d^3 a_1 + 2a_2 d^2 + 3d a_3 + 4a_4\right)},$$

$$A_0 = \frac{b d^4}{2\left(d^3 a_1 + 2a_2 d^2 + 3d a_3 + 4a_4\right)^2}(a_1(2c_0 + c_1) d^3 + 4a_2(c_0 + c_1) d^2 + 3a_3(2c_0 + 3c_1) d + 8a_4(c_0 + 2c_1)).$$

(iii): *Case* $r=2$ *(exactly two roots equal to* d*):*

$$W_n^C = n^2(A_1 n + A_0) d^n$$

where

$$\begin{aligned} A_1 &= -\frac{c_1bd^4}{3\left(a_1d^3+4a_2d^2+9a_3d+16a_4\right)}, \\ A_0 &= -\frac{1}{\left(a_1d^3+4a_2d^2+9a_3d+16a_4\right)}\left(c_0bd^4-\left(a_1d^3+8a_2d^2+27a_3d+64a_4\right)A_1\right), \end{aligned}$$

i.e.,

$$\begin{aligned} A_1 &= -\frac{c_1bd^4}{3\left(a_1d^3+4a_2d^2+9a_3d+16a_4\right)}, \\ A_0 &= -\frac{bd^4}{3\left(a_1d^3+4a_2d^2+9a_3d+16a_4\right)^2}\left(a_1(3c_0+c_1)d^3+4a_2(3c_0+2c_1)d^2+27a_3(c_0+c_1)d+16a_4(3c_0+4c_1)\right). \end{aligned}$$

(iv): *Case* $r=3$ *(exactly three roots equal to d):*

$$W_n^C = n^3(A_1n+A_0)d^n$$

where

$$\begin{aligned} A_1 &= \frac{c_1bd^4}{4\left(a_1d^3+8a_2d^2+27a_3d+64a_4\right)}, \\ A_0 &= \frac{1}{\left(a_1d^3+8a_2d^2+27a_3d+64a_4\right)}\left(c_0bd^4+(a_1d^3+16a_2d^2+81a_3d+256a_4)A_1\right), \end{aligned}$$

i.e.,

$$\begin{aligned} A_1 &= \frac{c_1bd^4}{4(a_1d^3+8a_2d^2+27a_3d+64a_4)}, \\ A_0 &= \frac{bd^4}{4(a_1d^3+8a_2d^2+27a_3d+64a_4)^2}\left(a_1(4c_0+c_1)d^3+16a_2(2c_0+c_1)d^2+27a_3(4c_0+3c_1)d+256a_4(c_0+c_1)\right). \end{aligned}$$

(v): *Case* $r=4$ *(all four roots equal to d):*

$$W_n^C = n^4(A_1n+A_0)d^n$$

where

$$\begin{aligned} A_1 &= \frac{1}{120}bc_1 \\ A_0 &= \frac{1}{24}b(c_0+2c_1) \end{aligned}$$

(c): $m=4,\ s=2$. *Consider the sequence* (W_n) *defined by*

$$W_n = a_1W_{n-1}+a_2W_{n-2}+a_3W_{n-3}+a_4W_{n-4}+(c_2n^2+c_1n+c_0)bd^n.$$

(i): *Case* $r=0$ *(no root equal to d):*

$$W_n^C = (A_2n^2+A_1n+A_0)d^n$$

where

$$\begin{aligned} &A_2 = -\frac{c_2bd^4}{(a_1d^3+a_2d^2+a_3d+a_4-d^4)}, \\ &A_1 = -\frac{1}{(a_1d^3+a_2d^2+a_3d+a_4-d^4)}\left(c_1bd^4-2(a_1d^3+2a_2d^2+3a_3d+4a_4)A_2\right), \\ &A_0 = -\frac{1}{(a_1d^3+a_2d^2+a_3d+a_4-d^4)}\left(c_0bd^4-(a_1d^3+2a_2d^2+3a_3d+4a_4)A_1+(a_1d^3+4a_2d^2+\right. \\ &\left.9a_3d+16a_4)A_2\right), \end{aligned}$$

i.e.,

$A_2 = -\dfrac{c_2 b d^4}{(a_1 d^3 + a_2 d^2 + a_3 d + a_4 - d^4)}$,

$A_1 = -\dfrac{b d^4}{(a_1 d^3 + a_2 d^2 + a_3 d + a_4 - d^4)^2}(-c_1 d^4 + a_1(c_1 + 2c_2)d^3 + a_2(c_1 + 4c_2)d^2 + a_3(c_1 + 6c_2)d +$ $a_4(c_1 + 8c_2))$,

$A_0 = -\dfrac{b d^4}{(a_1 d^3 + a_2 d^2 + a_3 d + a_4 - d^4)^3}(c_0 d^8 + a_1^2(c_0 + c_1 + c_2)d^6 + a_2^2(c_0 + 2c_1 + 4c_2)d^4 + a_3^2(c_0 +$ $3c_1 + 9c_2)d^2 + a_4^2(c_0 + 4c_1 + 16c_2) - a_1(2c_0 + c_1 - c_2)d^7 - 2a_2(c_0 + c_1 - 2c_2)d^6 - a_3(2c_0 + 3c_1 - 9c_2)d^5 -$ $2a_4(c_0 + 2c_1 - 8c_2)d^4 + a_1a_2(2c_0 + 3c_1 + 3c_2)d^5 + 2a_1a_3(c_0 + 2c_1 + c_2)d^4 + a_1a_4(2c_0 + 5c_1 - c_2)d^3 +$ $a_2a_3(2c_0 + 5c_1 + 11c_2)d^3 + 2a_2a_4(c_0 + 3c_1 + 6c_2)d^2 + a_3a_4(2c_0 + 7c_1 + 23c_2)d)$.

(ii): *Case $r = 1$ (exactly one root equal to d):*

$$W_n^C = n(A_2 n^2 + A_1 n + A_0)d^n$$

where

$$\begin{aligned}
A_2 &= \frac{c_2 b d^4}{3(a_1 d^3 + 2a_2 d^2 + 3a_3 d + 4a_4)},\\
A_1 &= \frac{1}{2(a_1 d^3 + 2a_2 d^2 + 3a_3 d + 4a_4)}(c_1 b d^4 + 3\left(d^3 a_1 + 4d^2 a_2 + 9d a_3 + 16a_4\right)A_2),\\
A_0 &= \frac{1}{(a_1 d^3 + 2a_2 d^2 + 3a_3 d + 4a_4)}(c_0 b d^4 + (d^3 a_1 + 4d^2 a_2 + 9d a_3 + 16a_4)A_1 - (d^3 a_1 + 8d^2 a_2 + 27d a_3 + 64a_4)A_2),
\end{aligned}$$

i.e.,

$A_2 = \dfrac{c_2 b d^4}{3(a_1 d^3 + 2a_2 d^2 + 3a_3 d + 4a_4)}$,

$A_1 = \dfrac{b d^4}{2(a_1 d^3 + 2a_2 d^2 + 3a_3 d + 4a_4)^2}(a_1(c_1 + c_2)d^3 + 2a_2(c_1 + 2c_2)d^2 + 3a_3(c_1 + 3c_2)d + 4a_4(c_1 + 4c_2))$,

$A_0 = \dfrac{b d^4}{6(a_1 d^3 + 2a_2 d^2 + 3a_3 d + 4a_4)^3}(a_1^2(6c_0 + 3c_1 + c_2)d^6 + 8a_2^2(3c_0 + 3c_1 + 2c_2)d^4 + 27a_3^2(2c_0 + 3c_1 +$ $3c_2)d^2 + 32a_4^2(3c_0 + 6c_1 + 8c_2) + 2a_1a_2(12c_0 + 9c_1 + 2c_2)d^5 + 6a_1a_3(6c_0 + 6c_1 - c_2)d^4 + 4a_1a_4(12c_0 +$ $15c_1 - 10c_2)d^3 + 6a_2a_3(12c_0 + 15c_1 + 10c_2)d^3 + 16a_2a_4(6c_0 + 9c_1 + 4c_2)d^2 + 12a_3a_4(12c_0 + 21c_1 + 22c_2)d)$.

(iii): *Case $r = 2$ (exactly two roots equal to d):*

$$W_n^C = n^2(A_2 n^2 + A_1 n + A_0)d^n$$

where

$A_2 = -\dfrac{c_2 b d^4}{6(a_1 d^3 + 4a_2 d^2 + 9a_3 d + 16a_4)}$,

$A_1 = -\dfrac{1}{3(a_1 d^3 + 4a_2 d^2 + 9a_3 d + 16a_4)}(c_1 b d^4 - 4\left(a_1 d^3 + 8a_2 d^2 + 27a_3 d + 64a_4\right)A_2)$,

$A_0 = -\dfrac{1}{(a_1 d^3 + 4a_2 d^2 + 9a_3 d + 16a_4)}(c_0 b d^4 - (a_1 d^3 + 8a_2 d^2 + 27a_3 d + 64a_4)A_1 + (a_1 d^3 + 16a_2 d^2 +$ $81a_3 d + 256a_4)A_2)$,

i.e.,

$A_2 = -\dfrac{c_2 b d^4}{6(a_1 d^3 + 4a_2 d^2 + 9a_3 d + 16a_4)}$,

$A_1 = -\dfrac{b d^4}{9(16a_4 + 9d a_3 + 4d^2 a_2 + d^3 a_1)^2}(a_1(3c_1 + 2c_2)d^3 + 4a_2(3c_1 + 4c_2)d^2 + 27a_3(c_1 + 2c_2)d +$ $16a_4(3c_1 + 8c_2))$,

$$A_0 = -\frac{bd^4}{18\,(16a_4+9da_3+4d^2a_2+d^3a_1)^3}(a_1^2(18c_0+6c_1+c_2)d^6+32a_2^2(9c_0+6c_1+2c_2)d^4+729a_3^2(2c_0+2c_1+c_2)d^2+512a_4^2(9c_0+12c_1+8c_2)+4a_1a_2(36c_0+18c_1+c_2)d^5+54a_1a_3(6c_0+4c_1-c_2)d^4+16a_1a_4(36c_0+30c_1-19c_2)d^3+108a_2a_3(12c_0+10c_1+3c_2)d^3+256a_2a_4(9c_0+9c_1+c_2)d^2+432a_3a_4(12c_0+14c_1+7c_2)d).$$

(iv): *Case* $r=3$ *(exactly three roots equal to d):*

$$W_n^C = n^3(A_2n^2+A_1n+A_0)d^n$$

where

$$A_2 = \frac{c_2bd^4}{10\,(a_1d^3+8a_2d^2+27a_3d+64a_4)},$$

$$A_1 = \frac{1}{4\,(a_1d^3+8a_2d^2+27a_3d+64a_4)}(c_1bd^4+5(a_1d^3+16a_2d^2+81a_3d+256a_4)A_2),$$

$$A_0 = \frac{1}{(a_1d^3+8a_2d^2+27a_3d+64a_4)}(c_0bd^4+(a_1d^3+16a_2d^2+81a_3d+256a_4)A_1-(a_1d^3+32a_2d^2+243a_3d+1024a_4)A_2),$$

i.e.,

$$A_2 = \frac{c_2bd^4}{10\,(a_1d^3+8a_2d^2+27a_3d+64a_4)},$$

$$A_1 = \frac{bd^4}{8\,(a_1d^3+8a_2d^2+27a_3d+64a_4)^2}(a_1(2c_1+c_2)d^3+16a_2(c_1+c_2)d^2+27a_3(2c_1+3c_2)d+128a_4(c_1+2c_2)),$$

$$A_0 = \frac{bd^4}{40\,(a_1d^3+8a_2d^2+27a_3d+64a_4)^3}(a_1^2(40c_0+10c_1+c_2)d^6+256a_2^2(10c_0+5c_1+c_2)d^4+729a_3^2(40c_0+30c_1+9c_2)d^2+32768a_4^2(5c_0+5c_1+2c_2)+80a_1a_2(8c_0+3c_1)d^5+270a_1a_3(8c_0+4c_1-c_2)d^4+128a_1a_4(40c_0+25c_1-14c_2)d^3+432a_2a_3(40c_0+25c_1+4c_2)d^3+10240a_2a_4(4c_0+3c_1)d^2+17280a_3a_4(8c_0+7c_1+2c_2)d).$$

(v): *Case* $r=4$ *(all four roots equal to d):*

$$W_n^C = n^4(A_2n^2+A_1n+A_0)d^n$$

where

$$\begin{aligned} A_2 &= \frac{1}{360}bc_2, \\ A_1 &= \frac{1}{120}b(c_1+4c_2), \\ A_0 &= \frac{1}{72}b(3c_0+6c_1+11c_2). \end{aligned}$$

(d): $m=4$, $s=3$. *Consider the sequence* (W_n) *defined by*

$$W_n = a_1W_{n-1}+a_2W_{n-2}+a_3W_{n-3}+a_4W_{n-4}+(c_3n^3+c_2n^2+c_1n+c_0)bd^n.$$

(i): *Case* $r=0$ *(no root equal to d):*

$$W_n^C = (A_3n^3+A_2n^2+A_1n+A_0)d^n$$

where

$$A_3 = -\frac{c_3bd^4}{(a_1d^3+a_2d^2+a_3d+a_4-d^4)},$$

$$A_2=-\frac{1}{(a_1d^3+a_2d^2+a_3d+a_4-d^4)}(c_2bd^4-3(a_1d^3+2a_2d^2+3a_3d+4a_4)A_3),$$

$$A_1=-\frac{1}{(a_1d^3+a_2d^2+a_3d+a_4-d^4)}(c_1bd^4-2(a_1d^3+2a_2d^2+3a_3d+4a_4)A_2+3(a_1d^3+4a_2d^2+9a_3d+16a_4)A_3),$$

$$A_0=-\frac{1}{(a_1d^3+a_2d^2+a_3d+a_4-d^4)}(c_0bd^4-(a_1d^3+2a_2d^2+3a_3d+4a_4)A_1+(a_1d^3+4a_2d^2+9a_3d+16a_4)A_2-(a_1d^3+8a_2d^2+27a_3d+64a_4)A_3),$$

i.e.,

$$A_3=-\frac{c_3bd^4}{(a_1d^3+a_2d^2+a_3d+a_4-d^4)},$$

$$A_2=-\frac{bd^4}{(a_1d^3+a_2d^2+a_3d+a_4-d^4)^2}(-c_2d^4+a_1(c_2+3c_3)d^3+a_2(c_2+6c_3)d^2+a_3(c_2+9c_3)d+(c_2+12c_3)a_4),$$

$$A_1=-\frac{bd^4}{(a_1d^3+a_2d^2+a_3d+a_4-d^4)^3}(c_1d^8+a_1^2(c_1+2c_2+3c_3)d^6+a_2^2(c_1+4c_2+12c_3)d^4+a_3^2(c_1+6c_2+27c_3)d^2+a_4^2(c_1+8c_2+48c_3)-a_1(2c_1+2c_2-3c_3)d^7-2a_2(c_1+2c_2-6c_3)d^6-a_3(2c_1+6c_2-27c_3)d^5-2a_4(c_1+4c_2-24c_3)d^4+a_1a_2(2c_1+6c_2+9c_3)d^5+2a_1a_3(c_1+4c_2+3c_3)d^4+a_1a_4(2c_1+10c_2-3c_3)d^3+a_2a_3(2c_1+10c_2+33c_3)d^3+2a_2a_4(c_1+6c_2+18c_3)d^2+a_3a_4(2c_1+14c_2+69c_3)d),$$

$$A_0=-\frac{bd^4}{(a_1d^3+a_2d^2+a_3d+a_4-d^4)^4}(-c_0d^{12}+a_1^3(c_0+c_1+c_2+c_3)d^9+a_2^3(c_0+2c_1+4c_2+8c_3)d^6+a_3^3(c_0+3c_1+9c_2+27c_3)d^3+a_4^3(c_0+4c_1+16c_2+64c_3)-a_1^2(3c_0+2c_1-4c_3)d^{10}-a_2^2(3c_0+4c_1-32c_3)d^8-3a_3^2(c_0+2c_1-36c_3)d^6-a_4^2(3c_0+8c_1-256c_3)d^4+a_1(3c_0+c_1-c_2+c_3)d^{11}+a_2(3c_0+2c_1-4c_2+8c_3)d^{10}+3a_3(c_0+c_1-3c_2+9c_3)d^9+a_4(3c_0+4c_1-16c_2+64c_3)d^8-2a_1a_2(3c_0+3c_1-c_2-9c_3)d^9-2a_1a_3(3c_0+4c_1-4c_2-8c_3)d^8-2a_1a_4(3c_0+5c_1-9c_2+5c_3)d^7-2a_2a_3(3c_0+5c_1-c_2-55c_3)d^7-2a_2a_4(3c_0+6c_1-4c_2-72c_3)d^6-2a_3a_4(3c_0+7c_1-c_2-161c_3)d^5+a_1a_2^2(3c_0+5c_1+7c_2+5c_3)d^7+a_1^2a_2(3c_0+4c_1+4c_2+4c_3)d^8+a_1a_3^2(3c_0+7c_1+11c_2-17c_3)d^5+a_1^2a_3(3c_0+5c_1+3c_2+5c_3)d^7+3a_1a_4^2(c_0+3c_1+5c_2-29c_3)d^3+3a_1^2a_4(c_0+2c_1+4c_3)d^6+a_2a_3^2(3c_0+8c_1+20c_2+44c_3)d^4+a_2^2a_3(3c_0+7c_1+15c_2+31c_3)d^5+a_2a_4^2(3c_0+10c_1+28c_2+40c_3)d^2+a_2^2a_4(3c_0+8c_1+16c_2+32c_3)d^4+a_3a_4^2(3c_0+11c_1+39c_2+131c_3)d+a_3^2a_4(3c_0+10c_1+32c_2+100c_3)d^2+2a_1a_2a_3(3c_0+6c_1+8c_2)d^6+2a_1a_2a_4(3c_0+7c_1+7c_2-5c_3)d^5+2a_1a_3a_4(3c_0+8c_1+12c_2-40c_3)d^4+2a_2a_3a_4(3c_0+9c_1+23c_2+45c_3)d^3),$$

(ii): *Case $r=1$ (exactly one root equal to d):*

$$W_n^C=n(A_3n^3+A_2n^2+A_1n+A_0)d^n$$

where

$$A_3=\frac{c_3bd^4}{4\,(a_1d^3+2a_2d^2+3a_3d+4a_4)},$$

$$A_2=\frac{1}{3\,(a_1d^3+2a_2d^2+3a_3d+4a_4)}(c_2bd^4+6(d^3a_1+4d^2a_2+9da_3+16a_4)A_3),$$

$$A_1=\frac{1}{2\,(a_1d^3+2a_2d^2+3a_3d+4a_4)}(c_1bd^4+3(d^3a_1+4d^2a_2+9da_3+16a_4)A_2-4(a_1d^3+8a_2d^2+27a_3d+64a_4)A_3),$$

$$A_0=\frac{1}{(a_1d^3+2a_2d^2+3a_3d+4a_4)}(c_0bd^4+(d^3a_1+4d^2a_2+9da_3+16a_4)A_1-(a_1d^3+8a_2d^2+27a_3d+64a_4)A_2+(a_1d^3+16a_2d^2+81a_3d+256a_4)A_3),$$

i.e.,

$$A_3 = \frac{c_3bd^4}{4\left(a_1d^3+2a_2d^2+3a_3d+4a_4\right)},$$

$$A_2 = \frac{bd^4}{6\left(a_1d^3+2a_2d^2+3a_3d+4a_4\right)^2}(a_1(2c_2+3c_3)d^3+4a_2(c_2+3c_3)d^2+3a_3(2c_2+9c_3)d+8a_4(c_2+6c_3)),$$

$$A_1 = \frac{bd^4}{4\left(a_1d^3+2a_2d^2+3a_3d+4a_4\right)^3}(a_1^2(2c_1+2c_2+c_3)d^6+8a_2^2(c_1+2c_2+2c_3)d^4+9a_3^2(2c_1+6c_2+9c_3)d^2+32a_4^2(c_1+4c_2+8c_3)+4a_1a_2(2c_1+3c_2+c_3)d^5+6a_1a_3(2c_1+4c_2-c_3)d^4+8a_1a_4(2c_1+5c_2-5c_3)d^3+12a_2a_3(2c_1+5c_2+5c_3)d^3+32a_2a_4(c_1+3c_2+2c_3)d^2+24a_3a_4(2c_1+7c_2+11c_3)d),$$

$$A_0 = \frac{bd^4}{6\left(a_1d^3+2a_2d^2+3a_3d+4a_4\right)^4}(a_1^3(6c_0+3c_1+c_2)d^9+16a_2^3(3c_0+3c_1+2c_2)d^6+81a_3^3(2c_0+3c_1+3c_2)d^3+128a_4^3(3c_0+6c_1+8c_2)+6a_1a_2^2(12c_0+10c_1+4c_2-3c_3)d^7+6a_1^2a_2(6c_0+4c_1+c_2)d^8+9a_1a_3^2(18c_0+21c_1+7c_2-30c_3)d^5+3a_1^2a_3(18c_0+15c_1-c_2+6c_3)d^7+24a_1a_4^2(12c_0+18c_1+4c_2-63c_3)d^3+36a_1^2a_4(2c_0+2c_1-c_2+3c_3)d^6+18a_2a_3^2(18c_0+24c_1+19c_2-6c_3)d^4+6a_2^2a_3(36c_0+42c_1+28c_2-3c_3)d^5+192a_2a_4^2(3c_0+5c_1+4c_2-6c_3)d^2+96a_2^2a_4(3c_0+4c_1+2c_2)d^4+24a_3a_4^2(36c_0+66c_1+76c_2-15c_3)d+36a_3^2a_4(18c_0+30c_1+31c_2-3c_3)d^2+12a_1a_2a_3(18c_0+18c_1+5c_2-9c_3)d^6+24a_1a_2a_4(12c_0+14c_1-3c_3)d^5+24a_1a_3a_4(18c_0+24c_1+5c_2-48c_3)d^4+24a_2a_3a_4(36c_0+54c_1+40c_2-27c_3)d^3).$$

(iii): *Case $r = 2$ (exactly two roots equal to d):*

$$W_n^C = n^2(A_3n^3+A_2n^2+A_1n+A_0)d^n$$

where

$$A_3 = -\frac{c_3bd^4}{10\left(a_1d^3+4a_2d^2+9a_3d+16a_4\right)},$$

$$A_2 = -\frac{1}{6\left(a_1d^3+4a_2d^2+9a_3d+16a_4\right)}(c_2bd^4-10(a_1d^3+8a_2d^2+27a_3d+64a_4)A_3),$$

$$A_1 = -\frac{1}{3\left(a_1d^3+4a_2d^2+9a_3d+16a_4\right)}(c_1bd^4-4(a_1d^3+8a_2d^2+27a_3d+64a_4)A_2+5(a_1d^3+16a_2d^2+81a_3d+256a_4)A_3),$$

$$A_0 = -\frac{1}{\left(a_1d^3+4a_2d^2+9a_3d+16a_4\right)}(c_0bd^4-(a_1d^3+8a_2d^2+27a_3d+64a_4)A_1+(a_1d^3+16a_2d^2+81a_3d+256a_4)A_2-(a_1d^3+32a_2d^2+243a_3d+1024a_4)A_3),$$

i.e.,

$$A_3 = -\frac{c_3bd^4}{10\left(a_1d^3+4a_2d^2+9a_3d+16a_4\right)},$$

$$A_2 = -\frac{bd^4}{6\left(a_1d^3+4a_2d^2+9a_3d+16a_4\right)^2}(a_1(c_2+c_3)d^3+4a_2(c_2+2c_3)d^2+9a_3(c_2+3c_3)d+16a_4(c_2+4c_3)),$$

$$A_1 = -\frac{bd^4}{18\left(a_1d^3+4a_2d^2+9a_3d+16a_4\right)^3}(a_1^2(6c_1+4c_2+c_3)d^6+32a_2^2(3c_1+4c_2+2c_3)d^4+243a_3^2(2c_1+4c_2+3c_3)d^2+512a_4^2(3c_1+8c_2+8c_3)+4a_1a_2(12c_1+12c_2+c_3)d^5+18a_1a_3(6c_1+8c_2-3c_3)d^4+16a_1a_4(12c_1+20c_2-19c_3)d^3+36a_2a_3(12c_1+20c_2+9c_3)d^3+256a_2a_4(3c_1+6c_2+c_3)d^2+144a_3a_4(12c_1+28c_2+21c_3)d),$$

$$A_0 = -\frac{bd^4}{90\left(a_1d^3+4a_2d^2+9a_3d+16a_4\right)^4}(a_1^3(90c_0+30c_1+5c_2-c_3)d^9+128a_2^3(45c_0+30c_1+10c_2-4c_3)d^6+6561a_3^3(10c_0+10c_1+5c_2-3c_3)d^3+8192a_4^3(45c_0+60c_1+40c_2-32c_3)+16a_1a_2^2(270c_0+150c_1+25c_2-27c_3)d^7+40a_1^2a_2(27c_0+12c_1+c_2)d^8+1215a_1a_3^2(18c_0+14c_1+c_2-9c_3)d^5+9a_1^2a_3(270c_0+150c_1-25c_2+51c_3)d^7+768a_1a_4^2(90c_0+90c_1-5c_2-133c_3)d^3+96a_1^2a_4(45c_0+30c_1-15c_2+34c_3)d^6+1944a_2a_3^2(45c_0+40c_1+15c_2-12c_3)d^4+144a_2^2a_3(270c_0+210c_1+65c_2-33c_3)d^5+2048a_2a_4^2(135c_0+$$

$150c_1+50c_2-108c_3)d^2+2560a_2^2a_4(27c_0+24c_1+4c_2)d^4+11520a_3a_4^2(54c_0+66c_1+37c_2-33c_3)d+7776a_3^2a_4(45c_0+50c_1+25c_2-18c_3)d^2+144a_1a_2a_3(135c_0+90c_1+5c_2-18c_3)d^6+128a_1a_2a_4(270c_0+210c_1-35c_2+27c_3)d^5+576a_1a_3a_4(135c_0+120c_1-5c_2-96c_3)d^4+1152a_2a_3a_4(270c_0+270c_1+85c_2-99c_3)d^3).$

(iv): *Case $r=3$ (exactly three roots equal to d):*

$$W_n^C=n^3(A_3n^3+A_2n^2+A_1n+A_0)d^n$$

where

$A_3=\dfrac{c_3bd^4}{20\left(a_1d^3+8a_2d^2+27a_3d+64a_4\right)},$

$A_2=\dfrac{1}{10\left(a_1d^3+8a_2d^2+27a_3d+64a_4\right)}(c_2bd^4+15(a_1d^3+16a_2d^2+81a_3d+256a_4)A_3),$

$A_1=\dfrac{1}{4\left(a_1d^3+8a_2d^2+27a_3d+64a_4\right)}(c_1bd^4+5A_2(a_1d^3+16a_2d^2+81a_3d+256a_4)-6A_3(a_1d^3+32a_2d^2+243a_3d+1024a_4)),$

$A_0=\dfrac{1}{\left(a_1d^3+8a_2d^2+27a_3d+64a_4\right)}(c_0bd^4+(a_1d^3+16a_2d^2+81a_3d+256a_4)A_1-(a_1d^3+32a_2d^2+243a_3d+1024a_4)A_2+(a_1d^3+64a_2d^2+729a_3d+4096a_4)A_3),$

i.e.,

$A_3=\dfrac{c_3bd^4}{20\left(a_1d^3+8a_2d^2+27a_3d+64a_4\right)},$

$A_2=\dfrac{bd^4}{40\left(a_1d^3+8a_2d^2+27a_3d+64a_4\right)^2}(a_1(4c_2+3c_3)d^3+16a_2(2c_2+3c_3)d^2+27a_3(4c_2+9c_3)d+256a_4(c_2+3c_3)),$

$A_1=\dfrac{bd^4}{160\left(a_1d^3+8a_2d^2+27a_3d+64a_4\right)^3}(a_1^2(40c_1+20c_2+3c_3)d^6+256a_2^2(10c_1+10c_2+3c_3)d^4+729a_3^2(40c_1+60c_2+27c_3)d^2+32768a_4^2(5c_1+10c_2+6c_3)+160a_1a_2(4c_1+3c_2)d^5+270a_1a_3(8c_1+8c_2-3c_3)d^4+256a_1a_4(20c_1+25c_2-21c_3)d^3+864a_2a_3(20c_1+25c_2+6c_3)d^3+20480a_2a_4(2c_1+3c_2)d^2+34560a_3a_4(4c_1+7c_2+3c_3)d),$

$A_0=\dfrac{bd^4}{160\left(a_1d^3+8a_2d^2+27a_3d+64a_4\right)^4}(a_1^3(160c_0+40c_1+4c_2-c_3)d^9+4096a_2^3(20c_0+10c_1+2c_2-c_3)d^6+19683a_3^3(160c_0+120c_1+36c_2-27c_3)d^3+8388608a_4^3(5c_0+5c_1+2c_2-2c_3)+256a_1a_2^2(120c_0+50c_1+4c_2-5c_3)d^7+16a_1^2a_2(240c_0+80c_1+2c_2+c_3)d^8+729a_1a_3^2(480c_0+280c_1-4c_2-91c_3)d^5+27a_1^2a_3(480c_0+200c_1-36c_2+55c_3)d^7+98304a_1a_4^2(20c_0+15c_1-2c_2-11c_3)d^3+768a_1^2a_4(40c_0+20c_1-9c_2+17c_3)d^6+11664a_2a_3^2(240c_0+160c_1+34c_2-31c_3)d^4+6912a_2^2a_3(120c_0+70c_1+12c_2-7c_3)d^5+524288a_2a_4^2\left(30c_0+25c_1+4c_2-10c_3\right)d^2+65536a_2^2a_4\left(30c_0+20c_1+c_2+c_3\right)d^4+884736a_3a_4^2(60c_0+55c_1+18c_2-19c_3)d+186624a_3^2a_4(120c_0+100c_1+29c_2-25c_3)d^2+864a_1a_2a_3(240c_0+120c_1-2c_2-9c_3)d^6+8192a_1a_2a_4(60c_0+35c_1-7c_2+7c_3)d^5+13824a_1a_3a_4(120c_0+80c_1-9c_2-29c_3)d^4+221184a_2a_3a_4(60c_0+45c_1+7c_2-9c_3)d^3).$

(v): *Case $r=4$ (all four roots equal to d):*

$$W_n^C=n^4(A_3n^3+A_2n^2+A_1n+A_0)d^n$$

where

$$\begin{aligned}
A_3 &= \frac{1}{840} b c_3, \\
A_2 &= \frac{1}{360} b(c_2 + 6c_3), \\
A_1 &= \frac{1}{120} b(c_1 + 4c_2 + 11c_3), \\
A_0 &= \frac{1}{72} b(3c_0 + 6c_1 + 11c_2 + 18c_3).
\end{aligned}$$

Summary and Conclusion

This study examined explicit particular solutions of generalized Leonardo-type recurrence relations subject to polynomial-exponential inputs. By extending Theorem 1.2, we provided closed-form expressions for the low-order cases $m = 3, 4$, thereby demonstrating how the general framework specializes into concrete computational examples. These derivations emphasize the interaction between characteristic polynomials, root multiplicities, and resonance effects, and they illustrate how classical recurrence identities can be unified with modern symbolic approaches.

The relevance of recurrence sequences extends well beyond pure mathematics. As noted, they arise naturally in physics (wave propagation and resonance phenomena), engineering (signal processing and control systems), architecture (proportional design and fractal structures), biology (growth models and ecological dynamics), computer science (algorithmic complexity and combinatorial enumeration), and even in artistic domains such as music and visual design. Homogeneous relations capture intrinsic system dynamics, while non-homogeneous relations incorporate external influences, making them indispensable tools for modeling real-world processes.

The contribution of this manuscript lies in presenting clear and practical methods for constructing particular solutions of generalized recurrences with polynomial-exponential inputs. By offering explicit formulas for different cases, the work simplifies abstract theoretical ideas and enhances understanding of central notions such as characteristic roots and resonance. These results are not only mathematically significant but also applicable in areas such as computer science and engineering.

The explicit examples derived from Theorem 1.2 play a dual role. They serve as verification of the general theorem and as pedagogical illustrations that clarify symbolic formulas through step-by-step computations. Working through low-order recurrences reveals how resonance modifies solutions and how polynomial-exponential inputs interact with characteristic roots. In this way, the examples bridge theory and application: they confirm the robustness of the framework, highlight resonance phenomena, and provide templates for interdisciplinary modeling. Their clarity also makes them suitable for inclusion in textbooks, where they can guide learners from abstract theory to concrete applications.

Beyond their theoretical significance, the examples provide both didactic and practical value. For teaching, they offer accessible cases that allow students to engage directly with non-homogeneous recurrences without excessive computation. For research, they supply resonance-aware formulas and explicit derivations that can be adapted to new problems in mathematics, computer science, engineering, and physics. Thus, the results contribute simultaneously to pedagogy and applied research, strengthening the originality, accessibility, and impact of the manuscript across multiple domains.

While the iterative framework developed here yields explicit polynomial-exponential-type particular solutions for generalized Leonardo-type sequences, its scope remains limited to inputs of polynomial-exponential form. Extending the method to non-polynomial-exponential inputs, such as trigonometric functions, would require further refinement. Moreover, although the framework clarifies resonance phenomena and multiplicity corrections, the computational complexity grows rapidly for higher-order recurrences, which may restrict practical use without the aid of computer algebra systems.

At the same time, the methodology opens promising directions for future work. It can be integrated into symbolic computation platforms, employed to validate classical identities in recurrence theory, and applied in interdisciplinary contexts such as coding theory, cryptography, and discrete modeling in physics and biology. By acknowledging current limitations and outlining avenues for extension, the study provides a balanced perspective: it consolidates the contribution of the present results while pointing toward further exploration and development.

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