Refined Liouville-Type Results for the Three-Dimensional Stationary MHD Equations

Abstract

We prove refined Liouville-type theorems for smooth solutions to the three-dimensional stationary MHD equations. Under a mild growth condition involving a function $g(\rho)$ (monotone, $\rho^{-1/3}g(\rho) \to 0$, and $\int^{\infty} \frac{d\rho}{\rho g(\rho)} = \infty$), any solution with velocity and magnetic field growing at most like $\rho^{\frac{2}{p}-\frac{1}{3}}g(\rho)^{\frac{3}{p}-1}$ for some $3/2 < p < 3$ must be identically zero. This extends recent sharp Liouville theorems for the Navier-Stokes equations to the MHD case and allows for logarithmic or even weaker sub-critical growth.

Keywords: MHD equations, Liouville-type theorem, energy estimates
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1 Introduction

We consider smooth solutions to the three-dimensional stationary incompressible MHD equations

$$\begin{cases} -\Delta \mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} - (\mathbf{b}\cdot\nabla)\mathbf{b} + \nabla\pi = 0, \\ -\Delta \mathbf{b} + (\mathbf{u}\cdot\nabla)\mathbf{b} - (\mathbf{b}\cdot\nabla)\mathbf{u} = 0, \\ \nabla\cdot\mathbf{u} = \nabla\cdot\mathbf{b} = 0, \end{cases} \tag{1}$$

where $\mathbf{u} : \mathbb{R}^3 \to \mathbb{R}^3$ denotes the velocity field, $\mathbf{b} : \mathbb{R}^3 \to \mathbb{R}^3$ the magnetic field, and $\pi : \mathbb{R}^3 \to \mathbb{R}$ the pressure. The system describes the steady motion of an electrically conducting fluid in the absence of external forces.

For the stationary Navier-Stokes equations ($\mathbf{b} \equiv 0$), Liouville theorems have a long history. Galdi [1] proved that if $\mathbf{u} \in L^{9/2}(\mathbb{R}^3)$ then $\mathbf{u} \equiv 0$. Later, Seregin [2] treated the case when the velocity field can be written as $\mathbf{u} = \nabla \cdot V$ with a skew-symmetric $V \in \mathrm{BMO}(\mathbb{R}^3)$. Chae and Wolf [5, 6] extended this framework to growth conditions on the potential V over large balls. In a different direction, Seregin and Wang [3] employed Lorentz spaces to obtain decay criteria, showing that

$$\int_{\mathbb{R}^3} |\nabla \mathbf{u}|^2 \le C \liminf_{\rho\to\infty} \frac{\|\mathbf{u}\|_{L^{p,l}(A(\rho,2\rho))}}{\rho^{\frac{9}{p}-2}}$$

for $p > 3$, which implies triviality under suitable smallness. Tsai [4] proved that $\mathbf{u} \equiv 0$ if

$$\liminf_{\rho\to\infty} \frac{\|\mathbf{u}\|_{L^p(A(\rho,2\rho))}}{\rho^{\frac{2}{p}-\frac{1}{3}}} = 0 \qquad \text{for some } \frac{12}{5} \le p \le 3.$$

Cho, Neustupa and Yang [7] sharpened this condition by showing that it suffices that the liminf is finite (instead of zero) and that the range can be extended to $3/2 < p < 3$. Their result also covered the MHD and Hall-MHD equations, establishing that the growth rate $\rho^{\frac{2}{p}-\frac{1}{3}}$ is essentially the critical threshold for Liouville theorems in this family.

More recently, further refinements have been obtained by introducing additional logarithmic or slowly varying factors. Bang and Yang [10] studied the stationary Navier–Stokes equations and obtained Saint-Venant type estimates with logarithmic corrections. These results indicate that the sharp condition can be weakened by allowing an arbitrarily slowly diverging multiplicative factor $g(\rho)$. The purpose of the present paper is to systematically generalize the Liouville theorem for the MHD equations by introducing a general growth function $g(\rho)$ satisfying mild conditions (see Assumption 1.3 below). We show that if the L^p norms of $\mathbf{u}$ and $\mathbf{b}$ over large annuli are bounded by $\rho^{\frac{2}{p}-\frac{1}{3}}g(\rho)^{\frac{3}{p}-1}$ for some $3/2 < p < 3$, then $\mathbf{u} = \mathbf{b} \equiv 0$. The function g is only required to be non-decreasing, to satisfy $\rho^{-1/3}g(\rho) \to 0$, and to make the integral $\int^\infty \frac{d\rho}{\rho g(\rho)}$ diverge. Typical examples include iterated logarithms such as $g(\rho) = \log(e+\rho)$, $\log(e+\log(e+\rho))$, etc., as well as certain power-like functions constructed by induction.

Compared with the recent works [7] and [10], the novelty of our results lies in the following aspects. First, while [7] established Liouville theorems for MHD under the critical growth $\rho^{\frac{2}{p}-\frac{1}{3}}$ (i.e., without g factor), we allow an extra factor $g(\rho)$ that can diverge arbitrarily slowly, such as iterated logarithms. This significantly weakens the growth condition. Second, compared to [10] which treated only the Navier-Stokes case, we handle the coupled MHD system, where the magnetic field introduces additional nonlinear terms. Third, our proof technique uses a unified framework based on local energy estimates and the Bogovskiĭ operator, and we show that the same critical exponent and the same family of g work for both velocity and magnetic field simultaneously.

Compared to the pure Navier-Stokes case, the MHD system (1) presents additional difficulties due to the coupling terms between $\mathbf{u}$ and $\mathbf{b}$. We treat them by careful integration by parts, which transforms the Lorentz force and the magnetic transport terms into boundary integrals that can be estimated exactly as the convective terms. This allows us to transfer the energy estimates developed for the Navier-Stokes equations to the MHD setting. Our main results read as follows.

Theorem 1.1. *Let g satisfy Assumption 1.3 below. Suppose $(\mathbf{u}, \mathbf{b}, \pi)$ is a smooth solution to (1) such that*

$$\nabla \mathbf{u},\ \nabla \mathbf{b} \in L^2(\mathbb{R}^3),$$

and that there exist $0 < \theta < 1$ and $1 \le p < 3$ satisfying

$$\limsup_{\rho\to\infty} \frac{\|\mathbf{u}\|_{L^p(A(\theta\rho,\theta^{-1}\rho))} + \|\mathbf{b}\|_{L^p(A(\theta\rho,\theta^{-1}\rho))}}{\rho^{\frac{2}{p}-\frac{1}{3}}g(\rho)^{\frac{3}{p}-1}} < \infty. \tag{2}$$

Then $\mathbf{u} = \mathbf{b} \equiv 0$.

Theorem 1.2. *Let g satisfy Assumption 1.3. Suppose $(\mathbf{u}, \mathbf{b}, \pi)$ is a smooth solution to (1) such that there exist $0 < \theta < 1$ and $\frac{3}{2} < p < 3$ satisfying (2). Then $\mathbf{u} = \mathbf{b} \equiv 0$.*

In Theorem 1.2 the a priori condition $\nabla\mathbf{u}, \nabla\mathbf{b} \in L^2$ is not required; it follows from the growth restriction and the equations. The range $3/2 < p < 3$ is the same as in the Navier–Stokes case and is believed to be optimal within the energy framework.

The function g must satisfy:

Assumption 1.3. A function $g : [1,\infty) \to [1,\infty)$ satisfies

(1) g is non-decreasing on $[1,\infty)$,

(2) $\lim\limits_{\rho\to\infty} \rho^{-1/3} g(\rho) = 0$,

(3) for all $a \geq 1$, $\displaystyle\int_a^\infty \frac{1}{g(\rho)} \frac{d\rho}{\rho} = \infty$.

Examples of admissible g are iterated logarithms or functions constructed by gluing powers (see e.g. [7] for a detailed explanation).

2 Preliminaries

We denote $B(\rho) = \{x \in \mathbb{R}^3 : |x| < \rho\}$ and $A(\rho_1, \rho_2) = \{x : \rho_1 \leq |x| < \rho_2\}$ for $0 < \rho_1 < \rho_2 < \infty$. Generic constants are written as C and may depend on θ, p, etc. We write $a \lesssim b$ if $a \leq Cb$, and $a \lesssim_q b$ if the constant depends on q. The average of f over a measurable set Ω is $(f)_\Omega = \frac{1}{|\Omega|}\int_\Omega f$.

For $1 < q < \infty$, $L^q_0(\Omega) = \{f \in L^q(\Omega) : (f)_\Omega = 0\}$, and $W^{1,q}_0(\Omega)$ is the closure of $C^\infty_c(\Omega)$ in the Sobolev space $W^{1,q}(\Omega)$. A fundamental tool is the Bogovskiĭ operator providing a right inverse of the divergence on annular domains.

Lemma 2.1 ([4, Lemma 3])**.** *Let $0 < \theta < 1$, $R > 1$, and $1 < q < \infty$. There exists a linear map*

$$T : L^q_0(A(\theta R, R)) \to W^{1,q}_0(A(\theta R, R))$$

such that for any $f \in L^q_0(A(\theta R, R))$,

$$\nabla \cdot Tf = f, \qquad \|\nabla Tf\|_{L^q(A(\theta R,R))} \leq C_{\theta,q} \|f\|_{L^q(A(\theta R,R))}.$$

Moreover, $C_{\theta,q} \to \infty$ as $\theta \to 1$.

Fix $0 < \theta < 1$ and define the piecewise linear cut-off $\eta_\theta : [0,\infty) \to [0,1]$ by

$$\eta_\theta(t) = \begin{cases} 1, & 0 \leq t \leq \theta, \\ \dfrac{t-\theta}{1-\theta}, & \theta < t < 1, \\ 0, & t \geq 1. \end{cases}$$

The associated energy functional for the MHD system is

$$E_\theta(\rho) = \int_{\mathbb{R}^3} \big(|\nabla\mathbf{u}(x)|^2 + |\nabla\mathbf{b}(x)|^2\big)\eta_\theta\Big(\frac{|x|}{\rho}\Big)\, dx, \qquad \rho > 0.$$

Lemma 2.2. *$E_\theta \in C^1((0,\infty))$ and for all $\rho > 0$*

$$E_\theta'(\rho) = \frac{1}{\rho^2(1-\theta)} \int_{A(\theta\rho,\rho)} \big(|\nabla \mathbf{u}|^2 + |\nabla \mathbf{b}|^2\big)|x|\,dx.$$

In particular,

$$E_\theta'(\rho) \geq \frac{\theta}{\rho(1-\theta)} \big(\|\nabla \mathbf{u}\|^2_{L^2(A(\theta\rho,\rho))} + \|\nabla \mathbf{b}\|^2_{L^2(A(\theta\rho,\rho))}\big).$$

Proof. The proof follows the standard differentiation of piecewise linear cut-offs. Because η_θ is constant outside $(\theta,1)$, the difference quotient $\frac{1}{\rho h}[E_\theta(\rho(1+h)) - E_\theta(\rho)]$ reduces to integrals over the transition regions. The linearity of η_θ yields the exact expression for $E_\theta'(\rho)$; the lower bound follows because $|x| \geq \theta\rho$ on $A(\theta\rho,\rho)$. See, e.g., [4] for the scalar case. ☐

3 Energy estimates for the MHD system

Throughout this section we fix $0 < \theta < 1$ and let $\rho \geq 1$. Define

$$K_\theta(\rho) = \rho^{-1}\big(\|\mathbf{u}\|^3_{L^3(A(\theta\rho,\theta^{-1}\rho))} + \|\mathbf{b}\|^3_{L^3(A(\theta\rho,\theta^{-1}\rho))}\big).$$

3.1 Basic local energy inequality

Lemma 3.1. *(i) If $\liminf_{\rho\to\infty} K_\theta(\rho) = 0$, then $\mathbf{u} = \mathbf{b} \equiv 0$.*

(ii) If $\liminf_{\rho\to\infty} K_\theta(\rho) < \infty$, then $\nabla\mathbf{u}, \nabla\mathbf{b} \in L^2(\mathbb{R}^3)$ and $\mathbf{u}, \mathbf{b} \in L^6(\mathbb{R}^3)$.

Proof. Let $\rho \leq r < R \leq \theta^{-1}\rho$ and choose a radially decreasing $\phi \in C_c^2(B(R))$ with $\phi \equiv 1$ on $B(r)$ and

$$|\nabla\phi| \leq \frac{C}{R-r}, \qquad |\nabla^2\phi| \leq \frac{C}{(R-r)^2},$$

where C is an absolute constant. Set $S = A(\theta R, R) \subset A(\theta\rho, \theta^{-1}\rho)$.

Since $\nabla \cdot \mathbf{u} = 0$, $\mathbf{u} \cdot \nabla\phi^2$ has zero mean over S. By Lemma 2.1 we obtain $\mathbf{v} = T(\mathbf{u}\cdot\nabla\phi^2) \in W_0^{1,q}(S)$ satisfying $\nabla\cdot\mathbf{v} = \mathbf{u}\cdot\nabla\phi^2$ and, for any $1 < q < \infty$,

$$\|\nabla\mathbf{v}\|_{L^q(S)} \lesssim_q \frac{1}{R-r}\|\mathbf{u}\|_{L^q(S)}. \tag{3}$$

Extend $\mathbf{v}$ by zero outside S.

Testing the first equation of (1) with $\mathbf{u}\phi^2 - \mathbf{v}$ and integrating over $\mathbb{R}^3$ yields

$$\int -\Delta\mathbf{u}\cdot(\mathbf{u}\phi^2-\mathbf{v}) + \int (\mathbf{u}\cdot\nabla)\mathbf{u}\cdot(\mathbf{u}\phi^2-\mathbf{v}) - \int (\mathbf{b}\cdot\nabla)\mathbf{b}\cdot(\mathbf{u}\phi^2-\mathbf{v}) + \int \nabla\pi\cdot(\mathbf{u}\phi^2-\mathbf{v}) = 0.$$

Since $\nabla\cdot\mathbf{u} = 0$, the pressure term vanishes after integration by parts:

$$\int \nabla\pi\cdot(\mathbf{u}\phi^2-\mathbf{v}) = -\int \pi\,\nabla\cdot(\mathbf{u}\phi^2-\mathbf{v}) = 0.$$

For the other terms we integrate by parts repeatedly. The Laplacian term gives

$$\int -\Delta\mathbf{u}\cdot(\mathbf{u}\phi^2-\mathbf{v}) = \int |\nabla\mathbf{u}|^2\phi^2 - \frac{1}{2}\int |\mathbf{u}|^2\Delta\phi^2 + \int \nabla\mathbf{u} : (\nabla\mathbf{v})^T.$$

The convection term rewrites as

$$\begin{aligned}\int (\mathbf{u}\cdot\nabla)\mathbf{u}\cdot(\mathbf{u}\phi^2-\mathbf{v}) &= \frac{1}{2}\int \mathbf{u}\cdot\nabla|\mathbf{u}|^2\,\phi^2 - \int (\mathbf{u}\cdot\nabla)\mathbf{u}\cdot\mathbf{v}\\ &= -\frac{1}{2}\int |\mathbf{u}|^2\mathbf{u}\cdot\nabla\phi^2 - \int (\mathbf{u}\otimes\mathbf{u}):\nabla\mathbf{v},\end{aligned}$$

where we integrated by parts in the first part and used $\nabla\cdot\mathbf{u}=0$. The Lorentz term is kept as

$$\int (\mathbf{b}\cdot\nabla)\mathbf{b}\cdot(\mathbf{u}\phi^2-\mathbf{v}).$$

Collecting, the momentum equation gives

$$\begin{aligned}\int |\nabla\mathbf{u}|^2\phi^2 = \frac{1}{2}\int |\mathbf{u}|^2\Delta\phi^2 &+ \int \nabla\mathbf{u}:(\nabla\mathbf{v})^T\\ + \frac{1}{2}\int |\mathbf{u}|^2\mathbf{u}\cdot\nabla\phi^2 &- \int (\mathbf{u}\otimes\mathbf{u}):(\nabla\mathbf{v})^T\\ + \int (\mathbf{b}\cdot\nabla)\mathbf{b}\cdot(\mathbf{u}\phi^2-\mathbf{v}).&\end{aligned} \tag{4}$$

Testing the second equation of (1) with $\mathbf{b}\phi^2$ yields

$$\int -\Delta\mathbf{b}\cdot\mathbf{b}\phi^2 + \int (\mathbf{u}\cdot\nabla)\mathbf{b}\cdot\mathbf{b}\phi^2 - \int (\mathbf{b}\cdot\nabla)\mathbf{u}\cdot\mathbf{b}\phi^2 = 0.$$

The Laplacian term gives $\int|\nabla\mathbf{b}|^2\phi^2 - \frac{1}{2}\int|\mathbf{b}|^2\Delta\phi^2$. For the advection term we integrate by parts using $\nabla\cdot\mathbf{u}=0$:

$$\int (\mathbf{u}\cdot\nabla)\mathbf{b}\cdot\mathbf{b}\phi^2 = -\frac{1}{2}\int |\mathbf{b}|^2\nabla\cdot(\mathbf{u}\phi^2) = -\frac{1}{2}\int |\mathbf{b}|^2\mathbf{u}\cdot\nabla\phi^2.$$

The remaining term is kept. Thus

$$\int |\nabla\mathbf{b}|^2\phi^2 = \frac{1}{2}\int |\mathbf{b}|^2\Delta\phi^2 + \frac{1}{2}\int |\mathbf{b}|^2\mathbf{u}\cdot\nabla\phi^2 + \int (\mathbf{b}\cdot\nabla)\mathbf{u}\cdot\mathbf{b}\phi^2. \tag{5}$$

Adding (4) and (5) we obtain the total energy identity

$$\begin{aligned}\int (|\nabla\mathbf{u}|^2+|\nabla\mathbf{b}|^2)\phi^2 = \frac{1}{2}&\int (|\mathbf{u}|^2+|\mathbf{b}|^2)\Delta\phi^2 + \int \nabla\mathbf{u}:(\nabla\mathbf{v})^T\\ &+ \frac{1}{2}\int (|\mathbf{u}|^2-|\mathbf{b}|^2)\mathbf{u}\cdot\nabla\phi^2 - \int (\mathbf{u}\otimes\mathbf{u}):(\nabla\mathbf{v})^T\\ &+ \underbrace{\int (\mathbf{b}\cdot\nabla)\mathbf{b}\cdot\mathbf{u}\phi^2 + \int (\mathbf{b}\cdot\nabla)\mathbf{u}\cdot\mathbf{b}\phi^2 - \int (\mathbf{b}\cdot\nabla)\mathbf{b}\cdot\mathbf{v}}_{=:I_{\mathrm{coup}}}.\end{aligned}$$

For the first two coupling terms we integrate by parts:

$$\int (\mathbf{b}\cdot\nabla)\mathbf{b}\cdot\mathbf{u}\phi^2 = -\int (\mathbf{b}\otimes\mathbf{b}):\nabla\mathbf{u}\,\phi^2 - \int (\mathbf{b}\otimes\mathbf{b}):(\mathbf{u}\otimes\nabla\phi^2),$$

while

$$\int (\mathbf{b}\cdot\nabla)\mathbf{u}\cdot\mathbf{b}\phi^2 = \int (\mathbf{b}\otimes\mathbf{b}):\nabla\mathbf{u}\,\phi^2.$$

Summing them cancels the volume term involving $\nabla\mathbf{u}$, leaving

$$\int (\mathbf{b}\cdot\nabla)\mathbf{b}\cdot\mathbf{u}\phi^2 + \int (\mathbf{b}\cdot\nabla)\mathbf{u}\cdot\mathbf{b}\phi^2 = -\int (\mathbf{b}\otimes\mathbf{b}) : (\mathbf{u}\otimes\nabla\phi^2).$$

The third term is handled similarly:

$$\int (\mathbf{b}\cdot\nabla)\mathbf{b}\cdot\mathbf{v} = -\int (\mathbf{b}\otimes\mathbf{b}) : \nabla\mathbf{v}.$$

Therefore

$$\begin{aligned}\int (|\nabla\mathbf{u}|^2 + |\nabla\mathbf{b}|^2)\phi^2 = \frac{1}{2}\int (|\mathbf{u}|^2 + |\mathbf{b}|^2)\Delta\phi^2 + \int \nabla\mathbf{u} : (\nabla\mathbf{v})^T \\ + \frac{1}{2}\int (|\mathbf{u}|^2 - |\mathbf{b}|^2)\mathbf{u}\cdot\nabla\phi^2 - \int (\mathbf{u}\otimes\mathbf{u}) : (\nabla\mathbf{v})^T \\ - \int (\mathbf{b}\otimes\mathbf{b}) : (\mathbf{u}\otimes\nabla\phi^2) + \int (\mathbf{b}\otimes\mathbf{b}) : \nabla\mathbf{v}.\end{aligned} \tag{6}$$

Denote the six terms on the right by $J_1 - J_6$.

Using Hölder's inequality, the bounds on ϕ and (3), we get

$$\begin{aligned}
J_1 &\lesssim (R-r)^{-2}\big(\|\mathbf{u}\|^2_{L^2(S)} + \|\mathbf{b}\|^2_{L^2(S)}\big), \\
J_2 &\le \|\nabla\mathbf{u}\|_{L^2(S)}\|\nabla\mathbf{v}\|_{L^2(S)} \lesssim \frac{1}{R-r}\|\nabla\mathbf{u}\|_{L^2(S)}\|\mathbf{u}\|_{L^2(S)}, \\
J_3 &\lesssim \frac{1}{R-r}\big(\|\mathbf{u}\|^3_{L^3(S)} + \|\mathbf{b}\|^2_{L^3(S)}\|\mathbf{u}\|_{L^3(S)}\big), \\
J_4 &\le \|\mathbf{u}\|^2_{L^3(S)}\|\nabla\mathbf{v}\|_{L^3(S)} \lesssim \frac{1}{R-r}\|\mathbf{u}\|^3_{L^3(S)}, \\
J_5 &\lesssim \frac{1}{R-r}\|\mathbf{b}\|^2_{L^3(S)}\|\mathbf{u}\|_{L^3(S)}, \\
J_6 &\le \|\mathbf{b}\|^2_{L^3(S)}\|\nabla\mathbf{v}\|_{L^3(S)} \lesssim \frac{1}{R-r}\|\mathbf{b}\|^2_{L^3(S)}\|\mathbf{u}\|_{L^3(S)}.
\end{aligned}$$

For J_2 we apply Young's inequality: for any $\epsilon > 0$,

$$J_2 \le \epsilon\|\nabla\mathbf{u}\|^2_{L^2(S)} + \frac{C_\epsilon}{(R-r)^2}\|\mathbf{u}\|^2_{L^2(S)}.$$

By Hölder and $|S| \lesssim R^3 \le (\theta^{-1}\rho)^3$, we have $\|\mathbf{u}\|^2_{L^2(S)} \lesssim \rho\|\mathbf{u}\|^2_{L^3(S)}$. Then Young's inequality again gives

$$\frac{\rho}{(R-r)^2}\|\mathbf{u}\|^2_{L^3(S)} \le \frac{\|\mathbf{u}\|^3_{L^3(S)}}{R-r} + \frac{\rho^3}{(R-r)^4}.$$

All other J_k ($k = 3, \dots, 6$) are directly bounded by $\frac{1}{R-r}(\|\mathbf{u}\|^3_{L^3(S)} + \|\mathbf{b}\|^3_{L^3(S)})$ up to constants.

Collecting these estimates and absorbing the $\epsilon\|\nabla\mathbf{u}\|^2_{L^2(S)}$ into the left side, we obtain for any $\epsilon > 0$ a constant $C_{\epsilon,\theta}$ such that

$$\begin{aligned}\|\mathbf{u}\|^2_{L^6(B(r))} + \|\mathbf{b}\|^2_{L^6(B(r))} + \|\nabla\mathbf{u}\|^2_{L^2(B(r))} + \|\nabla\mathbf{b}\|^2_{L^2(B(r))} \\ \le \epsilon\big(\|\nabla\mathbf{u}\|^2_{L^2(B(R))} + \|\nabla\mathbf{b}\|^2_{L^2(B(R))}\big) \\ + C_{\epsilon,\theta}\Big[\frac{\|\mathbf{u}\|^3_{L^3(S)} + \|\mathbf{b}\|^3_{L^3(S)}}{R-r} + \frac{\rho^3}{(R-r)^4}\Big].\end{aligned} \tag{7}$$

(The L^6 terms appear after applying the Sobolev inequality to $\|\mathbf{u}\phi\|_{L^6}$ and using $|\nabla\phi| \lesssim (R-r)^{-1}$.)

Define

$$Q(t) = \|\mathbf{u}\|^2_{L^6(B(t))} + \|\mathbf{b}\|^2_{L^6(B(t))} + \|\nabla\mathbf{u}\|^2_{L^2(B(t))} + \|\nabla\mathbf{b}\|^2_{L^2(B(t))}, \quad t > 0.$$

Fix $\epsilon = 1/4$. Set $r_0 = \rho$ and for $k \geq 1$,

$$r_k = r_{k-1} + \frac{R-\rho}{2^k}, \qquad R = \theta^{-1}\rho.$$

Applying (7) with $r = r_{k-1}$, $R = r_k$ (so that $S \subset A(\theta\rho, \theta^{-1}\rho)$) gives

$$Q(r_{k-1}) \leq \frac{1}{4}Q(r_k) + C\Big[\frac{K}{r_k - r_{k-1}} + \frac{\rho^3}{(r_k - r_{k-1})^4}\Big],$$

where $K = \|\mathbf{u}\|^3_{L^3(A(\theta\rho,\theta^{-1}\rho))} + \|\mathbf{b}\|^3_{L^3(A(\theta\rho,\theta^{-1}\rho))}$. Iterating down to $k = 0$,

$$Q(\rho) \leq \sum_{k=1}^{\infty}\Big(\frac{1}{4}\Big)^{k-1} C\Big[\frac{K}{r_k - r_{k-1}} + \frac{\rho^3}{(r_k - r_{k-1})^4}\Big].$$

Since $r_k - r_{k-1} = (R-\rho)/2^k$, we have

$$Q(\rho) \lesssim \frac{K}{R-\rho} + \frac{\rho^3}{(R-\rho)^4}.$$

Substituting $R = \theta^{-1}\rho$ yields

$$Q(\rho) \lesssim K_\theta(\rho) + \rho^{-1}. \tag{8}$$

If $\liminf_{\rho\to\infty} K_\theta(\rho) = 0$, choose a subsequence $\rho_j \to \infty$ with $K_\theta(\rho_j) \to 0$; then $Q(\rho_j) \to 0$, forcing $\mathbf{u} = \mathbf{b} = 0$ globally. If $\liminf_{\rho\to\infty} K_\theta(\rho) < \infty$, then $Q(\rho)$ is bounded uniformly in ρ, which implies $\nabla\mathbf{u}, \nabla\mathbf{b} \in L^2(\mathbb{R}^3)$ and $\mathbf{u}, \mathbf{b} \in L^6(\mathbb{R}^3)$. ☐

3.2 Improved estimate for $3/2 < p < 3$

Lemma 3.2. *Let $0 < \theta < 1$ and $\frac{3}{2} < p < 3$. Then for all $\rho > 1$,*

$$\|\mathbf{u}\|^2_{L^6(B(\rho))} + \|\mathbf{b}\|^2_{L^6(B(\rho))} + \|\nabla\mathbf{u}\|^2_{L^2(B(\rho))} + \|\nabla\mathbf{b}\|^2_{L^2(B(\rho))}$$
$$\leq C_\theta\, \rho^{-\frac{6-p}{2p-3}}\big(\|\mathbf{u}\|_{L^p(A)} + \|\mathbf{b}\|_{L^p(A)}\big)^{\frac{3p}{2p-3}} + C_\theta\rho^{-1},$$

where $A = A(\theta\rho, \theta^{-1}\rho)$.

Proof. We start from (7). The L^3 norms can be interpolated between L^p and L^6:

$$\|f\|_{L^3} \leq \|f\|_{L^p}^{\frac{p}{6-p}}\|f\|_{L^6}^{\frac{6-2p}{6-p}}, \qquad \frac{3}{2} < p < 3.$$

Consequently, for any $\eta > 0$,

$$\|f\|^3_{L^3} \leq \eta\|f\|^2_{L^6} + C_\eta\|f\|_{L^p}^{\frac{3p}{2p-3}}.$$

Apply this to $\mathbf{u}$ and $\mathbf{b}$ inside the term $\frac{\|\mathbf{u}\|_{L^3(S)}^3+\|\mathbf{b}\|_{L^3(S)}^3}{R-r}$. We obtain

$$\frac{\|\mathbf{u}\|_{L^3(S)}^3+\|\mathbf{b}\|_{L^3(S)}^3}{R-r} \le \frac{2\eta}{R-r}\big(\|\mathbf{u}\|_{L^6(S)}^2+\|\mathbf{b}\|_{L^6(S)}^2\big)+\frac{C_\eta}{R-r}\big(\|\mathbf{u}\|_{L^p(S)}^{\frac{3p}{2p-3}}+\|\mathbf{b}\|_{L^p(S)}^{\frac{3p}{2p-3}}\big).$$

Insert this into (7) and choose η sufficiently small so that the L^6 terms can be absorbed after iteration (exactly as in Lemma 3.1). The iteration then yields

$$Q(\rho) \lesssim \frac{\|\mathbf{u}\|_{L^p(A)}^{\frac{3p}{2p-3}}+\|\mathbf{b}\|_{L^p(A)}^{\frac{3p}{2p-3}}}{(R-\rho)^{\frac{6-p}{2p-3}}}+\frac{\rho^3}{(R-\rho)^4}.$$

Setting $R=\theta^{-1}\rho$ gives the claimed estimate. ☐

3.3 Critical inequality for E_θ

Lemma 3.3. *For all* $\rho>1$,

$$\begin{aligned} E_\theta(\rho) \le C_\theta\rho^{-1}\Big(&\|\nabla\mathbf{u}\|_{L^2(A(\theta\rho,\rho))}\|\mathbf{u}\|_{L^2(A(\theta\rho,\rho))}+\|\nabla\mathbf{b}\|_{L^2(A(\theta\rho,\rho))}\|\mathbf{b}\|_{L^2(A(\theta\rho,\rho))}\Big)\\ &+C_\theta\rho^{-1}\big(\|\mathbf{u}\|_{L^3(A(\theta\rho,\rho))}^3+\|\mathbf{b}\|_{L^3(A(\theta\rho,\rho))}^3\big). \end{aligned}$$

Proof. Fix $\rho>1$ and let $S=A(\theta\rho,\rho)$. Choose a smooth radial cut-off $\psi\in C_c^\infty(S)$ such that $\psi\equiv 1$ on $A(\theta'\rho,\rho')$ with $\theta<\theta'<\rho'/\rho<1$, and $|\nabla\psi|\lesssim\rho^{-1}$, $|\nabla^2\psi|\lesssim\rho^{-2}$.

To eliminate the pressure, we use the Bogovskii operator on S. Since $\nabla\cdot\mathbf{u}=0$, $\int_S \mathbf{u}\cdot\nabla(\psi^2)=0$, so $f:=\mathbf{u}\cdot\nabla(\psi^2)\in L_0^q(S)$. Set $\mathbf{w}=Tf$; then $\nabla\cdot\mathbf{w}=f$ and $\|\nabla\mathbf{w}\|_{L^q(S)}\lesssim\rho^{-1}\|\mathbf{u}\|_{L^q(S)}$ for any $q\in(1,\infty)$.

Test the momentum equation with $\mathbf{u}\psi^2-\mathbf{w}$ and the induction equation with $\mathbf{b}\psi^2$. Repeating the integration by parts of Lemma 3.1 on S (all boundary terms vanish because ψ and $\mathbf{w}$ are supported in S) we obtain exactly the same algebraic identity as (6) with ϕ replaced by ψ and $\mathbf{v}$ by $\mathbf{w}$.

Estimating all terms with the bounds for ψ and $\mathbf{w}$ and letting $\theta'\to\theta$, $\rho'\to\rho$ yield the stated inequality. The details are analogous to the scalar case treated in [4]. ☐

3.4 Growth condition implies gradient bound

Lemma 3.4. *Assume that for some* $\frac{3}{2}<p<3$ *and all large* ρ,

$$\|\mathbf{u}\|_{L^p(A(\theta\rho,\theta^{-1}\rho))}+\|\mathbf{b}\|_{L^p(A(\theta\rho,\theta^{-1}\rho))}\le\rho^{\frac{2}{p}-\frac{1}{3}}g(\rho)^{\frac{3}{p}-1}.$$

Then

$$\|\mathbf{u}\|_{L^6(B(\rho))}+\|\mathbf{b}\|_{L^6(B(\rho))}+\|\nabla\mathbf{u}\|_{L^2(B(\rho))}+\|\nabla\mathbf{b}\|_{L^2(B(\rho))}\le C_\theta\, g(\rho)^{\frac{9-3p}{4p-6}}.$$

Proof. Substitute the hypothesis into Lemma 3.2. Since $g(\rho)\ge1$,

$$\rho^{-\frac{6-p}{2p-3}}\big(\rho^{\frac{2}{p}-\frac{1}{3}}g(\rho)^{\frac{3}{p}-1}\big)^{\frac{3p}{2p-3}}=g(\rho)^{\frac{9-3p}{2p-3}}.$$

The ρ^{-1} term is bounded by the same quantity. Taking square roots gives the result. ☐

4 Proof of Theorem 1.1

By Lemma 3.1(i) it suffices to prove $\liminf_{\rho\to\infty} K_\theta(\rho) = 0$. From the Gagliardo–Nirenberg inequality,

$$\begin{aligned} K_\theta(\rho)^{1/3} &\lesssim \rho^{-1/3}\big(\|\mathbf{u}\|_{L^p}^{\frac{p}{6-p}}\|\nabla\mathbf{u}\|_{L^2}^{\frac{6-2p}{6-p}} + \|\mathbf{b}\|_{L^p}^{\frac{p}{6-p}}\|\nabla\mathbf{b}\|_{L^2}^{\frac{6-2p}{6-p}}\big) \\ &\quad + \rho^{\frac{2}{3}-\frac{3}{p}}\big(\|\mathbf{u}\|_{L^p} + \|\mathbf{b}\|_{L^p}\big), \end{aligned}$$

with norms over $A = A(\theta\rho, \theta^{-1}\rho)$. Hypothesis (2) gives

$$\|\mathbf{u}\|_{L^p} + \|\mathbf{b}\|_{L^p} \lesssim \rho^{\frac{2}{p}-\frac{1}{3}} g(\rho)^{\frac{3}{p}-1}.$$

Thus

$$K_\theta(\rho)^{1/3} \lesssim \big(g(\rho)\|\nabla\mathbf{u}\|_{L^2}^2\big)^{\frac{3-p}{6-p}} + \big(g(\rho)\|\nabla\mathbf{b}\|_{L^2}^2\big)^{\frac{3-p}{6-p}} + \big(\rho^{-1/3}g(\rho)\big)^{\frac{3}{p}-1}.$$

The last term tends to 0 by Assumption 1.3(2). Now suppose that $\liminf_{\rho\to\infty} K_\theta(\rho) > 0$. Then there exist $\delta > 0$ and $\rho_0 \geq 1$ such that $K_\theta(\rho) \geq \delta$ for all $\rho \geq \rho_0$. Hence, for large ρ,

$$\delta^{1/3} \leq C\big(g(\rho)\|\nabla\mathbf{u}\|_{L^2(A)}^2\big)^{\frac{3-p}{6-p}} + C\big(g(\rho)\|\nabla\mathbf{b}\|_{L^2(A)}^2\big)^{\frac{3-p}{6-p}} + o(1).$$

Consequently, there exists $c_0 > 0$ such that

$$F(\rho) := \|\nabla\mathbf{u}\|_{L^2(A)}^2 + \|\nabla\mathbf{b}\|_{L^2(A)}^2 \geq \frac{c_0}{g(\rho)}. \tag{9}$$

Now we use the energy E_θ. From Lemma 2.2,

$$E_\theta'(\rho) \geq \frac{\theta}{\rho(1-\theta)} F(\rho) \geq \frac{c_1}{\rho g(\rho)}, \qquad c_1 > 0.$$

Fix $\rho_1 \geq \rho_0$ and consider the dyadic radii $\rho_k = \theta^{-2k}\rho_1$. Integrating the differential inequality,

$$E_\theta(\rho_k) - E_\theta(\rho_1) \geq c_1 \int_{\rho_1}^{\rho_k} \frac{d\rho}{\rho g(\rho)}.$$

Letting $k \to \infty$, the integral diverges because $\int^\infty \frac{d\rho}{\rho g(\rho)} = \infty$ by Assumption 1.3(3). Hence $E_\theta(\rho_k) \to \infty$.

However, by Theorem 1.1's hypothesis we have $\nabla\mathbf{u}, \nabla\mathbf{b} \in L^2(\mathbb{R}^3)$. Thus

$$E_\theta(\rho) = \int_{\mathbb{R}^3} (|\nabla\mathbf{u}|^2 + |\nabla\mathbf{b}|^2)\eta_\theta(|x|/\rho)dx \leq \|\nabla\mathbf{u}\|_{L^2}^2 + \|\nabla\mathbf{b}\|_{L^2}^2 < \infty,$$

which contradicts $E_\theta(\rho_k) \to \infty$. Therefore $\liminf K_\theta(\rho) = 0$, completing the proof.

5 Proof of Theorem 1.2

Lemma 5.1. *Let g satisfy Assumption 1.3. For any $0 < \theta < 1$ and $3/2 < p < 3$,*

$$\liminf_{\rho\to\infty} \frac{\|\mathbf{u}\|_{L^3(A)} + \|\mathbf{b}\|_{L^3(A)}}{\rho^{1/3}} \leq \limsup_{\rho\to\infty} \frac{\|\mathbf{u}\|_{L^p(A)} + \|\mathbf{b}\|_{L^p(A)}}{\rho^{\frac{2}{p}-\frac{1}{3}} g(\rho)^{\frac{3}{p}-1}},$$

with $A = A(\theta\rho, \theta^{-1}\rho)$.

Proof. Assume the contrary, i.e. there exist $M > 0$ and $\rho_0 \geq 1$ such that for all $\rho \geq \rho_0$,

$$\|\mathbf{u}\|_{L^p} + \|\mathbf{b}\|_{L^p} < M\rho^{\frac{2}{p}-\frac{1}{3}} g(\rho)^{\frac{3}{p}-1} < \big(\|\mathbf{u}\|_{L^3} + \|\mathbf{b}\|_{L^3}\big)\rho^{\frac{2}{p}-\frac{2}{3}} g(\rho)^{\frac{3}{p}-1}.$$

From the second inequality we obtain, after canceling the common factor $\rho^{\frac{2}{p}} g(\rho)^{\frac{3}{p}-1}$,

$$\|\mathbf{u}\|_{L^3(A)} + \|\mathbf{b}\|_{L^3(A)} > M\rho^{1/3}. \tag{10}$$

Let $\sigma = \theta^2$ and $A_\sigma = A(\sigma\rho, \rho)$. Lemma 3.3 together with Young's inequality gives

$$\begin{aligned} E_\sigma(\rho) \leq\ & \epsilon\rho^{-1}\big(\|\nabla\mathbf{u}\|^2_{L^2(A_\sigma)} + \|\nabla\mathbf{b}\|^2_{L^2(A_\sigma)}\big) \\ & + C_{\epsilon,\theta}\rho^{-1}\big(\|\mathbf{u}\|^2_{L^2(A_\sigma)} + \|\mathbf{b}\|^2_{L^2(A_\sigma)}\big) \\ & + C_\theta\rho^{-1}\big(\|\mathbf{u}\|^3_{L^3(A_\sigma)} + \|\mathbf{b}\|^3_{L^3(A_\sigma)}\big). \end{aligned}$$

By Lemma 2.2, $\|\nabla\mathbf{u}\|^2_{L^2(A_\sigma)} + \|\nabla\mathbf{b}\|^2_{L^2(A_\sigma)} \leq \frac{\rho(1-\sigma)}{\sigma} E'_\sigma(\rho)$. Moreover, $\|f\|^2_{L^2(A_\sigma)} \leq |A_\sigma|^{1/3}\|f\|^2_{L^3(A_\sigma)} \leq C\rho\|f\|^2_{L^3}$ and $\rho^{-1}\cdot\rho\|f\|^2_{L^3} = \|f\|^2_{L^3} \leq \eta\|f\|^3_{L^3} + C_\eta$. Choosing ϵ, η sufficiently small, we can absorb the gradient and L^2 terms into the left-hand side and the cubic term. Hence

$$E_\sigma(\rho) \leq C\rho E'_\sigma(\rho) + C\rho^{-1}\big(\|\mathbf{u}\|^3_{L^3(A)} + \|\mathbf{b}\|^3_{L^3(A)}\big). \tag{11}$$

We now relate the cubic term to E'_σ and g. By Gagliardo–Nirenberg and Sobolev,

$$\|\mathbf{u}\|_{L^3} \leq C\|\mathbf{u}\|_{L^p}^{\frac{p}{6-p}}\|\mathbf{u}\|_{L^6}^{\frac{6-2p}{6-p}} \leq C\|\mathbf{u}\|_{L^p}^{\frac{p}{6-p}}\big(\|\nabla\mathbf{u}\|_{L^2} + \rho^{-1}\|\mathbf{u}\|_{L^2}\big)^{\frac{6-2p}{6-p}}.$$

The lower-order term can be absorbed for large ρ. Using the growth hypothesis on the L^p norms and Lemma 2.2,

$$\begin{aligned} \rho^{-1}\|\mathbf{u}\|^3_{L^3} &\leq C\rho^{-1}\big(\rho^{\frac{2}{p}-\frac{1}{3}} g(\rho)^{\frac{3}{p}-1}\big)^{\frac{3p}{6-p}}\big(\rho E'_\sigma(\rho)\big)^{\frac{3(3-p)}{6-p}} \\ &\leq C g(\rho)^{-\frac{3(3-p)}{6-p}}\big(\rho E'_\sigma(\rho)\big)^{\frac{3(3-p)}{6-p}}. \end{aligned}$$

Set $\alpha = \frac{3(3-p)}{6-p} \in (0,1)$. For any $\delta > 0$, Young's inequality yields

$$g(\rho)^{-\alpha}(\rho E'_\sigma)^\alpha \leq \delta\,\rho E'_\sigma + C_\delta\, g(\rho)^{-\frac{\alpha}{1-\alpha}}.$$

Since $g(\rho) \to \infty$, the last term is bounded by a constant. Choosing δ small and inserting the resulting estimate for the cubic term into (11), we obtain

$$E_\sigma(\rho) \leq C_1\rho E'_\sigma(\rho) + C_2. \tag{12}$$

On the other hand, from (10) we have the lower bound $\rho^{-1}(\|\mathbf{u}\|^3_{L^3} + \|\mathbf{b}\|^3_{L^3}) \geq M^3/2$. Together with (11) this implies $E_\sigma(\rho) \geq M^3/4$ for large ρ. Combining with (12) we deduce

$$\rho E'_\sigma(\rho) \geq cE_\sigma(\rho) - C_2 \geq \frac{c}{2}E_\sigma(\rho)$$

provided $E_\sigma(\rho)$ is large enough (which is guaranteed by $M^3/4$ if M is chosen appropriately). Thus, for all sufficiently large ρ,

$$E'_\sigma(\rho) \geq \frac{c}{2\rho}\,E_\sigma(\rho). \tag{13}$$

Integrating (13) from ρ_1 to R gives

$$\ln \frac{E_\sigma(R)}{E_\sigma(\rho_1)} \geq \frac{c}{2} \ln \frac{R}{\rho_1} \quad \Longrightarrow \quad E_\sigma(R) \geq E_\sigma(\rho_1)\Big(\frac{R}{\rho_1}\Big)^{c/2}.$$

Hence $E_\sigma(R)$ grows at least polynomially.

But we can also bound E_σ from above using Lemma 3.4 together with the L^p growth condition:

$$E_\sigma(\rho) \leq C g(\rho)^{\frac{9-3p}{2p-3}}.$$

Since g satisfies $\lim_{\rho\to\infty} \rho^{-1/3} g(\rho) = 0$, we have $g(\rho) = o(\rho^{1/3})$, and therefore $g(\rho)^{\frac{9-3p}{2p-3}} = o\big(\rho^{\frac{9-3p}{3(2p-3)}}\big)$. Because $\frac{9-3p}{3(2p-3)}$ is a fixed positive number (for $p \in (3/2, 3)$), the polynomial growth $\rho^{c/2}$ will eventually dominate, contradicting the upper bound. This completes the proof. ☐

Proof of Theorem 1.2. Because the right-hand side of Lemma 5.1 is finite by (2), we have $\liminf K_\theta(\rho) < \infty$. Lemma 3.1(ii) gives $\nabla \mathbf{u}, \nabla \mathbf{b} \in L^2(\mathbb{R}^3)$, and Theorem 1.1 forces $\mathbf{u} = \mathbf{b} = 0$. ☐

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