Closed-form Solutions of Third-Order Generalized Leonardo Sequences with Polynomial Input

Abstract. In this paper, we present closed-form solutions to third-order nonhomogeneous linear recurrence relations, referred to as generalized Leonardo-type sequences, where the input function is a polynomial.

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Keywords. Leonardo numbers, Leonardo polynomials, nonhomogeneous linear recurrence relations, homogeneous recurrence relations, closed-form solutions.

1. Introduction

Sequences defined by recurrence relations have long stood at the heart of mathematics, branching into diverse disciplines such as physics, engineering, architecture, biology, computer science, and even the arts. Though their definitions may appear elementary, they conceal remarkable depth: modeling growth, oscillations, and symbolic structures. Classical second-order families-most notably the Fibonacci, Lucas, Pell, and Jacobsthal sequences-remain central examples of this tradition.

The scope, however, extends well beyond second-order cases. Higher-order recurrence sequences enrich both theory and practice, broadening the classical framework and uncovering intricate algebraic and analytic patterns. The Tribonacci (third-order), Tetranacci (fourth-order), and Pentanacci (fifth-order) sequences exemplify this expansion, each governed by characteristic polynomials whose root configurations dictate closed-form expressions. Homogeneous recurrences emphasize the interplay of characteristic polynomials and root multiplicities, while non-homogeneous forms introduce symbolic terms that interact with root structures to generate resonance phenomena. Together, these families establish a coherent framework that unites classical recurrence identities with the evolving field of symbolic recurrence theory.

Let the third order nonhomogeneous linear recurrence relation, referred to as generalized Leonardo-type sequences, be given by

$$W_n = a_1 W_{n-1} + a_2 W_{n-2} + a_3 W_{n-3} + p(n) \tag{1.1}$$

with initial conditions $W_0 = k_0, W_1 = k_1, W_2 = k_2$ where $p(n)$ is the polynomial with degree s, with coefficients in $\mathbb{C}[x]$ or $\mathbb{C}$:

$$p(n) = \sum_{i=0}^{s} c_i n^i,$$

and the recurrence coefficients a_1, a_2, a_3 are complex scalars or polynomials in $\mathbb{C}[x]$. For more information on generalized Leonardo-type sequences, see Soykan [2] and [1].

Let the homogeneous relation corresponding to (1.1) be written as

$$V_n = a_1 V_{n-1} + a_2 V_{n-2} + a_3 V_{n-3} \tag{1.2}$$

with the same initial conditions as W_n, i.e.,

$$V_0 = W_0, V_1 = W_1, V_2 = W_2.$$

Suppose that θ_1, θ_2 and θ_3 are the roots of the characteristic equation

$$z^3 - a_1 z^2 - a_2 z - a_3 = 0 \tag{1.3}$$

of (1.2).

Note that if all the roots of (1.3) are equal to 1 then

$$z^3 - a_1 z^2 - a_2 z - a_3 = (z-1)^3 = z^3 - 3z^2 + 3z - 1 = 0$$

so that $a_1 = 3,\ a_2 = -3,\ a_3 = 1$ and (1.2) reduces to

$$V_n = 3V_{n-1} - 3V_{n-2} + V_{n-3}.$$

In our earlier work, particular solutions to third-order nonhomogeneous linear recurrence relations (1.1) with polynomial inputs were established for the cases $s = 0, 1, 2, 3$; see Soykan [3]. In the present paper, we build upon those results to derive closed-form solutions by systematically applying Theorem 1.1. Within this framework, the closed-form expressions are obtained by decomposing each recurrence into its homogeneous and particular components, with the latter determined through an iterative scheme for the coefficients. The analysis is organized according to the multiplicity r of 1 as a root of the characteristic equation (1.2) and the degree s of the polynomial $p(n)$, with explicit formulas derived for all cases $r = 0, 1, 2, 3$ and $s = 0, 1, 2, 3$. This approach highlights how the root multiplicity and polynomial degree jointly determine the structure of the solution, while ensuring that explicit formulas are available for all cases considered. By integrating the previously obtained particular solutions into a unified framework, we provide complete closed-form expressions that extend the classical theory of recurrence relations and advance the study of generalized Leonardo-type sequences in the nonhomogeneous setting.

Theorem 1.1.

(a): *[1, Theorem 7.7. (a)] The case $r=0$, i.e., all three roots of the characteristic equation of (1.2) is distinct from 1.*

The solution of (1.1) is given by

$$\begin{aligned} W_n(W_0,W_1,W_2) &= W_n^{(h)}+W_n^{(p)} \\ &= V_n(W_0,W_1,W_2)-V_n(W_0^{(p)},W_1^{(p)},W_2^{(p)})+W_n^{(p)} \end{aligned}$$

where $W_n^{(h)}=V_n(W_0,W_1,W_2)-V_n(W_0^{(p)},W_1^{(p)},W_2^{(p)})$ *is the solution of (1.2) and*

$$W_n^{(p)}=\sum_{i=0}^{s}A_i n^i=A_0+\sum_{i=1}^{s}A_i n^i$$

is the particular solution of (1.1). For each $0\leq i\leq s$, A_i can be calculated with the iteration

$$A_s=-\frac{c_s}{a_1+a_2+a_3-1},\ \textit{for } n=s$$

and

$$A_n=-\frac{1}{a_1+a_2+a_3-1}(c_n-\sum_{k=n+1}^{s}(-1)^{k-n+1}\binom{k}{n}(a_1+2^{k-n}a_2+3^{k-n}a_3)A_k),\ \textit{for } n=s-1,s-2,...,2,1,0.$$

Here

$$\begin{aligned} W_0^{(p)} &= A_0, \\ W_1^{(p)} &= \sum_{i=0}^{s}A_i, \\ W_2^{(p)} &= \sum_{i=0}^{s}2^iA_i, \end{aligned}$$

and

$$\begin{aligned} V_n(W_0^{(p)},W_1^{(p)},W_2^{(p)}) &= V_n(A_0,\sum_{i=0}^{s}A_i,\sum_{i=0}^{s}2^iA_i) \\ &= B_1V_n(W_0,W_1,W_2)+B_2V_{n-1}(W_0,W_1,W_2)+B_3V_{n-2}(W_0,W_1,W_2) \end{aligned}$$

where

$$B_1=\frac{Y_1}{\Delta},\ B_2=\frac{Y_2}{\Delta},\ B_3=\frac{Y_3}{\Delta}$$

and

$Y_1=(W_2^2+a_1^2W_1^2+a_1a_3W_0^2-2a_1W_1W_2-a_2W_0W_2+(a_1a_2-a_3)W_0W_1)W_2^{(p)}+((a_3+a_1a_2)W_1^2+a_2a_3W_0^2-a_2W_1W_2-a_3W_0W_2+a_2^2W_0W_1)W_1^{(p)}+a_3(a_1W_1^2+a_3W_0^2-W_1W_2+a_2W_0W_1)W_0^{(p)}$

$Y_2=((a_3+a_1a_2)W_1^2-a_2W_2W_1+(a_2^2-2a_3a_1)W_0W_1+a_3a_2W_0^2+2a_3a_1W_0W_1-a_3W_0W_2)W_2^{(p)}+(a_2W_2^2+2a_3W_1W_2+a_3^2W_0^2-(3a_3+a_1a_2)W_1W_2-(a_2^2-2a_3a_1)W_0W_2-a_3a_1W_0W_2)W_1^{(p)}+(a_3W_2^2-a_3a_2W_0W_2-a_3a_1W_1W_2-a_3^2W_0W_1)W_0^{(p)}$

$Y_3=a_3(-W_1W_2+a_1W_1^2+a_2W_0W_1+a_3W_0^2)W_2^{(p)}+a_3(W_2^2-a_1W_1W_2-a_2W_0W_2-a_3W_0W_1)W_1^{(p)}+a_3(a_3W_1^2-a_3W_0W_2)W_0^{(p)}$

$\Delta = W_2^3 + (a_3 + a_1 a_2)W_1^3 + a_3^2 W_0^3 - 2a_1 W_1 W_2^2 + (a_1^2 - a_2)W_1^2 W_2 - a_2 W_0 W_2^2 + a_3 a_1 W_0^2 W_2 + (a_2^2 + a_1 a_3)W_0 W_1^2 + 2a_2 a_3 W_0^2 W_1 + (-3a_3 + a_1 a_2)W_0 W_1 W_2$

i.e.,

$Y_1 = (W_2^2 + a_1^2 W_1^2 + a_1 a_3 W_0^2 - 2a_1 W_1 W_2 - a_2 W_0 W_2 + (a_1 a_2 - a_3)W_0 W_1)\sum_{i=0}^{s} 2^i A_i + ((a_3 + a_1 a_2)W_1^2 + a_2 a_3 W_0^2 - a_2 W_1 W_2 - a_3 W_0 W_2 + a_2^2 W_0 W_1)\sum_{i=0}^{s} A_i + a_3(a_1 W_1^2 + a_3 W_0^2 - W_1 W_2 + a_2 W_0 W_1)A_0$

$Y_2 = ((a_3 + a_1 a_2)W_1^2 - a_2 W_2 W_1 + (a_2^2 - 2a_3 a_1)W_0 W_1 + a_3 a_2 W_0^2 + 2a_3 a_1 W_0 W_1 - a_3 W_0 W_2)\sum_{i=0}^{s} 2^i A_i + (a_2 W_2^2 + 2a_3 W_1 W_2 + a_3^2 W_0^2 - (3a_3 + a_1 a_2)W_1 W_2 - (a_2^2 - 2a_3 a_1)W_0 W_2 - a_3 a_1 W_0 W_2)\sum_{i=0}^{s} A_i + (a_3 W_2^2 - a_3 a_2 W_0 W_2 - a_3 a_1 W_1 W_2 - a_3^2 W_0 W_1)A_0$

$Y_3 = a_3(-W_1 W_2 + a_1 W_1^2 + a_2 W_0 W_1 + a_3 W_0^2)\sum_{i=0}^{s} 2^i A_i + a_3(W_2^2 - a_1 W_1 W_2 - a_2 W_0 W_2 - a_3 W_0 W_1)\sum_{i=0}^{s} A_i + a_3(a_3 W_1^2 - a_3 W_0 W_2)A_0$

$\Delta = W_2^3 + (a_3 + a_1 a_2)W_1^3 + a_3^2 W_0^3 - 2a_1 W_1 W_2^2 + (a_1^2 - a_2)W_1^2 W_2 - a_2 W_0 W_2^2 + a_3 a_1 W_0^2 W_2 + (a_2^2 + a_1 a_3)W_0 W_1^2 + 2a_2 a_3 W_0^2 W_1 + (-3a_3 + a_1 a_2)W_0 W_1 W_2$

In summary, the solution of (1.1) is given by

$$W_n(W_0, W_1, W_2) = (-B_1 + 1)V_n(W_0, W_1, W_2) - B_2 V_{n-1}(W_0, W_1, W_2) - B_3 V_{n-2}(W_0, W_1, W_2) + \sum_{i=0}^{s} A_i n^i$$

(b): *The case $r = 1$, i.e., 1 is a simple root of the characteristic equation of (1.2). The solution of (1.1) is*

$$W_n(W_0, W_1, W_2) = W_n^{(h)} + W_n^{(p)} = V_n(W_0, W_1, W_2) - V_n(W_0^{(p)}, W_1^{(p)}, W_2^{(p)}) + W_n^{(p)},$$

where the particular solution of (1.1) is

$$W_n^{(p)} = n\sum_{i=0}^{s} A_i n^i = n(A_0 + \sum_{i=1}^{s} A_i n^i).$$

For each $0 \leq i \leq s$, the coefficients A_i are obtained iteratively:

$$A_s = (-1)^2 \frac{c_s}{(\sum_{j=1}^{3} j a_j)\binom{s+1}{1}}, \quad n = s,$$

and

$$A_n = (-1)^2 \frac{1}{(\sum_{j=1}^{3} j a_j)\binom{n+1}{1}} \left(c_n - \sum_{k=n+1}^{s} (-1)^{k-n+2} \binom{k+1}{n} (a_1 + 2^{k-n+1} a_2 + 3^{k-n+1} a_3) A_k \right),$$

for $n = s-1, s-2, \ldots, 1, 0$. The expressions for $W_0^{(p)}, W_1^{(p)}, W_2^{(p)}$ and the representation of $V_n(W_0^{(p)}, W_1^{(p)}, W_2^{(p)})$ in terms of B_1, B_2, B_3 follow analogously to case (a).

Here

$$W_0^{(p)} = 0, \quad W_1^{(p)} = \sum_{i=0}^{s} A_i, \quad W_2^{(p)} = 2\sum_{i=0}^{s} 2^i A_i,$$

and

$$\begin{aligned} V_n(W_0^{(p)}, W_1^{(p)}, W_2^{(p)}) &= V_n(0, \sum_{i=0}^{s} A_i, 2\sum_{i=0}^{s} 2^i A_i) \\ &= B_1 V_n(W_0, W_1, W_2) + B_2 V_{n-1}(W_0, W_1, W_2) + B_3 V_{n-2}(W_0, W_1, W_2) \end{aligned}$$

where

$$B_1 = \frac{Y_1}{\Delta},\ B_2 = \frac{Y_2}{\Delta},\ B_3 = \frac{Y_3}{\Delta}$$

and

$Y_1 = (W_2^2+a_1^2W_1^2+a_1a_3W_0^2-2a_1W_1W_2-a_2W_0W_2+(a_1a_2-a_3)W_0W_1)W_2^{(p)}+((a_3+a_1a_2)W_1^2+a_2a_3W_0^2-a_2W_1W_2-a_3W_0W_2+a_2^2W_0W_1)W_1^{(p)}+a_3(a_1W_1^2+a_3W_0^2-W_1W_2+a_2W_0W_1)W_0^{(p)}$

$Y_2 = ((a_3+a_1a_2)W_1^2-a_2W_2W_1+(a_2^2-2a_3a_1)W_0W_1+a_3a_2W_0^2+2a_3a_1W_0W_1-a_3W_0W_2)W_2^{(p)}+(a_2W_2^2+2a_3W_1W_2+a_3^2W_0^2-(3a_3+a_1a_2)W_1W_2-(a_2^2-2a_3a_1)W_0W_2-a_3a_1W_0W_2)W_1^{(p)}+(a_3W_2^2-a_3a_2W_0W_2-a_3a_1W_1W_2-a_3^2W_0W_1)W_0^{(p)}$

$Y_3 = a_3(-W_1W_2+a_1W_1^2+a_2W_0W_1+a_3W_0^2)W_2^{(p)}+a_3(W_2^2-a_1W_1W_2-a_2W_0W_2-a_3W_0W_1)W_1^{(p)}+a_3(a_3W_1^2-a_3W_0W_2)W_0^{(p)}$

$\Delta = W_2^3+(a_3+a_1a_2)W_1^3+a_3^2W_0^3-2a_1W_1W_2^2+(a_1^2-a_2)W_1^2W_2-a_2W_0W_2^2+a_3a_1W_0^2W_2+(a_2^2+a_1a_3)W_0W_1^2+2a_2a_3W_0^2W_1+(-3a_3+a_1a_2)W_0W_1W_2$

In summary, the solution of (1.1) is given by

$$W_n(W_0,W_1,W_2) = (-B_1+1)V_n(W_0,W_1,W_2) - B_2V_{n-1}(W_0,W_1,W_2) - B_3V_{n-2}(W_0,W_1,W_2) + n\sum_{i=0}^{s} A_in^i$$

(c): *The case $r = 2$, i.e., 1 is a double root of the characteristic equation of (1.2). The solution of (1.1) is*

$$W_n(W_0,W_1,W_2) = W_n^{(h)} + W_n^{(p)} = V_n(W_0,W_1,W_2) - V_n(W_0^{(p)},W_1^{(p)},W_2^{(p)}) + W_n^{(p)},$$

where the particular solution of (1.1) is

$$W_n^{(p)} = n^2\sum_{i=0}^{s} A_in^i.$$

The coefficients A_i are determined by

$$A_s = (-1)^3\frac{c_s}{(\sum_{j=1}^{3} j^2a_j)\binom{s+2}{2}}, \quad n = s,$$

and

$$A_n = (-1)^3\frac{1}{(\sum_{j=1}^{3} j^2a_j)\binom{n+2}{2}}\left(c_n - \sum_{k=n+1}^{s}(-1)^{k-n+3}\binom{k+2}{n}(a_1+2^{\,k-n+2}a_2+3^{\,k-n+2}a_3)A_k\right),$$

for $n = s-1, s-2, \ldots, 1, 0$. Again, $W_0^{(p)}, W_1^{(p)}, W_2^{(p)}$ and the B_1, B_2, B_3 representation follow the same scheme.

Here

$$W_0^{(p)} = 0, \quad W_1^{(p)} = \sum_{i=0}^{s} A_i, \quad W_2^{(p)} = 2^2\sum_{i=0}^{s} 2^iA_i,$$

and

$$\begin{aligned} V_n(W_0^{(p)},W_1^{(p)},W_2^{(p)}) &= V_n(0,\sum_{i=0}^{s} A_i, 2^2\sum_{i=0}^{s} 2^iA_i) \\ &= B_1V_n(W_0,W_1,W_2) + B_2V_{n-1}(W_0,W_1,W_2) + B_3V_{n-2}(W_0,W_1,W_2) \end{aligned}$$

where

$$B_1 = \frac{Y_1}{\Delta},\ B_2 = \frac{Y_2}{\Delta},\ B_3 = \frac{Y_3}{\Delta}$$

and

$Y_1 = (W_2^2 + a_1^2 W_1^2 + a_1 a_3 W_0^2 - 2a_1 W_1 W_2 - a_2 W_0 W_2 + (a_1 a_2 - a_3) W_0 W_1) W_2^{(p)} + ((a_3 + a_1 a_2) W_1^2 + a_2 a_3 W_0^2 - a_2 W_1 W_2 - a_3 W_0 W_2 + a_2^2 W_0 W_1) W_1^{(p)} + a_3 (a_1 W_1^2 + a_3 W_0^2 - W_1 W_2 + a_2 W_0 W_1) W_0^{(p)}$

$Y_2 = ((a_3 + a_1 a_2) W_1^2 - a_2 W_2 W_1 + (a_2^2 - 2a_3 a_1) W_0 W_1 + a_3 a_2 W_0^2 + 2a_3 a_1 W_0 W_1 - a_3 W_0 W_2) W_2^{(p)} + (a_2 W_2^2 + 2a_3 W_1 W_2 + a_3^2 W_0^2 - (3a_3 + a_1 a_2) W_1 W_2 - (a_2^2 - 2a_3 a_1) W_0 W_2 - a_3 a_1 W_0 W_2) W_1^{(p)} + (a_3 W_2^2 - a_3 a_2 W_0 W_2 - a_3 a_1 W_1 W_2 - a_3^2 W_0 W_1) W_0^{(p)}$

$Y_3 = a_3(-W_1 W_2 + a_1 W_1^2 + a_2 W_0 W_1 + a_3 W_0^2) W_2^{(p)} + a_3 (W_2^2 - a_1 W_1 W_2 - a_2 W_0 W_2 - a_3 W_0 W_1) W_1^{(p)} + a_3 (a_3 W_1^2 - a_3 W_0 W_2) W_0^{(p)}$

$\Delta = W_2^3 + (a_3 + a_1 a_2) W_1^3 + a_3^2 W_0^3 - 2a_1 W_1 W_2^2 + (a_1^2 - a_2) W_1^2 W_2 - a_2 W_0 W_2^2 + a_3 a_1 W_0^2 W_2 + (a_2^2 + a_1 a_3) W_0 W_1^2 + 2a_2 a_3 W_0^2 W_1 + (-3a_3 + a_1 a_2) W_0 W_1 W_2$

In summary, the solution of (1.1) is given by

$$W_n(W_0, W_1, W_2) = (-B_1 + 1) V_n(W_0, W_1, W_2) - B_2 V_{n-1}(W_0, W_1, W_2) - B_3 V_{n-2}(W_0, W_1, W_2) + n^2 \sum_{i=0}^{s} A_i n^i$$

(d): *The case $r = 3$, i.e., 1 is a triple root of the characteristic equation of (1.2). The solution of (1.1) is*

$$W_n(W_0, W_1, W_2) = W_n^{(h)} + W_n^{(p)} = V_n(W_0, W_1, W_2) - V_n(W_0^{(p)}, W_1^{(p)}, W_2^{(p)}) + W_n^{(p)},$$

where the particular solution of (1.1) is

$$W_n^{(p)} = n^3 \sum_{i=0}^{s} A_i n^i.$$

The coefficients A_i are determined by

$$A_s = (-1)^4 \frac{c_s}{(\sum_{j=1}^{3} j^3 a_j)\binom{s+3}{3}}, \quad n = s,$$

and

$$A_n = (-1)^4 \frac{1}{(\sum_{j=1}^{3} j^3 a_j)\binom{n+3}{3}} \left(c_n - \sum_{k=n+1}^{s} (-1)^{k-n+4} \binom{k+3}{n} (a_1 + 2^{k-n+3} a_2 + 3^{k-n+3} a_3) A_k \right),$$

for $n = s-1, s-2, \ldots, 1, 0$. The evaluation of $W_0^{(p)}, W_1^{(p)}, W_2^{(p)}$ and the decomposition of $V_n(W_0^{(p)}, W_1^{(p)}, W_2^{(p)})$ into B_1, B_2, B_3 terms are analogous to case (a).

Here

$$W_0^{(p)} = 0, \quad W_1^{(p)} = \sum_{i=0}^{s} A_i, \quad W_2^{(p)} = 2^3 \sum_{i=0}^{s} 2^i A_i,$$

and

$$\begin{aligned} V_n(W_0^{(p)}, W_1^{(p)}, W_2^{(p)}) &= V_n(0, \sum_{i=0}^{s} A_i, 2^3 \sum_{i=0}^{s} 2^i A_i) \\ &= B_1 V_n(W_0, W_1, W_2) + B_2 V_{n-1}(W_0, W_1, W_2) + B_3 V_{n-2}(W_0, W_1, W_2) \end{aligned}$$

where

$$B_1 = \frac{Y_1}{\Delta},\ B_2 = \frac{Y_2}{\Delta},\ B_3 = \frac{Y_3}{\Delta}$$

and

$Y_1 = (W_2^2+a_1^2W_1^2+a_1a_3W_0^2-2a_1W_1W_2-a_2W_0W_2+(a_1a_2-a_3)W_0W_1)W_2^{(p)}+((a_3+a_1a_2)W_1^2+a_2a_3W_0^2-a_2W_1W_2-a_3W_0W_2+a_2^2W_0W_1)W_1^{(p)}+a_3(a_1W_1^2+a_3W_0^2-W_1W_2+a_2W_0W_1)W_0^{(p)}$

$Y_2 = ((a_3+a_1a_2)W_1^2-a_2W_2W_1+(a_2^2-2a_3a_1)W_0W_1+a_3a_2W_0^2+2a_3a_1W_0W_1-a_3W_0W_2)W_2^{(p)}+(a_2W_2^2+2a_3W_1W_2+a_3^2W_0^2-(3a_3+a_1a_2)W_1W_2-(a_2^2-2a_3a_1)W_0W_2-a_3a_1W_0W_2)W_1^{(p)}+(a_3W_2^2-a_3a_2W_0W_2-a_3a_1W_1W_2-a_3^2W_0W_1)W_0^{(p)}$

$Y_3 = a_3(-W_1W_2+a_1W_1^2+a_2W_0W_1+a_3W_0^2)W_2^{(p)}+a_3(W_2^2-a_1W_1W_2-a_2W_0W_2-a_3W_0W_1)W_1^{(p)}+a_3(a_3W_1^2-a_3W_0W_2)W_0^{(p)}$

$\Delta = W_2^3+(a_3+a_1a_2)W_1^3+a_3^2W_0^3-2a_1W_1W_2^2+(a_1^2-a_2)W_1^2W_2-a_2W_0W_2^2+a_3a_1W_0^2W_2+(a_2^2+a_1a_3)W_0W_1^2+2a_2a_3W_0^2W_1+(-3a_3+a_1a_2)W_0W_1W_2$

In summary, the solution of (1.1) is given by

$$W_n(W_0,W_1,W_2) = (-B_1+1)V_n(W_0,W_1,W_2) - B_2V_{n-1}(W_0,W_1,W_2) - B_3V_{n-2}(W_0,W_1,W_2) + n^3\sum_{i=0}^{s} A_i n^i$$

Proof. The result is obtained by combining Theorem 5.5 (p. 104), Theorem 5.6 (pp. 104–105), and Theorem 3.1 (pp. 88–89) for the case $m = 3$, as established in Soykan [1]. □

2. Closed-Form Solutions via Theorem 1.1 for Special Cases of $p(n)$

2.1. The Case: $\theta_1 \neq 1$ and $\theta_2 \neq 1$ and $\theta_3 \neq 1$, i.e., All of the Roots of the Characteristic Equation of (1.2) is distinct from 1. In the following example, we assume that all of the roots of the characteristic equation of (1.2) is distinct from 1 and we consider special cases of $p(n)$ defined in (1.1) by using Theorem 1.1.

Example 2.1. *In this example, in each case, the homogeneous relation corresponding to the given non-homogeneous recurrence relation is expressed as in (1.2), i.e.,*

$$V_n = a_1V_{n-1} + a_2V_{n-2} + a_3V_{n-3}.$$

(a): *The case $s = 0$: ($a_3 \neq 0$, $c_0 \neq 0$).*

Let the sequence $\{W_n\}$ defined by nonhomogeneous linear recurrence relation

$$W_n = a_1W_{n-1} + a_2W_{n-2} + a_3W_{n-3} + c_0. \tag{2.1}$$

Then the solution of (2.1) is given by

$$\begin{aligned} W_n(W_0,W_1,W_2) &= W_n^{(h)} + W_n^{(p)} \\ &= V_n(W_0,W_1,W_2) - V_n(W_0^{(p)},W_1^{(p)},W_2^{(p)}) + W_n^{(p)} \end{aligned}$$

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where $W_n^{(h)} = V_n(W_0, W_1, W_2) - V_n(W_0^{(p)}, W_1^{(p)}, W_2^{(p)})$ *is the solution of (1.2) and*

$$W_n^{(p)} = A_0 = -\frac{c_0}{a_1 + a_2 + a_3 - 1}$$

is the particular solution of (2.1). Here,

$$W_0^{(p)} = A_0, \; W_1^{(p)} = A_0, \; W_2^{(p)} = A_0,$$

and

$$\begin{aligned} V_n(W_0^{(p)}, W_1^{(p)}, W_2^{(p)}) &= V_n(A_0, A_0, A_0) \\ &= B_1 V_n(W_0, W_1, W_2) + B_2 V_{n-1}(W_0, W_1, W_2) + B_3 V_{n-2}(W_0, W_1, W_2) \end{aligned}$$

where

$$B_1 = \frac{Y_1}{\Delta}, \; B_2 = \frac{Y_2}{\Delta}, \; B_3 = \frac{Y_3}{\Delta}$$

and

$Y_1 = (W_2^2 + a_1^2 W_1^2 + a_1 a_3 W_0^2 - 2a_1 W_1 W_2 - a_2 W_0 W_2 + (a_1 a_2 - a_3) W_0 W_1) A_0 + ((a_3 + a_1 a_2) W_1^2 + a_2 a_3 W_0^2 - a_2 W_1 W_2 - a_3 W_0 W_2 + a_2^2 W_0 W_1) A_0 + a_3 (a_1 W_1^2 + a_3 W_0^2 - W_1 W_2 + a_2 W_0 W_1) A_0$

$Y_2 = ((a_3 + a_1 a_2) W_1^2 - a_2 W_2 W_1 + (a_2^2 - 2a_3 a_1) W_0 W_1 + a_3 a_2 W_0^2 + 2a_3 a_1 W_0 W_1 - a_3 W_0 W_2) A_0 + (a_2 W_2^2 + 2a_3 W_1 W_2 + a_3^2 W_0^2 - (3a_3 + a_1 a_2) W_1 W_2 - (a_2^2 - 2a_3 a_1) W_0 W_2 - a_3 a_1 W_0 W_2) A_0 + (a_3 W_2^2 - a_3 a_2 W_0 W_2 - a_3 a_1 W_1 W_2 - a_3^2 W_0 W_1) A_0$

$Y_3 = a_3(-W_1 W_2 + a_1 W_1^2 + a_2 W_0 W_1 + a_3 W_0^2) A_0 + a_3 (W_2^2 - a_1 W_1 W_2 - a_2 W_0 W_2 - a_3 W_0 W_1) A_0 + a_3 (a_3 W_1^2 - a_3 W_0 W_2) A_0$

$\Delta = W_2^3 + (a_3 + a_1 a_2) W_1^3 + a_3^2 W_0^3 - 2a_1 W_1 W_2^2 + (a_1^2 - a_2) W_1^2 W_2 - a_2 W_0 W_2^2 + a_3 a_1 W_0^2 W_2 + (a_2^2 + a_1 a_3) W_0 W_1^2 + 2a_2 a_3 W_0^2 W_1 + (-3a_3 + a_1 a_2) W_0 W_1 W_2$

In summary, the solution of (2.1) is given by

$$W_n(W_0, W_1, W_2) = (-B_1 + 1) V_n(W_0, W_1, W_2) - B_2 V_{n-1}(W_0, W_1, W_2) - B_3 V_{n-2}(W_0, W_1, W_2) + A_0$$

(b): *The case* $s = 1 :$ *(*$a_3 \neq 0, \; c_1 \neq 0$*).*

Let the sequence $\{W_n\}$ *defined by nonhomogeneous linear recurrence relation*

$$W_n = a_1 W_{n-1} + a_2 W_{n-2} + a_3 W_{n-3} + c_1 n + c_0. \tag{2.2}$$

Then the solution of (2.2) is given by

$$\begin{aligned} W_n(W_0, W_1, W_2) &= W_n^{(h)} + W_n^{(p)} \\ &= V_n(W_0, W_1, W_2) - V_n(W_0^{(p)}, W_1^{(p)}, W_2^{(p)}) + W_n^{(p)} \end{aligned}$$

where $W_n^{(h)} = V_n(W_0, W_1, W_2) - V_n(W_0^{(p)}, W_1^{(p)}, W_2^{(p)})$ *is the solution of (1.2) and*

$$W_n^{(p)} = A_1 n + A_0$$

is the particular solution of (2.2). Here,

$$
\begin{aligned}
A_1 &= -\frac{c_1}{a_1+a_2+a_3-1} \\
A_0 &= -\frac{1}{a_1+a_2+a_3-1}\left(c_0-(a_1+2a_2+3a_3)\,A_1\right)
\end{aligned}
$$

i.e.,

$$
\begin{aligned}
A_1 &= -\frac{c_1}{a_1+a_2+a_3-1} \\
A_0 &= -\frac{1}{(a_1+a_2+a_3-1)^2}(-c_0+a_1(c_0+c_1)+a_2(c_0+2c_1)+a_3(c_0+3c_1))
\end{aligned}
$$

i.e.,

$$
\begin{aligned}
A_1 &= -\frac{c_1}{a_1+a_2+a_3-1} \\
A_0 &= -\frac{1}{(a_1+a_2+a_3-1)^2}((a_1+a_2+a_3-1)\,c_0+(a_1+2a_2+3a_3)c_1)
\end{aligned}
$$

and

$$W_0^{(p)} = A_0,\ W_1^{(p)} = A_0+A_1,\ W_2^{(p)} = A_0+2A_1,$$

and

$$
\begin{aligned}
V_n(W_0^{(p)},W_1^{(p)},W_2^{(p)}) &= V_n(A_0,A_0+A_1,A_0+2A_1) \\
&= B_1V_n(W_0,W_1,W_2)+B_2V_{n-1}(W_0,W_1,W_2)+B_3V_{n-2}(W_0,W_1,W_2)
\end{aligned}
$$

where

$$B_1 = \frac{Y_1}{\Delta},\ B_2 = \frac{Y_2}{\Delta},\ B_3 = \frac{Y_3}{\Delta}$$

and

$Y_1 = (W_2^2 + a_1^2W_1^2 + a_1a_3W_0^2 - 2a_1W_1W_2 - a_2W_0W_2 + (a_1a_2 - a_3)W_0W_1)(A_0 + 2A_1) + ((a_3 + a_1a_2)W_1^2 + a_2a_3W_0^2 - a_2W_1W_2 - a_3W_0W_2 + a_2^2W_0W_1)(A_0 + A_1) + a_3(a_1W_1^2 + a_3W_0^2 - W_1W_2 + a_2W_0W_1)A_0$

$Y_2 = ((a_3+a_1a_2)W_1^2 - a_2W_2W_1 + (a_2^2 - 2a_3a_1)W_0W_1 + a_3a_2W_0^2 + 2a_3a_1W_0W_1 - a_3W_0W_2)(A_0 + 2A_1) + (a_2W_2^2 + 2a_3W_1W_2 + a_3^2W_0^2 - (3a_3 + a_1a_2)W_1W_2 - (a_2^2 - 2a_3a_1)W_0W_2 - a_3a_1W_0W_2)(A_0 + A_1) + (a_3W_2^2 - a_3a_2W_0W_2 - a_3a_1W_1W_2 - a_3^2W_0W_1)A_0$

$Y_3 = a_3(-W_1W_2 + a_1W_1^2 + a_2W_0W_1 + a_3W_0^2)(A_0 + 2A_1) + a_3(W_2^2 - a_1W_1W_2 - a_2W_0W_2 - a_3W_0W_1)(A_0 + A_1) + a_3(a_3W_1^2 - a_3W_0W_2)A_0$

$\Delta = W_2^3 + (a_3 + a_1a_2)W_1^3 + a_3^2W_0^3 - 2a_1W_1W_2^2 + (a_1^2 - a_2)W_1^2W_2 - a_2W_0W_2^2 + a_3a_1W_0^2W_2 + (a_2^2 + a_1a_3)W_0W_1^2 + 2a_2a_3W_0^2W_1 + (-3a_3 + a_1a_2)W_0W_1W_2$

In summary, the solution of (2.2) is given by

$$W_n(W_0,W_1,W_2) = (-B_1+1)V_n(W_0,W_1,W_2) - B_2V_{n-1}(W_0,W_1,W_2) - B_3V_{n-2}(W_0,W_1,W_2) + A_1n + A_0$$

(c): *The case $s = 2$: ($a_3 \neq 0$, $c_2 \neq 0$).*

Let the sequence $\{W_n\}$ defined by nonhomogeneous linear recurrence relation

$$W_n = a_1 W_{n-1} + a_2 W_{n-2} + a_3 W_{n-3} + c_2 n^2 + c_1 n + c_0. \tag{2.3}$$

Then the solution of (2.3) is given by

$$\begin{aligned} W_n(W_0, W_1, W_2) &= W_n^{(h)} + W_n^{(p)} \\ &= V_n(W_0, W_1, W_2) - V_n(W_0^{(p)}, W_1^{(p)}, W_2^{(p)}) + W_n^{(p)} \end{aligned}$$

where $W_n^{(h)} = V_n(W_0, W_1, W_2) - V_n(W_0^{(p)}, W_1^{(p)}, W_2^{(p)})$ is the solution of (1.2) and

$$W_n^{(p)} = A_2 n^2 + A_1 n + A_0$$

is the particular solution of (2.3). Here,

$$\begin{aligned} A_2 &= -\frac{c_2}{a_1 + a_2 + a_3 - 1} \\ A_1 &= -\frac{1}{a_1 + a_2 + a_3 - 1}(c_1 - 2(a_1 + 2a_2 + 3a_3) A_2) \\ A_0 &= -\frac{1}{a_1 + a_2 + a_3 - 1}(c_0 - (a_1 + 2a_2 + 3a_3)A_1 + (a_1 + 4a_2 + 9a_3)A_2) \end{aligned}$$

i.e.,

$$A_2 = -\frac{c_2}{a_1 + a_2 + a_3 - 1}$$

$$A_1 = -\frac{1}{(a_1 + a_2 + a_3 - 1)^2}(-c_1 + a_1(2c_2 + c_1) + a_2(4c_2 + c_1) + a_3(6c_2 + c_1))$$

$$A_0 = -\frac{1}{(a_1 + a_2 + a_3 - 1)^3}(c_0 + a_1(c_2 - c_1 - 2c_0) + 2a_2(2c_2 - c_1 - c_0) + a_3(9c_2 - 3c_1 - 2c_0) + a_1^2(c_2 + c_1 + c_0) + a_2^2(4c_2 + 2c_1 + c_0) + a_3^2(9c_2 + 3c_1 + c_0) + a_1 a_2(3c_2 + 3c_1 + 2c_0) + 2a_1 a_3(c_2 + 2c_1 + c_0) + a_2 a_3(11c_2 + 5c_1 + 2c_0))$$

i.e.,

$$A_2 = -\tfrac{c_2}{a_1+a_2+a_3-1}$$

$$A_1 = -\tfrac{1}{(a_1+a_2+a_3-1)^2}((a_1 + a_2 + a_3 - 1)c_1 + 2(a_1 + 2a_2 + 3a_3)c_2)$$

$$A_0 = -\tfrac{1}{(a_1+a_2+a_3-1)^3}((a_1 + a_2 + a_3 - 1)^2 c_0 + (a_1 + a_2 + a_3 - 1)(a_1 + 2a_2 + 3a_3)c_1 + ((a_1 + 2a_2 + 3a_3)^2 + (a_1 + 4a_2 + 9a_3) - (a_1 a_2 + 4a_1 a_3 + a_2 a_3))c_2)$$

and

$$W_0^{(p)} = A_0,\ W_1^{(p)} = A_0 + A_1 + A_2,\ W_2^{(p)} = A_0 + 2A_1 + 4A_2,$$

and

$$\begin{aligned} V_n(W_0^{(p)}, W_1^{(p)}, W_2^{(p)}) &= V_n(A_0, A_0 + A_1 + A_2, A_0 + 2A_1 + 4A_2) \\ &= B_1 V_n(W_0, W_1, W_2) + B_2 V_{n-1}(W_0, W_1, W_2) + B_3 V_{n-2}(W_0, W_1, W_2) \end{aligned}$$

where

$$B_1 = \frac{Y_1}{\Delta},\ B_2 = \frac{Y_2}{\Delta},\ B_3 = \frac{Y_3}{\Delta}$$

and

$Y_1 = (W_2^2 + a_1^2 W_1^2 + a_1 a_3 W_0^2 - 2a_1 W_1 W_2 - a_2 W_0 W_2 + (a_1 a_2 - a_3) W_0 W_1)(A_0 + 2A_1 + 4A_2) + ((a_3 + a_1 a_2) W_1^2 + a_2 a_3 W_0^2 - a_2 W_1 W_2 - a_3 W_0 W_2 + a_2^2 W_0 W_1)(A_0 + A_1 + A_2) + a_3(a_1 W_1^2 + a_3 W_0^2 - W_1 W_2 + a_2 W_0 W_1) A_0$

$Y_2 = ((a_3 + a_1 a_2) W_1^2 - a_2 W_2 W_1 + (a_2^2 - 2a_3 a_1) W_0 W_1 + a_3 a_2 W_0^2 + 2a_3 a_1 W_0 W_1 - a_3 W_0 W_2)(A_0 + 2A_1 + 4A_2) + (a_2 W_2^2 + 2a_3 W_1 W_2 + a_3^2 W_0^2 - (3a_3 + a_1 a_2) W_1 W_2 - (a_2^2 - 2a_3 a_1) W_0 W_2 - a_3 a_1 W_0 W_2)(A_0 + A_1 + A_2) + (a_3 W_2^2 - a_3 a_2 W_0 W_2 - a_3 a_1 W_1 W_2 - a_3^2 W_0 W_1) A_0$

$Y_3 = a_3(-W_1 W_2 + a_1 W_1^2 + a_2 W_0 W_1 + a_3 W_0^2)(A_0 + 2A_1 + 4A_2) + a_3(W_2^2 - a_1 W_1 W_2 - a_2 W_0 W_2 - a_3 W_0 W_1)(A_0 + A_1 + A_2) + a_3(a_3 W_1^2 - a_3 W_0 W_2) A_0$

$\Delta = W_2^3 + (a_3 + a_1 a_2) W_1^3 + a_3^2 W_0^3 - 2a_1 W_1 W_2^2 + (a_1^2 - a_2) W_1^2 W_2 - a_2 W_0 W_2^2 + a_3 a_1 W_0^2 W_2 + (a_2^2 + a_1 a_3) W_0 W_1^2 + 2a_2 a_3 W_0^2 W_1 + (-3a_3 + a_1 a_2) W_0 W_1 W_2$

In summary, the solution of (2.3) is given by

$$W_n(W_0, W_1, W_2) = (-B_1 + 1) V_n(W_0, W_1, W_2) - B_2 V_{n-1}(W_0, W_1, W_2) - B_3 V_{n-2}(W_0, W_1, W_2) + A_2 n^2 + A_1 n + A_0$$

(d): *The case $s = 3$: ($a_3 \neq 0$, $c_3 \neq 0$).*

Let the sequence $\{W_n\}$ defined by nonhomogeneous linear recurrence relation

$$W_n = a_1 W_{n-1} + a_2 W_{n-2} + a_3 W_{n-3} + c_3 n^3 + c_2 n^2 + c_1 n + c_0. \tag{2.4}$$

Then the solution of (2.4) is given by

$$\begin{aligned} W_n(W_0, W_1, W_2) &= W_n^{(h)} + W_n^{(p)} \\ &= V_n(W_0, W_1, W_2) - V_n(W_0^{(p)}, W_1^{(p)}, W_2^{(p)}) + W_n^{(p)} \end{aligned}$$

where $W_n^{(h)} = V_n(W_0, W_1, W_2) - V_n(W_0^{(p)}, W_1^{(p)}, W_2^{(p)})$ is the solution of (1.2) and

$$W_n^{(p)} = A_3 n^3 + A_2 n^2 + A_1 n + A_0$$

is the particular solution of (2.4). Here,

$$\begin{aligned} A_3 &= -\frac{c_3}{a_1 + a_2 + a_3 - 1} \\ A_2 &= -\frac{1}{a_1 + a_2 + a_3 - 1}(c_2 - 3(a_1 + 2a_2 + 3a_3) A_3) \\ A_1 &= -\frac{1}{a_1 + a_2 + a_3 - 1}(c_1 - 2(a_1 + 2a_2 + 3a_3) A_2 + 3(a_1 + 4a_2 + 9a_3) A_3) \\ A_0 &= -\frac{1}{a_1 + a_2 + a_3 - 1}(c_0 - (a_1 + 2a_2 + 3a_3) A_1 + (a_1 + 4a_2 + 9a_3) A_2 - (a_1 + 8a_2 + 27a_3) A_3) \end{aligned}$$

i.e.,

$$A_3 = -\frac{c_3}{a_1 + a_2 + a_3 - 1}$$

$$A_2 = -\frac{1}{(a_1 + a_2 + a_3 - 1)^2}(-c_2 + a_1(c_2 + 3c_3) + a_2(c_2 + 6c_3) + a_3(c_2 + 9c_3))$$

$A_1 = -\frac{1}{(a_1+a_2+a_3-1)^3}(c_1 - a_1(2c_1 + 2c_2 - 3c_3) + a_1^2(c_1 + 2c_2 + 3c_3) - 2a_2(c_1 + 2c_2 - 6c_3) + a_2^2(c_1 + 4c_2 + 12c_3) - a_3(2c_1 + 6c_2 - 27c_3) + a_3^2(c_1 + 6c_2 + 27c_3) + a_1a_2(2c_1 + 6c_2 + 9c_3) + 2a_1a_3(c_1 + 4c_2 + 3c_3) + a_2a_3(2c_1 + 10c_2 + 33c_3))$

$A_0 = -\dfrac{1}{(a_1 + a_2 + a_3 - 1)^4}(-c_0 + a_1(3c_0 + c_1 - c_2 + c_3) - a_1^2(3c_0 + 2c_1 - 4c_3) + a_1^3(c_0 + c_1 + c_2 + c_3) + a_2(3c_0 + 2c_1 - 4c_2 + 8c_3) - a_2^2(3c_0 + 4c_1 - 32c_3) + a_2^3(c_0 + 2c_1 + 4c_2 + 8c_3) + 3a_3(c_0 + c_1 - 3c_2 + 9c_3) - 3a_3^2(c_0 + 2c_1 - 36c_3) + a_3^3(c_0 + 3c_1 + 9c_2 + 27c_3) + a_1a_2^2(3c_0 + 5c_1 + 7c_2 + 5c_3) + a_1^2a_2(3c_0 + 4c_1 + 4c_2 + 4c_3) + a_1^2a_3(3c_0 + 5c_1 + 3c_2 + 5c_3) + a_1a_3^2(3c_0 + 7c_1 + 11c_2 - 17c_3) + a_2^2a_3(3c_0 + 7c_1 + 15c_2 + 31c_3) + a_2a_3^2(3c_0 + 8c_1 + 20c_2 + 44c_3) - 2a_1a_2(3c_0 + 3c_1 - c_2 - 9c_3) - 2a_1a_3(3c_0 + 4c_1 - 4c_2 - 8c_3) - 2a_2a_3(3c_0 + 5c_1 - c_2 - 55c_3) + 2a_1a_2a_3(3c_0 + 6c_1 + 8c_2))$

and

$$W_0^{(p)} = A_0,\ W_1^{(p)} = A_0 + A_1 + A_2 + A_3,\ W_2^{(p)} = A_0 + 2A_1 + 4A_2 + 8A_3,$$

and

$$\begin{aligned} V_n(W_0^{(p)}, W_1^{(p)}, W_2^{(p)}, W_3^{(p)}) &= V_n(A_0, A_0 + A_1 + A_2 + A_3, A_0 + 2A_1 + 4A_2 + 8A_3) \\ &= B_1V_n(W_0, W_1, W_2) + B_2V_{n-1}(W_0, W_1, W_2) + B_3V_{n-2}(W_0, W_1, W_2) \end{aligned}$$

where

$$B_1 = \frac{Y_1}{\Delta},\ B_2 = \frac{Y_2}{\Delta},\ B_3 = \frac{Y_3}{\Delta}$$

and

$Y_1 = (W_2^2 + a_1^2W_1^2 + a_1a_3W_0^2 - 2a_1W_1W_2 - a_2W_0W_2 + (a_1a_2 - a_3)W_0W_1)(A_0 + 2A_1 + 4A_2 + 8A_3) + ((a_3 + a_1a_2)W_1^2 + a_2a_3W_0^2 - a_2W_1W_2 - a_3W_0W_2 + a_2^2W_0W_1)(A_0 + A_1 + A_2 + A_3) + a_3(a_1W_1^2 + a_3W_0^2 - W_1W_2 + a_2W_0W_1)A_0$

$Y_2 = ((a_3 + a_1a_2)W_1^2 - a_2W_2W_1 + (a_2^2 - 2a_3a_1)W_0W_1 + a_3a_2W_0^2 + 2a_3a_1W_0W_1 - a_3W_0W_2)(A_0 + 2A_1 + 4A_2 + 8A_3) + (a_2W_2^2 + 2a_3W_1W_2 + a_3^2W_0^2 - (3a_3 + a_1a_2)W_1W_2 - (a_2^2 - 2a_3a_1)W_0W_2 - a_3a_1W_0W_2)(A_0 + A_1 + A_2 + A_3) + (a_3W_2^2 - a_3a_2W_0W_2 - a_3a_1W_1W_2 - a_3^2W_0W_1)A_0$

$Y_3 = a_3(-W_1W_2 + a_1W_1^2 + a_2W_0W_1 + a_3W_0^2)(A_0 + 2A_1 + 4A_2 + 8A_3) + a_3(W_2^2 - a_1W_1W_2 - a_2 W_0W_2 - a_3W_0W_1)(A_0 + A_1 + A_2 + A_3) + a_3(a_3W_1^2 - a_3W_0W_2)A_0$

$\Delta = W_2^3 + (a_3 + a_1a_2)W_1^3 + a_3^2W_0^3 - 2a_1W_1W_2^2 + (a_1^2 - a_2)W_1^2W_2 - a_2W_0W_2^2 + a_3a_1W_0^2W_2 + (a_2^2 + a_1a_3)W_0W_1^2 + 2a_2a_3W_0^2W_1 + (-3a_3 + a_1a_2)W_0W_1W_2$

In summary, the solution of (2.4) is given by

$$W_n(W_0, W_1, W_2) = (-B_1+1)V_n(W_0, W_1, W_2) - B_2V_{n-1}(W_0, W_1, W_2) - B_3V_{n-2}(W_0, W_1, W_2) + A_3n^3 + A_2n^2 + A_1n + A_0$$

2.2. The Case: $\theta_1 = 1$ and $\theta_2 \neq 1$ and $\theta_3 \neq 1$, i.e., Only one of the Roots of the Characteristic Equation of (1.2) is 1. In the following example, we assume that only one of the roots of the characteristic equation of (1.2) is equal to 1 and we consider special cases of $p(n)$ defined in (1.1) by using Theorem 1.1.

Example 2.2. *In this example, in each case, the homogeneous relation corresponding to the given non-homogeneous recurrence relation is expressed as in (1.2), i.e.,*

$$V_n = a_1 V_{n-1} + a_2 V_{n-2} + a_3 V_{n-3}.$$

(a): *The case $s = 0$: ($a_3 \neq 0$, $c_0 \neq 0$).*

Let the sequence $\{W_n\}$ defined by nonhomogeneous linear recurrence relation

$$W_n = a_1 W_{n-1} + a_2 W_{n-2} + a_3 W_{n-3} + c_0. \tag{2.5}$$

Then the solution of (2.5) is given by

$$\begin{aligned} W_n(W_0, W_1, W_2) &= W_n^{(h)} + W_n^{(p)} \\ &= V_n(W_0, W_1, W_2) - V_n(W_0^{(p)}, W_1^{(p)}, W_2^{(p)}) + W_n^{(p)} \end{aligned}$$

where $W_n^{(h)} = V_n(W_0, W_1, W_2) - V_n(W_0^{(p)}, W_1^{(p)}, W_2^{(p)})$ is the solution of (1.2) and

$$W_n^{(p)} = nA_0$$

is the particular solution of (2.5). Here,

$$A_0 = \frac{c_0}{a_1 + 2a_2 + 3a_3}$$

and

$$W_0^{(p)} = 0, \ W_1^{(p)} = A_0, \ W_2^{(p)} = 2A_0,$$

$$\begin{aligned} V_n(W_0^{(p)}, W_1^{(p)}, W_2^{(p)}) &= V_n(0, A_0, 2A_0) \\ &= B_1 V_n(W_0, W_1, W_2) + B_2 V_{n-1}(W_0, W_1, W_2) + B_3 V_{n-2}(W_0, W_1, W_2) \end{aligned}$$

where

$$B_1 = \frac{Y_1}{\Delta}, \ B_2 = \frac{Y_2}{\Delta}, \ B_3 = \frac{Y_3}{\Delta}$$

and

$Y_1 = (W_2^2 + a_1^2 W_1^2 + a_1 a_3 W_0^2 - 2a_1 W_1 W_2 - a_2 W_0 W_2 + (a_1 a_2 - a_3) W_0 W_1) \times 2A_0 + ((a_3 + a_1 a_2) W_1^2 + a_2 a_3 W_0^2 - a_2 W_1 W_2 - a_3 W_0 W_2 + a_2^2 W_0 W_1) A_0$

$Y_2 = ((a_3 + a_1 a_2) W_1^2 - a_2 W_2 W_1 + (a_2^2 - 2a_3 a_1) W_0 W_1 + a_3 a_2 W_0^2 + 2a_3 a_1 W_0 W_1 - a_3 W_0 W_2) \times 2A_0 + (a_2 W_2^2 + 2a_3 W_1 W_2 + a_3^2 W_0^2 - (3a_3 + a_1 a_2) W_1 W_2 - (a_2^2 - 2a_3 a_1) W_0 W_2 - a_3 a_1 W_0 W_2) A_0$

$Y_3 = a_3(-W_1 W_2 + a_1 W_1^2 + a_2 W_0 W_1 + a_3 W_0^2) \times 2A_0 + a_3(W_2^2 - a_1 W_1 W_2 - a_2 W_0 W_2 - a_3 W_0 W_1) A_0$

$\Delta = W_2^3 + (a_3 + a_1 a_2) W_1^3 + a_3^2 W_0^3 - 2a_1 W_1 W_2^2 + (a_1^2 - a_2) W_1^2 W_2 - a_2 W_0 W_2^2 + a_3 a_1 W_0^2 W_2 + (a_2^2 + a_1 a_3) W_0 W_1^2 + 2a_2 a_3 W_0^2 W_1 + (-3a_3 + a_1 a_2) W_0 W_1 W_2$

In summary, the solution of (2.5) is given by

$$W_n(W_0, W_1, W_2) = (-B_1 + 1) V_n(W_0, W_1, W_2) - B_2 V_{n-1}(W_0, W_1, W_2) - B_3 V_{n-2}(W_0, W_1, W_2) + nA_0$$

(b): *The case* $s=1:$ *(*$a_3\neq 0,\ c_1\neq 0$*).*

Let the sequence $\{W_n\}$ *defined by nonhomogeneous linear recurrence relation*

$$W_n = a_1W_{n-1}+a_2W_{n-2}+a_3W_{n-3}+c_1n+c_0. \tag{2.6}$$

Then the solution of (2.6) is given by

$$\begin{aligned} W_n(W_0,W_1,W_2) &= W_n^{(h)}+W_n^{(p)} \\ &= V_n(W_0,W_1,W_2)-V_n(W_0^{(p)},W_1^{(p)},W_2^{(p)})+W_n^{(p)} \end{aligned}$$

where $W_n^{(h)}=V_n(W_0,W_1,W_2)-V_n(W_0^{(p)},W_1^{(p)},W_2^{(p)})$ *is the solution of (1.2) and*

$$W_n^{(p)} = n(A_1n+A_0)=A_1n^2+A_0n$$

is the particular solution of (2.6). Here,

$$\begin{aligned} A_1 &= \frac{c_1}{2\left(a_1+2a_2+3a_3\right)}, \\ A_0 &= \frac{1}{\left(a_1+2a_2+3a_3\right)}(c_0+\left(a_1+4a_2+9a_3\right)A_1), \end{aligned}$$

i.e.,

$$\begin{aligned} A_1 &= \frac{c_1}{2\left(a_1+2a_2+3a_3\right)}, \\ A_0 &= \frac{1}{2\left(a_1+2a_2+3a_3\right)^2}(a_1(2c_0+c_1)+4a_2(c_0+c_1)+3a_3(2c_0+3c_1)), \end{aligned}$$

and

$$W_0^{(p)}=0,\ W_1^{(p)}=A_1+A_0,\ W_2^{(p)}=4A_1+2A_0,$$

and

$$\begin{aligned} V_n(W_0^{(p)},W_1^{(p)},W_2^{(p)}) &= V_n(0,A_1+A_0,4A_1+2A_0) \\ &= B_1V_n(W_0,W_1,W_2)+B_2V_{n-1}(W_0,W_1,W_2)+B_3V_{n-2}(W_0,W_1,W_2) \end{aligned}$$

where

$$B_1=\frac{Y_1}{\Delta},\ B_2=\frac{Y_2}{\Delta},\ B_3=\frac{Y_3}{\Delta}$$

and

$Y_1=(W_2^2+a_1^2W_1^2+a_1a_3W_0^2-2a_1W_1W_2-a_2W_0W_2+(a_1a_2-a_3)W_0W_1)(4A_1+2A_0)+((a_3+a_1a_2)W_1^2+a_2a_3W_0^2-a_2W_1W_2-a_3W_0W_2+a_2^2W_0W_1)(A_1+A_0)$

$Y_2=((a_3+a_1a_2)W_1^2-a_2W_2W_1+(a_2^2-2a_3a_1)W_0W_1+a_3a_2W_0^2+2a_3a_1W_0W_1-a_3W_0W_2)(4A_1+2A_0)+(a_2W_2^2+2a_3W_1W_2+a_3^2W_0^2-(3a_3+a_1a_2)W_1W_2-(a_2^2-2a_3a_1)W_0W_2-a_3a_1W_0W_2)(A_1+A_0)$

$Y_3=a_3(-W_1W_2+a_1W_1^2+a_2W_0W_1+a_3W_0^2)(4A_1+2A_0)+a_3(W_2^2-a_1W_1W_2-a_2W_0W_2-a_3W_0W_1)(A_1+A_0)$

$$\Delta = W_2^3 + (a_3 + a_1a_2)W_1^3 + a_3^2W_0^3 - 2a_1W_1W_2^2 + (a_1^2 - a_2)W_1^2W_2 - a_2W_0W_2^2 + a_3a_1W_0^2W_2 + (a_2^2 + a_1a_3)W_0W_1^2 + 2a_2a_3W_0^2W_1 + (-3a_3 + a_1a_2)W_0W_1W_2$$

In summary, the solution of (2.6) is given by

$$W_n(W_0, W_1, W_2) = (-B_1 + 1)V_n(W_0, W_1, W_2) - B_2V_{n-1}(W_0, W_1, W_2) - B_3V_{n-2}(W_0, W_1, W_2) + A_1n^2 + A_0n$$

(c): *The case $s = 2 :$ ($a_3 \neq 0$, $c_2 \neq 0$).*

Let the sequence $\{W_n\}$ defined by nonhomogeneous linear recurrence relation

$$W_n = a_1W_{n-1} + a_2W_{n-2} + a_3W_{n-3} + c_2n^2 + c_1n + c_0. \tag{2.7}$$

Then the solution of (2.7) is given by

$$\begin{aligned} W_n(W_0, W_1, W_2) &= W_n^{(h)} + W_n^{(p)} \\ &= V_n(W_0, W_1, W_2) - V_n(W_0^{(p)}, W_1^{(p)}, W_2^{(p)}) + W_n^{(p)} \end{aligned}$$

where $W_n^{(h)} = V_n(W_0, W_1, W_2) - V_n(W_0^{(p)}, W_1^{(p)}, W_2^{(p)})$ is the solution of (1.2) and

$$W_n^{(p)} = n(A_2n^2 + A_1n + A_0) = A_2n^3 + A_1n^2 + A_0n$$

is the particular solution of (2.7). Here,

$$\begin{aligned} A_2 &= \frac{c_2}{3\,(a_1 + 2a_2 + 3a_3)}, \\ A_1 &= \frac{1}{2\,(a_1 + 2a_2 + 3a_3)}(c_1 + 3\,(a_1 + 4a_2 + 9a_3)\,A_2), \\ A_0 &= \frac{1}{(a_1 + 2a_2 + 3a_3)}(c_0 + (a_1 + 4a_2 + 9a_3)\,A_1 - (a_1 + 8a_2 + 27a_3)\,A_2), \end{aligned}$$

i.e.,

$$A_2 = \frac{c_2}{3\,(a_1 + 2a_2 + 3a_3)},$$

$$A_1 = \frac{1}{2\,(a_1 + 2a_2 + 3a_3)^2}(a_1(c_1 + c_2) + 2a_2(c_1 + 2c_2) + 3a_3(c_1 + 3c_2)),$$

$$A_0 = \frac{1}{6\,(a_1 + 2a_2 + 3a_3)^3}(a_1^2(6c_0 + 3c_1 + c_2) + 8a_2^2(3c_0 + 3c_1 + 2c_2) + 27a_3^2(2c_0 + 3c_1 + 3c_2) + 2a_1a_2(12c_0 + 9c_1 + 2c_2) + 6a_1a_3(6c_0 + 6c_1 - c_2) + 6a_2a_3(12c_0 + 15c_1 + 10c_2)),$$

and

$$W_0^{(p)} = 0,\ W_1^{(p)} = A_2 + A_1 + A_0,\ W_2^{(p)} = 8A_2 + 4A_1 + 2A_0,$$

and

$$\begin{aligned} V_n(W_0^{(p)}, W_1^{(p)}, W_2^{(p)}) &= V_n(0, A_2 + A_1 + A_0, 8A_2 + 4A_1 + 2A_0) \\ &= B_1V_n(W_0, W_1, W_2) + B_2V_{n-1}(W_0, W_1, W_2) + B_3V_{n-2}(W_0, W_1, W_2) \end{aligned}$$

where

$$B_1 = \frac{Y_1}{\Delta},\ B_2 = \frac{Y_2}{\Delta},\ B_3 = \frac{Y_3}{\Delta}$$

and

$Y_1 = (W_2^2 + a_1^2 W_1^2 + a_1 a_3 W_0^2 - 2a_1 W_1 W_2 - a_2 W_0 W_2 + (a_1 a_2 - a_3) W_0 W_1)(8A_2 + 4A_1 + 2A_0) + ((a_3 + a_1 a_2) W_1^2 + a_2 a_3 W_0^2 - a_2 W_1 W_2 - a_3 W_0 W_2 + a_2^2 W_0 W_1)(A_2 + A_1 + A_0)$

$Y_2 = ((a_3 + a_1 a_2) W_1^2 - a_2 W_2 W_1 + (a_2^2 - 2a_3 a_1) W_0 W_1 + a_3 a_2 W_0^2 + 2a_3 a_1 W_0 W_1 - a_3 W_0 W_2)(8A_2 + 4A_1 + 2A_0) + (a_2 W_2^2 + 2a_3 W_1 W_2 + a_3^2 W_0^2 - (3a_3 + a_1 a_2) W_1 W_2 - (a_2^2 - 2a_3 a_1) W_0 W_2 - a_3 a_1 W_0 W_2)(A_2 + A_1 + A_0)$

$Y_3 = a_3(-W_1 W_2 + a_1 W_1^2 + a_2 W_0 W_1 + a_3 W_0^2)(8A_2 + 4A_1 + 2A_0) + a_3(W_2^2 - a_1 W_1 W_2 - a_2 W_0 W_2 - a_3 W_0 W_1)(A_2 + A_1 + A_0)$

$\Delta = W_2^3 + (a_3 + a_1 a_2) W_1^3 + a_3^2 W_0^3 - 2a_1 W_1 W_2^2 + (a_1^2 - a_2) W_1^2 W_2 - a_2 W_0 W_2^2 + a_3 a_1 W_0^2 W_2 + (a_2^2 + a_1 a_3) W_0 W_1^2 + 2a_2 a_3 W_0^2 W_1 + (-3a_3 + a_1 a_2) W_0 W_1 W_2$

In summary, the solution of (2.7) is given by

$$W_n(W_0, W_1, W_2) = (-B_1 + 1) V_n(W_0, W_1, W_2) - B_2 V_{n-1}(W_0, W_1, W_2) - B_3 V_{n-2}(W_0, W_1, W_2) + A_2 n^3 + A_1 n^2 + A_0 n$$

(d): *The case* $s = 3 : (a_3 \neq 0,\ c_3 \neq 0)$.

Let the sequence $\{W_n\}$ *defined by nonhomogeneous linear recurrence relation*

$$W_n = a_1 W_{n-1} + a_2 W_{n-2} + a_3 W_{n-3} + c_3 n^3 + c_2 n^2 + c_1 n + c_0. \tag{2.8}$$

Then the solution of (2.8) is given by

$$\begin{aligned} W_n(W_0, W_1, W_2) &= W_n^{(h)} + W_n^{(p)} \\ &= V_n(W_0, W_1, W_2) - V_n(W_0^{(p)}, W_1^{(p)}, W_2^{(p)}) + W_n^{(p)} \end{aligned}$$

where $W_n^{(h)} = V_n(W_0, W_1, W_2) - V_n(W_0^{(p)}, W_1^{(p)}, W_2^{(p)})$ *is the solution of (1.2) and*

$$W_n^{(p)} = n(A_3 n^3 + A_2 n^2 + A_1 n + A_0) = A_3 n^4 + A_2 n^3 + A_1 n^2 + A_0 n$$

is the particular solution of (2.8). Here,

$$\begin{aligned} A_3 &= \frac{c_3}{4(a_1 + 2a_2 + 3a_3)}, \\ A_2 &= \frac{1}{3(a_1 + 2a_2 + 3a_3)}(c_2 + 6(a_1 + 4a_2 + 9a_3) A_3), \\ A_1 &= \frac{1}{2(a_1 + 2a_2 + 3a_3)}(c_1 + 3(a_1 + 4a_2 + 9a_3) A_2 - 4(a_1 + 8a_2 + 27a_3) A_3), \\ A_0 &= \frac{1}{(a_1 + 2a_2 + 3a_3)}(c_0 + (a_1 + 4a_2 + 9a_3) A_1 - (a_1 + 8a_2 + 27a_3) A_2 + (a_1 + 16a_2 + 81a_3) A_3), \end{aligned}$$

i.e.,

$A_3 = \dfrac{c_3}{4(a_1 + 2a_2 + 3a_3)},$

$A_2 = \dfrac{1}{6(a_1 + 2a_2 + 3a_3)^2}(a_1(2c_2 + 3c_3) + 4a_2(c_2 + 3c_3) + 3a_3(2c_2 + 9c_3)),$

$A_1 = \dfrac{1}{4(a_1 + 2a_2 + 3a_3)^3}(a_1^2(2c_1 + 2c_2 + c_3) + 8a_2^2(c_1 + 2c_2 + 2c_3) + 9a_3^2(2c_1 + 6c_2 + 9c_3) + 4a_1 a_2(2c_1 + 3c_2 + c_3) + 6a_1 a_3(2c_1 + 4c_2 - c_3) + 12a_2 a_3(2c_1 + 5c_2 + 5c_3)),$

$$A_0 = \frac{1}{6\left(a_1+2a_2+3a_3\right)^4}(a_1^3\left(6c_0+3c_1+c_2\right)+16a_2^3\left(3c_0+3c_1+2c_2\right)+81a_3^3\left(2c_0+3c_1+3c_2\right)+$$
$$6a_1a_2^2\left(12c_0+10c_1+4c_2-3c_3\right)+9a_1a_3^2\left(18c_0+21c_1+7c_2-30c_3\right)+6a_1^2a_2\left(6c_0+4c_1+c_2\right)+3a_1^2a_3(18c_0+$$
$$15c_1-c_2+6c_3)+18a_2a_3^2\left(18c_0+24c_1+19c_2-6c_3\right)+6a_2^2a_3\left(36c_0+42c_1+28c_2-3c_3\right)+12a_1a_2a_3(18c_0+$$
$$18c_1+5c_2-9c_3)),$$

and

$$W_0^{(p)}=0,\ W_1^{(p)}=A_3+A_2+A_1+A_0,\ W_2^{(p)}=16A_3+8A_2+4A_1+2A_0,$$

and

$$\begin{aligned} V_n(W_0^{(p)},W_1^{(p)},W_2^{(p)}) &= V_n(0,A_3+A_2+A_1+A_0,16A_3+8A_2+4A_1+2A_0) \\ &= B_1V_n(W_0,W_1,W_2)+B_2V_{n-1}(W_0,W_1,W_2)+B_3V_{n-2}(W_0,W_1,W_2) \end{aligned}$$

where

$$B_1=\frac{Y_1}{\Delta},\ B_2=\frac{Y_2}{\Delta},\ B_3=\frac{Y_3}{\Delta}$$

and

$$Y_1 = (W_2^2+a_1^2W_1^2+a_1a_3W_0^2-2a_1W_1W_2-a_2W_0W_2+(a_1a_2-a_3)W_0W_1)(16A_3+8A_2+4A_1+$$
$$2A_0)+((a_3+a_1a_2)W_1^2+a_2a_3W_0^2-a_2W_1W_2-a_3W_0W_2+a_2^2W_0W_1)(A_3+A_2+A_1+A_0)$$

$$Y_2 = ((a_3+a_1a_2)W_1^2-a_2W_2W_1+(a_2^2-2a_3a_1)W_0W_1+a_3a_2W_0^2+2a_3a_1W_0W_1-a_3W_0W_2)$$
$$(16A_3+8A_2+4A_1+2A_0)+(a_2W_2^2+2a_3W_1W_2+a_3^2W_0^2-(3a_3+a_1a_2)W_1W_2-(a_2^2-2a_3a_1)$$
$$W_0W_2-a_3a_1W_0W_2)(A_3+A_2+A_1+A_0)$$

$$Y_3 = a_3(-W_1W_2+a_1W_1^2+a_2W_0W_1+a_3W_0^2)(16A_3+8A_2+4A_1+2A_0)+a_3(W_2^2-a_1W_1W_2-a_2$$
$$W_0W_2-a_3W_0W_1)(A_3+A_2+A_1+A_0)$$

$$\Delta = W_2^3+(a_3+a_1a_2)W_1^3+a_3^2W_0^3-2a_1W_1W_2^2+(a_1^2-a_2)W_1^2W_2-a_2W_0W_2^2+a_3a_1W_0^2W_2+$$
$$(a_2^2+a_1a_3)W_0W_1^2+2a_2a_3W_0^2W_1+(-3a_3+a_1a_2)W_0W_1W_2$$

In summary, the solution of (2.8) is given by

$$W_n(W_0,W_1,W_2)=(-B_1+1)V_n(W_0,W_1,W_2)-B_2V_{n-1}(W_0,W_1,W_2)-B_3V_{n-2}(W_0,W_1,W_2)+A_3n^4+A_2n^3+A_1n^2+A_0n$$

2.3. The Case: $\theta_1=1$ and $\theta_2=1$ and $\theta_3\neq 1$, i.e., Only Two of the Roots of the Characteristic Equation of (1.2) is 1. In the following example, we assume that only two of the roots of the characteristic equation of (1.2) is equal to 1 and we consider special cases of $p(n)$ defined in (1.1) by using Theorem 1.1.

Example 2.3. *In this example, in each case, the homogeneous relation corresponding to the given non-homogeneous recurrence relation is expressed as in (1.2), i.e.,*

$$V_n = a_1V_{n-1}+a_2V_{n-2}+a_3V_{n-3}.$$

(a): *The case $s=0$: ($a_3\neq 0$, $c_0\neq 0$).*

Let the sequence $\{W_n\}$ defined by nonhomogeneous linear recurrence relation

$$W_n = a_1W_{n-1} + a_2W_{n-2} + a_3W_{n-3} + c_0. \tag{2.9}$$

Then the solution of (2.9) is given by

$$\begin{aligned} W_n(W_0, W_1, W_2) &= W_n^{(h)} + W_n^{(p)} \\ &= V_n(W_0, W_1, W_2) - V_n(W_0^{(p)}, W_1^{(p)}, W_2^{(p)}) + W_n^{(p)} \end{aligned}$$

where $W_n^{(h)} = V_n(W_0, W_1, W_2) - V_n(W_0^{(p)}, W_1^{(p)}, W_2^{(p)})$ *is the solution of (1.2) and*

$$W_n^{(p)} = n^2 A_0$$

is the particular solution of (2.9). Here,

$$A_0 = -\frac{c_0}{a_1 + 4a_2 + 9a_3}$$

and

$$W_0^{(p)} = 0,\ W_1^{(p)} = A_0,\ W_2^{(p)} = 4A_0,$$

$$\begin{aligned} V_n(W_0^{(p)}, W_1^{(p)}, W_2^{(p)}) &= V_n(0, A_0, 4A_0) \\ &= B_1V_n(W_0, W_1, W_2) + B_2V_{n-1}(W_0, W_1, W_2) + B_3V_{n-2}(W_0, W_1, W_2) \end{aligned}$$

where

$$B_1 = \frac{Y_1}{\Delta},\ B_2 = \frac{Y_2}{\Delta},\ B_3 = \frac{Y_3}{\Delta}$$

and

$Y_1 = (W_2^2 + a_1^2W_1^2 + a_1a_3W_0^2 - 2a_1W_1W_2 - a_2W_0W_2 + (a_1a_2 - a_3)W_0W_1) \times 4A_0 + ((a_3 + a_1a_2)W_1^2 + a_2a_3W_0^2 - a_2W_1W_2 - a_3W_0W_2 + a_2^2W_0W_1)A_0$

$Y_2 = ((a_3 + a_1a_2)W_1^2 - a_2W_2W_1 + (a_2^2 - 2a_3a_1)W_0W_1 + a_3a_2W_0^2 + 2a_3a_1W_0W_1 - a_3W_0W_2) \times 4A_0 + (a_2W_2^2 + 2a_3W_1W_2 + a_3^2W_0^2 - (3a_3 + a_1a_2)W_1W_2 - (a_2^2 - 2a_3a_1)W_0W_2 - a_3a_1W_0W_2)A_0$

$Y_3 = a_3(-W_1W_2 + a_1W_1^2 + a_2W_0W_1 + a_3W_0^2) \times 4A_0 + a_3(W_2^2 - a_1W_1W_2 - a_2W_0W_2 - a_3W_0W_1)A_0$

$\Delta = W_2^3 + (a_3 + a_1a_2)W_1^3 + a_3^2W_0^3 - 2a_1W_1W_2^2 + (a_1^2 - a_2)W_1^2W_2 - a_2W_0W_2^2 + a_3a_1W_0^2W_2 + (a_2^2 + a_1a_3)W_0W_1^2 + 2a_2a_3W_0^2W_1 + (-3a_3 + a_1a_2)W_0W_1W_2$

In summary, the solution of (2.9) is given by

$$W_n(W_0, W_1, W_2) = (-B_1 + 1)V_n(W_0, W_1, W_2) - B_2V_{n-1}(W_0, W_1, W_2) - B_3V_{n-2}(W_0, W_1, W_2) + n^2A_0$$

(b): *The case $s = 1 :$ ($a_3 \neq 0,\ c_1 \neq 0$).*

Let the sequence $\{W_n\}$ defined by nonhomogeneous linear recurrence relation

$$W_n = a_1W_{n-1} + a_2W_{n-2} + a_3W_{n-3} + c_1n + c_0. \tag{2.10}$$

Then the solution of (2.10) is given by

$$\begin{aligned} W_n(W_0, W_1, W_2) &= W_n^{(h)} + W_n^{(p)} \\ &= V_n(W_0, W_1, W_2) - V_n(W_0^{(p)}, W_1^{(p)}, W_2^{(p)}) + W_n^{(p)} \end{aligned}$$

where $W_n^{(h)} = V_n(W_0, W_1, W_2) - V_n(W_0^{(p)}, W_1^{(p)}, W_2^{(p)})$ *is the solution of (1.2) and*

$$W_n^{(p)} = n^2(A_1 n + A_0) = A_1 n^3 + A_0 n^2$$

is the particular solution of (2.10). Here,

$$\begin{aligned} A_1 &= -\frac{c_1}{3\left(a_1 + 4a_2 + 9a_3\right)}, \\ A_0 &= -\frac{1}{\left(a_1 + 4a_2 + 9a_3\right)}(c_0 - \left(a_1 + 8a_2 + 27a_3\right) A_1), \end{aligned}$$

i.e.,

$$\begin{aligned} A_1 &= -\frac{c_1}{3\left(a_1 + 4a_2 + 9a_3\right)}, \\ A_0 &= -\frac{1}{3\left(a_1 + 4a_2 + 9a_3\right)^2}(a_1(3c_0 + c_1) + 4a_2(3c_0 + 2c_1) + 27a_3(c_0 + c_1)), \end{aligned}$$

and

$$W_0^{(p)} = 0,\ W_1^{(p)} = A_1 + A_0,\ W_2^{(p)} = 8A_1 + 4A_0,$$

and

$$\begin{aligned} V_n(W_0^{(p)}, W_1^{(p)}, W_2^{(p)}) &= V_n(0, A_1 + A_0, 8A_1 + 4A_0) \\ &= B_1 V_n(W_0, W_1, W_2) + B_2 V_{n-1}(W_0, W_1, W_2) + B_3 V_{n-2}(W_0, W_1, W_2) \end{aligned}$$

where

$$B_1 = \frac{Y_1}{\Delta},\ B_2 = \frac{Y_2}{\Delta},\ B_3 = \frac{Y_3}{\Delta}$$

and

$Y_1 = (W_2^2 + a_1^2 W_1^2 + a_1 a_3 W_0^2 - 2a_1 W_1 W_2 - a_2 W_0 W_2 + (a_1 a_2 - a_3) W_0 W_1)(8A_1 + 4A_0) + ((a_3 + a_1 a_2) W_1^2 + a_2 a_3 W_0^2 - a_2 W_1 W_2 - a_3 W_0 W_2 + a_2^2 W_0 W_1)(A_1 + A_0)$

$Y_2 = ((a_3 + a_1 a_2) W_1^2 - a_2 W_2 W_1 + (a_2^2 - 2a_3 a_1) W_0 W_1 + a_3 a_2 W_0^2 + 2a_3 a_1 W_0 W_1 - a_3 W_0 W_2)(8A_1 + 4A_0) + (a_2 W_2^2 + 2a_3 W_1 W_2 + a_3^2 W_0^2 - (3a_3 + a_1 a_2) W_1 W_2 - (a_2^2 - 2a_3 a_1) W_0 W_2 - a_3 a_1 W_0 W_2)(A_1 + A_0)$

$Y_3 = a_3(-W_1 W_2 + a_1 W_1^2 + a_2 W_0 W_1 + a_3 W_0^2)(8A_1 + 4A_0) + a_3(W_2^2 - a_1 W_1 W_2 - a_2 W_0 W_2 - a_3 W_0 W_1)(A_1 + A_0)$

$\Delta = W_2^3 + (a_3 + a_1 a_2) W_1^3 + a_3^2 W_0^3 - 2a_1 W_1 W_2^2 + (a_1^2 - a_2) W_1^2 W_2 - a_2 W_0 W_2^2 + a_3 a_1 W_0^2 W_2 + (a_2^2 + a_1 a_3) W_0 W_1^2 + 2a_2 a_3 W_0^2 W_1 + (-3a_3 + a_1 a_2) W_0 W_1 W_2$

In summary, the solution of (2.10) is given by

$$W_n(W_0, W_1, W_2) = (-B_1 + 1) V_n(W_0, W_1, W_2) - B_2 V_{n-1}(W_0, W_1, W_2) - B_3 V_{n-2}(W_0, W_1, W_2) + A_1 n^3 + A_0 n^2$$

(c): *The case* $s=2:$ *(*$a_3\neq 0,\ c_2\neq 0$*).*

Let the sequence $\{W_n\}$ *defined by nonhomogeneous linear recurrence relation*

$$W_n = a_1W_{n-1} + a_2W_{n-2} + a_3W_{n-3} + c_2n^2 + c_1n + c_0. \tag{2.11}$$

Then the solution of (2.11) is given by

$$\begin{aligned} W_n(W_0,W_1,W_2) &= W_n^{(h)} + W_n^{(p)} \\ &= V_n(W_0,W_1,W_2) - V_n(W_0^{(p)},W_1^{(p)},W_2^{(p)}) + W_n^{(p)} \end{aligned}$$

where $W_n^{(h)} = V_n(W_0,W_1,W_2) - V_n(W_0^{(p)},W_1^{(p)},W_2^{(p)})$ *is the solution of (1.2) and*

$$W_n^{(p)} = n^2(A_2n^2 + A_1n + A_0) = A_2n^4 + A_1n^3 + A_0n^2$$

is the particular solution of (2.11). Here,

$$\begin{aligned} A_2 &= -\frac{c_2}{6\left(a_1+4a_2+9a_3\right)}, \\ A_1 &= -\frac{1}{3\left(a_1+4a_2+9a_3\right)}(c_1 - 4\left(a_1+8a_2+27a_3\right)A_2), \\ A_0 &= -\frac{1}{\left(a_1+4a_2+9a_3\right)}(c_0 - \left(a_1+8a_2+27a_3\right)A_1 + \left(a_1+16a_2+81a_3\right)A_2), \end{aligned}$$

i.e.,

$$A_2 = -\frac{c_2}{6\left(a_1+4a_2+9a_3\right)},$$
$$A_1 = -\frac{1}{9\left(a_1+4a_2+9a_3\right)^2}(a_1(3c_1+2c_2) + 4a_2(3c_1+4c_2) + 27a_3(c_1+2c_2)),$$
$$A_0 = -\frac{1}{18\left(a_1+4a_2+9a_3\right)^3}(a_1^2(18c_0+6c_1+c_2) + 32a_2^2(9c_0+6c_1+2c_2) + 729a_3^2(2c_0+2c_1+c_2) +$$
$$4a_1a_2(36c_0+18c_1+c_2) + 54a_1a_3(6c_0+4c_1-c_2) + 108a_2a_3(12c_0+10c_1+3c_2)),$$

and

$$W_0^{(p)} = 0,\ W_1^{(p)} = A_2 + A_1 + A_0,\ W_2^{(p)} = 16A_2 + 8A_1 + 4A_0,$$

and

$$\begin{aligned} V_n(W_0^{(p)},W_1^{(p)},W_2^{(p)}) &= V_n(0, A_2+A_1+A_0, 16A_2+8A_1+4A_0) \\ &= B_1V_n(W_0,W_1,W_2) + B_2V_{n-1}(W_0,W_1,W_2) + B_3V_{n-2}(W_0,W_1,W_2) \end{aligned}$$

where

$$B_1 = \frac{Y_1}{\Delta},\ B_2 = \frac{Y_2}{\Delta},\ B_3 = \frac{Y_3}{\Delta}$$

and

$$Y_1 = (W_2^2 + a_1^2W_1^2 + a_1a_3W_0^2 - 2a_1W_1W_2 - a_2W_0W_2 + (a_1a_2 - a_3)W_0W_1)(16A_2 + 8A_1 + 4A_0) +$$
$$((a_3 + a_1a_2)W_1^2 + a_2a_3W_0^2 - a_2W_1W_2 - a_3W_0W_2 + a_2^2W_0W_1)(A_2 + A_1 + A_0)$$

$Y_2 = ((a_3+a_1a_2)W_1^2 - a_2W_2W_1 + (a_2^2 - 2a_3a_1)W_0W_1 + a_3a_2W_0^2 + 2a_3a_1W_0W_1 - a_3W_0W_2)(16A_2 + 8A_1 + 4A_0) + (a_2W_2^2 + 2a_3W_1W_2 + a_3^2W_0^2 - (3a_3 + a_1a_2)W_1W_2 - (a_2^2 - 2a_3a_1)W_0W_2 - a_3a_1W_0W_2)(A_2 + A_1 + A_0)$

$Y_3 = a_3(-W_1W_2 + a_1W_1^2 + a_2W_0W_1 + a_3W_0^2)(16A_2 + 8A_1 + 4A_0) + a_3(W_2^2 - a_1W_1W_2 - a_2W_0W_2 - a_3W_0W_1)(A_2 + A_1 + A_0)$

$\Delta = W_2^3 + (a_3 + a_1a_2)W_1^3 + a_3^2W_0^3 - 2a_1W_1W_2^2 + (a_1^2 - a_2)W_1^2W_2 - a_2W_0W_2^2 + a_3a_1W_0^2W_2 + (a_2^2 + a_1a_3)W_0W_1^2 + 2a_2a_3W_0^2W_1 + (-3a_3 + a_1a_2)W_0W_1W_2$

In summary, the solution of (2.11) is given by

$$W_n(W_0, W_1, W_2) = (-B_1+1)V_n(W_0, W_1, W_2) - B_2V_{n-1}(W_0, W_1, W_2) - B_3V_{n-2}(W_0, W_1, W_2) + A_2n^4 + A_1n^3 + A_0n^2$$

(d): *The case $s = 3 :$ ($a_3 \neq 0$, $c_3 \neq 0$).*

Let the sequence $\{W_n\}$ defined by nonhomogeneous linear recurrence relation

$$W_n = a_1W_{n-1} + a_2W_{n-2} + a_3W_{n-3} + c_3n^3 + c_2n^2 + c_1n + c_0. \tag{2.12}$$

Then the solution of (2.12) is given by

$$\begin{aligned} W_n(W_0, W_1, W_2) &= W_n^{(h)} + W_n^{(p)} \\ &= V_n(W_0, W_1, W_2) - V_n(W_0^{(p)}, W_1^{(p)}, W_2^{(p)}) + W_n^{(p)} \end{aligned}$$

where $W_n^{(h)} = V_n(W_0, W_1, W_2) - V_n(W_0^{(p)}, W_1^{(p)}, W_2^{(p)})$ is the solution of (1.2) and

$$W_n^{(p)} = n^2(A_3n^3 + A_2n^2 + A_1n + A_0) = A_3n^5 + A_2n^4 + A_1n^3 + A_0n^2$$

is the particular solution of (2.12). Here,

$$\begin{aligned} A_3 &= -\frac{c_3}{10\,(a_1 + 4a_2 + 9a_3)}, \\ A_2 &= -\frac{1}{6\,(a_1 + 4a_2 + 9a_3)}(c_2 - 10\,(a_1 + 8a_2 + 27a_3)\,A_3), \\ A_1 &= -\frac{1}{3\,(a_1 + 4a_2 + 9a_3)}(c_1 - 4\,(a_1 + 8a_2 + 27a_3)\,A_2 + 5\,(a_1 + 16a_2 + 81a_3)\,A_3), \\ A_0 &= -\frac{1}{(a_1 + 4a_2 + 9a_3)}(c_0 - (a_1 + 8a_2 + 27a_3)\,A_1 + (a_1 + 16a_2 + 81a_3)\,A_2 - (a_1 + 32a_2 + 243a_3)\,A_3), \end{aligned}$$

i.e.,

$A_3 = -\dfrac{c_3}{10\,(a_1 + 4a_2 + 9a_3)},$

$A_2 = -\dfrac{1}{6\,(a_1 + 4a_2 + 9a_3)^2}(a_1(c_2 + c_3) + 4a_2(c_2 + 2c_3) + 9a_3(c_2 + 3c_3)),$

$A_1 = -\frac{1}{18(a_1+4a_2+9a_3)^3}(a_1^2(6c_1 + 4c_2 + c_3) + 32a_2^2(3c_1 + 4c_2 + 2c_3) + 243a_3^2(2c_1 + 4c_2 + 3c_3) + 4a_1a_2(12c_1 + 12c_2 + c_3) + 18a_1a_3(6c_1 + 8c_2 - 3c_3) + 36a_2a_3(12c_1 + 20c_2 + 9c_3)),$

$A_0 = -\dfrac{1}{90\,(a_1 + 4a_2 + 9a_3)^4}(a_1^3(90c_0 + 30c_1 + 5c_2 - c_3) + 128a_2^3(45c_0 + 30c_1 + 10c_2 - 4c_3) + 6561a_3^3(10c_0 + 10c_1 + 5c_2 - 3c_3) + 16a_1a_2^2(270c_0 + 150c_1 + 25c_2 - 27c_3) + 5a_1^2(216a_2c_0 + 96a_2c_1 +$

$8a_2c_2 - 45a_3c_2) + 1215a_1a_3^2(18c_0 + 14c_1 + c_2 - 9c_3) + 27a_1^2a_3(90c_0 + 50c_1 + 17c_3) + 1944a_2a_3^2(45c_0 + 40c_1 + 15c_2 - 12c_3) + 144a_2^2a_3(270c_0 + 210c_1 + 65c_2 - 33c_3) + 144a_1a_2a_3(135c_0 + 90c_1 + 5c_2 - 18c_3)),$

and

$$W_0^{(p)} = 0,\ W_1^{(p)} = A_3 + A_2 + A_1 + A_0,\ W_2^{(p)} = 32A_3 + 16A_2 + 8A_1 + 4A_0,$$

and

$$\begin{aligned} V_n(W_0^{(p)}, W_1^{(p)}, W_2^{(p)}) &= V_n(0, A_3 + A_2 + A_1 + A_0, 32A_3 + 16A_2 + 8A_1 + 4A_0) \\ &= B_1V_n(W_0, W_1, W_2) + B_2V_{n-1}(W_0, W_1, W_2) + B_3V_{n-2}(W_0, W_1, W_2) \end{aligned}$$

where

$$B_1 = \frac{Y_1}{\Delta},\ B_2 = \frac{Y_2}{\Delta},\ B_3 = \frac{Y_3}{\Delta}$$

and

$Y_1 = (W_2^2 + a_1^2W_1^2 + a_1a_3W_0^2 - 2a_1W_1W_2 - a_2W_0W_2 + (a_1a_2 - a_3)W_0W_1)(32A_3 + 16A_2 + 8A_1 + 4A_0) + ((a_3 + a_1a_2)W_1^2 + a_2a_3W_0^2 - a_2W_1W_2 - a_3W_0W_2 + a_2^2W_0W_1)(A_3 + A_2 + A_1 + A_0)$

$Y_2 = ((a_3 + a_1a_2)W_1^2 - a_2W_2W_1 + (a_2^2 - 2a_3a_1)W_0W_1 + a_3a_2W_0^2 + 2a_3a_1W_0W_1 - a_3W_0W_2)(32A_3 + 16A_2 + 8A_1 + 4A_0) + (a_2W_2^2 + 2a_3W_1W_2 + a_3^2W_0^2 - (3a_3 + a_1a_2)W_1W_2 - (a_2^2 - 2a_3a_1)W_0W_2 - a_3a_1W_0W_2)(A_3 + A_2 + A_1 + A_0)$

$Y_3 = a_3(-W_1W_2 + a_1W_1^2 + a_2W_0W_1 + a_3W_0^2)(32A_3 + 16A_2 + 8A_1 + 4A_0) + a_3(W_2^2 - a_1W_1W_2 - a_2W_0W_2 - a_3W_0W_1)(A_3 + A_2 + A_1 + A_0)$

$\Delta = W_2^3 + (a_3 + a_1a_2)W_1^3 + a_3^2W_0^3 - 2a_1W_1W_2^2 + (a_1^2 - a_2)W_1^2W_2 - a_2W_0W_2^2 + a_3a_1W_0^2W_2 + (a_2^2 + a_1a_3)W_0W_1^2 + 2a_2a_3W_0^2W_1 + (-3a_3 + a_1a_2)W_0W_1W_2$

In summary, the solution of (2.12) is given by

$$W_n(W_0, W_1, W_2) = (-B_1+1)V_n(W_0, W_1, W_2) - B_2V_{n-1}(W_0, W_1, W_2) - B_3V_{n-2}(W_0, W_1, W_2) + A_3n^5 + A_2n^4 + A_1n^3 + A_0n^2$$

2.4. The Case: $\theta_1 = 1$ and $\theta_2 = 1$ and $\theta_3 = 1$, i.e., All of the Roots of the Characteristic Equation of (1.2) is 1. In the following example, we assume that all of the roots of the characteristic equation of (1.2) is equal to 1 and we consider special cases of $p(n)$ defined in (1.1) by using Theorem 1.1.

EXAMPLE 2.4. *In this example, in each case, the homogeneous relation corresponding to the given non-homogeneous recurrence relation is expressed as in (1.2), i.e.,*

$$V_n = 3V_{n-1} - 3V_{n-2} + V_{n-3}.$$

so that $a_1 = 3,\ a_2 = -3,\ a_3 = 1.$

(a): *The case* $s = 0 :$ *(*$a_3 \neq 0,\ c_0 \neq 0$*).*

Let the sequence $\{W_n\}$ *defined by nonhomogeneous linear recurrence relation*

$$W_n = 3W_{n-1} - 3W_{n-2} + W_{n-3} + c_0. \tag{2.13}$$

Then the solution of (2.13) is given by

$$\begin{aligned} W_n(W_0, W_1, W_2) &= W_n^{(h)} + W_n^{(p)} \\ &= V_n(W_0, W_1, W_2) - V_n(W_0^{(p)}, W_1^{(p)}, W_2^{(p)}) + W_n^{(p)} \end{aligned}$$

where $W_n^{(h)} = V_n(W_0, W_1, W_2) - V_n(W_0^{(p)}, W_1^{(p)}, W_2^{(p)})$ *is the solution of (1.2) and*

$$W_n^{(p)} = n^3 A_0$$

is the particular solution of (2.13). Here,

$$A_0 = \tfrac{1}{6} c_0$$

and

$$W_0^{(p)} = 0, \; W_1^{(p)} = A_0, \; W_2^{(p)} = 8A_0$$

and

$$\begin{aligned} V_n(W_0^{(p)}, W_1^{(p)}, W_2^{(p)}) &= V_n(0, A_0, 8A_0) \\ &= B_1 V_n(W_0, W_1, W_2) + B_2 V_{n-1}(W_0, W_1, W_2) + B_3 V_{n-2}(W_0, W_1, W_2) \end{aligned}$$

where

$$B_1 = \frac{Y_1}{\Delta}, \; B_2 = \frac{Y_2}{\Delta}, \; B_3 = \frac{Y_3}{\Delta}$$

and

$Y_1 = (W_2^2 + 9W_1^2 + 3W_0^2 - 6W_1W_2 + 3W_0W_2 - 10W_0W_1) \times 8A_0 + (-8W_1^2 - 3W_0^2 + 3W_1W_2 - W_0W_2 + 9W_0W_1)A_0$

$Y_2 = (-8W_1^2 + 3W_2W_1 + 3W_0W_1 - 3W_0^2 + 6W_0W_1 - W_0W_2) \times 8A_0 + (-3W_2^2 + 2W_1W_2 + W_0^2 + 6W_1W_2 - 3W_0W_2 - 3W_0W_2)A_0$

$Y_3 = (-W_1W_2 + 3W_1^2 - 3W_0W_1 + W_0^2) \times 8A_0 + (W_2^2 - 3W_1W_2 + 3W_0W_2 - W_0W_1)A_0$

$\Delta = W_2^3 - 8W_1^3 + W_0^3 - 6W_1W_2^2 + 12W_1^2W_2 + 3W_0W_2^2 + 3W_0^2W_2 + 12W_0W_1^2 - 6W_0^2W_1 - 12W_0W_1W_2$

In summary, the solution of (2.13) is given by

$$W_n(W_0, W_1, W_2) = (-B_1 + 1)V_n(W_0, W_1, W_2) - B_2V_{n-1}(W_0, W_1, W_2) - B_3V_{n-2}(W_0, W_1, W_2) + n^3A_0$$

(b): *The case* $s = 1 : (a_3 \neq 0, \; c_1 \neq 0)$*.*

Let the sequence $\{W_n\}$ *defined by nonhomogeneous linear recurrence relation*

$$W_n = 3W_{n-1} - 3W_{n-2} + W_{n-3} + c_1 n + c_0. \tag{2.14}$$

Then the solution of (2.14) is given by

$$\begin{aligned} W_n(W_0, W_1, W_2) &= W_n^{(h)} + W_n^{(p)} \\ &= V_n(W_0, W_1, W_2) - V_n(W_0^{(p)}, W_1^{(p)}, W_2^{(p)}) + W_n^{(p)} \end{aligned}$$

where $W_n^{(h)} = V_n(W_0, W_1, W_2) - V_n(W_0^{(p)}, W_1^{(p)}, W_2^{(p)})$ *is the solution of (1.2) and*

$$W_n^{(p)} = n^3(A_1 n + A_0) = A_1 n^4 + A_0 n^3$$

is the particular solution of (2.14). Here,

$$A_1 = \frac{1}{24}c_1, \ A_0 = \frac{1}{12}(2c_0 + 3c_1),$$

and

$$W_0^{(p)} = 0, \ W_1^{(p)} = A_1 + A_0, \ W_2^{(p)} = 16A_1 + 8A_0$$

and

$$\begin{aligned} V_n(W_0^{(p)}, W_1^{(p)}, W_2^{(p)}) &= V_n(0, A_1 + A_0, 16A_1 + 8A_0) \\ &= B_1 V_n(W_0, W_1, W_2) + B_2 V_{n-1}(W_0, W_1, W_2) + B_3 V_{n-2}(W_0, W_1, W_2) \end{aligned}$$

where

$$B_1 = \frac{Y_1}{\Delta}, \ B_2 = \frac{Y_2}{\Delta}, \ B_3 = \frac{Y_3}{\Delta}$$

and

$Y_1 = (W_2^2 + 9W_1^2 + 3W_0^2 - 6W_1W_2 + 3W_0W_2 - 10W_0W_1)(16A_1 + 8A_0) + (-8W_1^2 - 3W_0^2 + 3W_1W_2 - W_0W_2 + 9W_0W_1)(A_1 + A_0)$

$Y_2 = (-8W_1^2 + 3W_2W_1 + 3W_0W_1 - 3W_0^2 + 6W_0W_1 - W_0W_2)(16A_1 + 8A_0) + (-3W_2^2 + 2W_1W_2 + W_0^2 + 6W_1W_2 - 3W_0W_2 - 3W_0W_2)(A_1 + A_0)$

$Y_3 = (-W_1W_2 + 3W_1^2 - 3W_0W_1 + W_0^2)(16A_1 + 8A_0) + (W_2^2 - 3W_1W_2 + 3W_0W_2 - W_0W_1)(A_1 + A_0)$

$\Delta = W_2^3 - 8W_1^3 + W_0^3 - 6W_1W_2^2 + 12W_1^2W_2 + 3W_0W_2^2 + 3W_0^2W_2 + 12W_0W_1^2 - 6W_0^2W_1 - 12W_0W_1W_2$

In summary, the solution of (2.14) is given by

$$W_n(W_0, W_1, W_2) = (-B_1 + 1)V_n(W_0, W_1, W_2) - B_2 V_{n-1}(W_0, W_1, W_2) - B_3 V_{n-2}(W_0, W_1, W_2) + A_1 n^4 + A_0 n^3$$

(c): *The case* $s = 2$: $(a_3 \neq 0, \ c_2 \neq 0)$.

Let the sequence $\{W_n\}$ *defined by nonhomogeneous linear recurrence relation*

$$W_n = 3W_{n-1} - 3W_{n-2} + W_{n-3} + c_2 n^2 + c_1 n + c_0. \tag{2.15}$$

Then the solution of (2.15) is given by

$$\begin{aligned} W_n(W_0, W_1, W_2) &= W_n^{(h)} + W_n^{(p)} \\ &= V_n(W_0, W_1, W_2) - V_n(W_0^{(p)}, W_1^{(p)}, W_2^{(p)}) + W_n^{(p)} \end{aligned}$$

where $W_n^{(h)} = V_n(W_0, W_1, W_2) - V_n(W_0^{(p)}, W_1^{(p)}, W_2^{(p)})$ *is the solution of (1.2) and*

$$W_n^{(p)} = n^3(A_2 n^2 + A_1 n + A_0) = A_2 n^5 + A_1 n^4 + A_0 n^3$$

is the particular solution of (2.15). Here,

$$A_2 = \frac{1}{60}c_2,\ A_1 = \frac{1}{24}(c_1 + 3c_2),\ A_0 = \frac{1}{12}(2c_0 + 3c_1 + 4c_2),$$

and

$$W_0^{(p)} = 0,\ W_1^{(p)} = A_2 + A_1 + A_0,\ W_2^{(p)} = 32A_2 + 16A_1 + 8A_0$$

and

$$\begin{aligned} V_n(W_0^{(p)}, W_1^{(p)}, W_2^{(p)}) &= V_n(0, A_2 + A_1 + A_0, 32A_2 + 16A_1 + 8A_0) \\ &= B_1 V_n(W_0, W_1, W_2) + B_2 V_{n-1}(W_0, W_1, W_2) + B_3 V_{n-2}(W_0, W_1, W_2) \end{aligned}$$

where

$$B_1 = \frac{Y_1}{\Delta},\ B_2 = \frac{Y_2}{\Delta},\ B_3 = \frac{Y_3}{\Delta}$$

and

$Y_1 = (W_2^2 + 9W_1^2 + 3W_0^2 - 6W_1W_2 + 3W_0W_2 - 10W_0W_1)(32A_2 + 16A_1 + 8A_0) + (-8W_1^2 - 3W_0^2 + 3W_1W_2 - W_0W_2 + 9W_0W_1)(A_2 + A_1 + A_0)$

$Y_2 = (-8W_1^2 + 3W_2W_1 + 3W_0W_1 - 3W_0^2 + 6W_0W_1 - W_0W_2)(32A_2 + 16A_1 + 8A_0) + (-3W_2^2 + 2W_1W_2 + W_0^2 + 6W_1W_2 - 3W_0W_2 - 3W_0W_2)(A_2 + A_1 + A_0)$

$Y_3 = (-W_1W_2 + 3W_1^2 - 3W_0W_1 + W_0^2)(32A_2 + 16A_1 + 8A_0) + (W_2^2 - 3W_1W_2 + 3W_0W_2 - W_0W_1)(A_2 + A_1 + A_0)$

$\Delta = W_2^3 - 8W_1^3 + W_0^3 - 6W_1W_2^2 + 12W_1^2W_2 + 3W_0W_2^2 + 3W_0^2W_2 + 12W_0W_1^2 - 6W_0^2W_1 - 12W_0W_1W_2$

In summary, the solution of (2.15) is given by

$$W_n(W_0, W_1, W_2) = (-B_1+1)V_n(W_0, W_1, W_2) - B_2V_{n-1}(W_0, W_1, W_2) - B_3V_{n-2}(W_0, W_1, W_2) + A_2n^5 + A_1n^4 + A_0n^3$$

(d): *The case $s = 3$: ($a_3 \neq 0$, $c_3 \neq 0$).*

Let the sequence $\{W_n\}$ defined by nonhomogeneous linear recurrence relation

$$W_n = 3W_{n-1} - 3W_{n-2} + W_{n-3} + c_3n^3 + c_2n^2 + c_1n + c_0. \tag{2.16}$$

Then the solution of (2.16) is given by

$$\begin{aligned} W_n(W_0, W_1, W_2) &= W_n^{(h)} + W_n^{(p)} \\ &= V_n(W_0, W_1, W_2) - V_n(W_0^{(p)}, W_1^{(p)}, W_2^{(p)}) + W_n^{(p)} \end{aligned}$$

where $W_n^{(h)} = V_n(W_0, W_1, W_2) - V_n(W_0^{(p)}, W_1^{(p)}, W_2^{(p)})$ is the solution of (1.2) and

$$W_n^{(p)} = n^3(A_3n^3 + A_2n^2 + A_1n + A_0) = A_3n^6 + A_2n^5 + A_1n^4 + A_0n^3$$

is the particular solution of (2.16). Here,

$$\begin{aligned} A_3 &= \frac{1}{120}c_3,\ A_2 = \frac{1}{120}(2c_2 + 9c_3), \\ A_1 &= \frac{1}{24}(c_1 + 3c_2 + 6c_3),\ A_0 = \frac{1}{24}(4c_0 + 6c_1 + 8c_2 + 9c_3), \end{aligned}$$

and

$$W_0^{(p)}=0,\ W_1^{(p)}=A_3+A_2+A_1+A_0,\ W_2^{(p)}=64A_3+32A_2+16A_1+8A_0$$

and

$$\begin{aligned} V_n(W_0^{(p)},W_1^{(p)},W_2^{(p)}) &= V_n(0,A_3+A_2+A_1+A_0,64A_3+32A_2+16A_1+8A_0) \\ &= B_1V_n(W_0,W_1,W_2)+B_2V_{n-1}(W_0,W_1,W_2)+B_3V_{n-2}(W_0,W_1,W_2) \end{aligned}$$

where

$$B_1=\frac{Y_1}{\Delta},\ B_2=\frac{Y_2}{\Delta},\ B_3=\frac{Y_3}{\Delta}$$

and

$Y_1=(W_2^2+9W_1^2+3W_0^2-6W_1W_2+3W_0W_2-10W_0W_1)(64A_3+32A_2+16A_1+8A_0)+(-8W_1^2-3W_0^2+3W_1W_2-W_0W_2+9W_0W_1)(A_3+A_2+A_1+A_0)$

$Y_2=(-8W_1^2+3W_2W_1+3W_0W_1-3W_0^2+6W_0W_1-W_0W_2)(64A_3+32A_2+16A_1+8A_0)+(-3W_2^2+2W_1W_2+W_0^2+6W_1W_2-3W_0W_2-3W_0W_2)(A_3+A_2+A_1+A_0)$

$Y_3=(-W_1W_2+3W_1^2-3W_0W_1+W_0^2)(64A_3+32A_2+16A_1+8A_0)+(W_2^2-3W_1W_2+3W_0W_2-W_0W_1)(A_3+A_2+A_1+A_0)$

$\Delta=W_2^3-8W_1^3+W_0^3-6W_1W_2^2+12W_1^2W_2+3W_0W_2^2+3W_0^2W_2+12W_0W_1^2-6W_0^2W_1-12W_0W_1W_2$

In summary, the solution of (2.16) is given by

$$W_n(W_0,W_1,W_2)=(-B_1+1)V_n(W_0,W_1,W_2)-B_2V_{n-1}(W_0,W_1,W_2)-B_3V_{n-2}(W_0,W_1,W_2)+A_3n^6+A_2n^5+A_1n^4+A_0n^3$$

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