Lambert W-function in mathematical models of intra-year pricing, storage and manipulation of perishable agricultural goods

Abstract

We present a didactically structured application of the Lambert W-function in modeling the pricing, storage, and handling of agricultural goods with lifespans extending over two intra-year periods. The original linear model of Helmberger is extended to two semi-exponential models, both explicitly solvable in terms of the Lambert W-function. We also provide the Lambert W-function solution to the problem of determining the optimal time to transfer harvested goods into cool storage under given agro-technical and managerial constraints.

Keywords: *Lambert W-function, Helmberger linear pricing model, semi-exponential models, optimal transfer time*

Mathematics Subject Classification 33F99, 00A69

1 Introduction

Over the course of the last couple of decades, we have become more and more acustomed to mathematical models that rely entirely on computers and numerical solutions. Exactly solvable models, having solutions in nice, closed, explicit formulas, now play less and less prominent role. One reason may be that we are modelling increasingly complex phenomena, which do not readily allow for simple and elegant explicit solutions. Such approach has been facilitated by the steadily increasing computing power at our disposal, along with increasingly powerful and sophisticated numerical methods. There is, however, also a certain degree of complacency, leading to a loss of interest in the finer aspects of mathematical modelling. Once upon a time, simple exactly solvable models were employed and subsequently gradually improved, with efforts made to maintain solvability, thereby expanding the repertoire of acceptable explicit solutions. Nowadays, these intermediary steps are often omitted. One of the (unintended) consequence is the loss of once familiar techniques and concepts that could still prove useful. In this note, we focus on one such concept, the Lambert W-function, and its potential uses in modelling certain aspects of intra-year pricing, storage, and handling of perishable and/or degradable agricultural products.

The Lambert W-function $W(z)$ is the function that satisfies $W(z)e^{W(z)} = z,\ z \in \mathbb{C}$. If we restrict ourselves to $x \in \mathbb{R}$, there are two real branches of $W(x)$, see Figure 1. (The coordinates of a given point are $A(-e^{-1}, -1)$.)

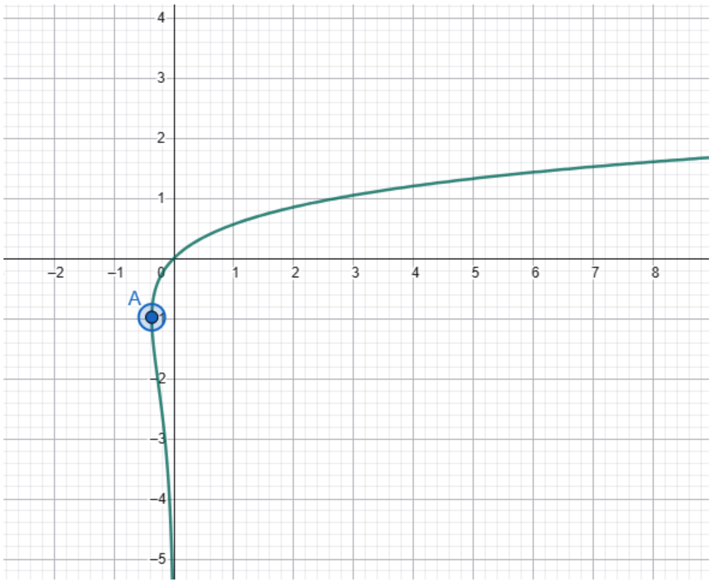

Figure 1: Real branches of the Lambert W- function

Both branches are strictly monotone and are usually denoted by

$$W_0(x) : [-e^{-1}, \infty\rangle \to [-1, \infty\rangle \quad \text{and} \quad W_1(x) : [-e^{-1}, 0\rangle \to \langle -\infty, -1].$$

Some specific values of $W_0(x)$ are: $W_0(-e^{-1}) = -1, W_0(0) = 0, W_0(e) = 1$.

Here, the Lambert W-function solutions to problems involve only the positive branch of $W_0(x)$, namely $W_0^+(x) : [0, \infty\rangle \to [0, , \infty\rangle$.

Despite its rather elementary definition as the (piecewise) inverse function of $x\,e^x$, the Lambert W-function is insufficiently known and used. It is not considered one of the elementary functions. However, this is merely a matter of convention and need not concern the end-user. Indeed, the Lambert W-function is available as a built-in function in all modern computer algebra systems, and its use is no more difficult than any of more familiar elementary functions such as, e.g., logarithms or trigonometric functions. For a brief review of its history and computational issues, we refer the reader to Dence [3] and Haque and Chakraborty [6]. For more advanced and detailed evaluation issues, the interested reader may consult Loczi [8]. This reference also contains an extensive bibliography on various applications of the Lambert W-function in science and everyday practical problems. In that list of references, its applications in agriculture appear to be absent. We aim to correct this with this note.

We first introduce the notation and revisit Helmberger's linear two-period model of intra-year pricing and storage. We then consider two semi-exponential extensions of Helmberger's model and provide their solutions in terms of the Lambert W-function. Next, we address the problem of determining the optimal time to transfer harvested agricultural goods into cool storage under a given set of agro-technical and managerial constraints. The concise solution is again given in terms of the Lambert W-function. In the concluding section, we suggest possible ways to upgrade and extend our models.

2 Models of intra-year pricing and storage

There are many specific aspects of the seasonal and perishable nature of agricultural production. These include pricing (Meng-Chuen Chen et al. [2]; Wang et al. [11]), transportation, storage, and supply chain models, all of which are widely researched by: Helmberger [7], Blackburn and Scudder [1], Disney and Warburton [4], Sarkar et al. [10] and Gijsenberg [5].

The model of intra-year pricing and storage originates from Helmberger's book ([7], Chapter 5), where he considered a linear two-period model. Blackburn and Scudder [1] reported on a semi-exponential model for melon supply chain design. The same model was revisited by Disney and Warburton [4], who found its Lambert W-function solution.

Let us introduce the notation. The model concerns agricultural goods harvested in a quantity of H_0 and stored for two intra-year periods. The harvested quantity H_0 is decomposed as $H_0 = Q_1 + I_1$, where Q_1 is the demand in the first post-harvest period and I_1 is the quantity stored in the first period for use in the second period. Let P_i, $i = 1, 2$, denote the prices of the goods in the respective periods. Let K be the storage cost per unit of goods, and i (%) the interest rate when the money spent on buying goods is invested in savings. Originally, Helmberger presented a linear model.

2.1 A linear model

A linear model assumes that demand in the first post-harvest period depends linearly on price, i.e., $Q_1 = a - bP_1 + u_1$, where a and b are positive constants, P_1 is the price of goods in the first period, and u_1 is the random, stochastic term. This term is known in the first period, but not before, so $E_0 u_1 = 0$ applies to its (mathematical) expectation before the first period.

A single buyer purchases goods in quantity q in the first period and then stores them with the intention of selling in the second period. His profit function is

$$\pi_2 = P_2 q - P_1 q - Kq - iP_1 q.$$

Here, P_2 is the price in the second period, Kq is the storage cost and the last term is the interest on the money spent on the purchase.

The subjectively expected profit $e_1\pi_2$ depends on the subjectively expected price $e_1 P_2$, i.e.,

$$e_1\pi_2 = q(e_1 P_2 - P_1(1+i) - K).$$

In a competitive market, $e_1\pi_2 = 0$, which leads to $e_1 P_2 = P_1(1+i) + K$.

Let us further assume a linear dependence of quantity on price in the second period, i.e., $Q_2 = c - dP_2 + u_2$, with positive constants c, d, $d \neq 0$, and a stochastic term u_2. Since $Q_2 = I_1$, we have $P_2 = \frac{1}{d}(c - I_1 + u_2)$. The fact that the mathematical expectation of the random term u_2 prior to the second period is 0, i.e. $E_1 u_2 = 0$, implies $E_1 P_2 = \frac{1}{d}(c - I_1)$. According to the Muth hypothesis on rational expectations, $e_1 P_2 = E_1 P_2$ ("expectations, being informed predictions of future events, are essentially the same as predictions", Muth [9]). With this in mind, the structural model for the first period is

$$\begin{aligned} &Q_1 = a - bP_1 + u_1, \\ &Q_1 + I_1 = H_0, \\ &I_1 = c - dK - d(1+i)P_1. \end{aligned}$$

This original model consists of three equations with three unknowns. Since linear, it can be solved easily. Its matrix form is $AX = b$, with

$$A = \begin{bmatrix} 1 & b & 0 \\ 1 & 0 & 1 \\ 0 & d(1+i) & 1 \end{bmatrix}, \; X = \begin{bmatrix} Q_1 \\ P_1 \\ I_1 \end{bmatrix}, \; b = \begin{bmatrix} a + u_1 \\ H_0 \\ c - dK \end{bmatrix}.$$

The solution is $X = A^{-1}b$ and explicit formulas for Q_1, P_1 and I_1 are

$$\begin{bmatrix} Q_1 \\ P_1 \\ I_1 \end{bmatrix} = \frac{1}{b+d(1+i)} \begin{bmatrix} d(1+i)(a+u_1) - b(c-dK-H_0) \\ a+u_1-H_0+c-dK \\ b(c-dK) - d(1+i)(a+u_1-H_0) \end{bmatrix}.$$

This solution gives a formula for the expected price in the second period

$$E_1P_2 = \frac{bK + (1+i)(a+u_1-H_0+c)}{b+d(1+i)}.$$

Helmberger provides a numerical example of his model. He assumes that for the first period, $Q_1 = 10 - P_1$, $H_0 = 8$, $K = 1$, and $i = 0$. Then $E_1P_2 = P_1 + 1$. For the second period, he takes $Q_2 = 12 - 2P_2 + u_2$ and assumes $u_2 = \pm 2$ with equal probabilities, so $E_1u_2 = 0$. Next, he sets $I_1 = 2$. The value $u_2 = 2$ results in $P_2 = 6$, and $u_2 = -2$ in $P_2 = 4$. Thus, with $I_1 = 2$, the expected price is $P_2 = 5$. Similarly, with $I_1 = 6$, the resulting $P_2 = 3$. Now, we can model a linear dependence of price P_2 on the quantity I_1: $E_1P_2 = 6 - 0.5I_1$. Next, the demand for storage in the first period is $I_1 = 10 - 2P_1$. The final solution of the given model is $P_1 = 4$, $Q_1 = 6$, $I_1 = 2$, and $E_1P_2 = 5$ which is in accordance with the above formula.

2.2 Semi-exponential models

Blackburn and Scudder [1] used one semi-exponential model to track the loss of value of melons. Picked in the field at high temperatures, melons lose value exponentially. Later, when refrigerated, the loss continues at a constant linear rate (Figure 4 in[1]). The authors provided bounds for the numerical solution to the problem of minimising costs and obtained a solution for the optimal picking batch quantity. Subsequently, Disney and Warburton [4] presented the exact solution using the Lambert W-function.

Similarly to the model given by Blackburn and Scudder, following the Helmberger idea, we assume that in the first period the quantity function Q_1 depends exponentially on its price P_1, while the dependence of quantity Q_2 on price P_2 is linear.

Similarly to the linear model case, we again have:

$$\begin{aligned} Q_1 &= a\mathrm{e}^{-bP_1} + u_1 \\ Q_1 + I_1 &= H_0 \\ I_1 &= c - dK - d(1+i)P_1. \end{aligned} \tag{1}$$

Theorem 1. *The solution of the system (1) for P_1 is given in terms of the Lambert W-function as*

$$P_1 = \frac{c-dK+u_1-H_0}{d(1+i)} + \frac{1}{b}W\left[\frac{ab}{d(1+i)}\mathrm{e}^{-\frac{b(c-dK+u_1-H_0)}{d(1+i)}}\right].$$

Proof. From the above model we obtain

$$d(1+i)P_1 - a\mathrm{e}^{-bP_1} = c - dK + u_1 - H_0,$$

i.e.

$$P_1 = \frac{a}{d(1+i)}\mathrm{e}^{-bP_1} + \frac{c-dK+u_1-H_0}{d(1+i)}.$$

Multiplying by e^{bP_1} gives

$$\left(P_1 - \frac{c - dK + u_1 - H_0}{d(1+i)}\right) \mathrm{e}^{bP_1} = \frac{a}{d(1+i)}.$$

Multiplying again, this time by $b\mathrm{e}^{-\frac{b(c-dK+u_1-H_0)}{d(1+i)}}$ results in

$$\left(bP_1 - \frac{b(c - dK + u_1 - H_0)}{d(1+i)}\right) \mathrm{e}^{bP_1 - \frac{b(c-dK+u_1-H_0)}{d(1+i)}} = \frac{ab}{d(1+i)}\mathrm{e}^{-\frac{b(c-dK+u_1-H_0)}{d(1+i)}}.$$

From this, we conclude that $bP_1 - \frac{b(c-dK+u_1-H_0)}{d(1+i)}$ is exactly the Lambert W-function

$$W\left[\frac{ab}{d(1+i)}\mathrm{e}^{-\frac{b(c-dK+u_1-H_0)}{d(1+i)}}\right].$$

Therefore,

$$P_1 = \frac{c - dK + u_1 - H_0}{d(1+i)} + \frac{1}{b}W\left[\frac{ab}{d(1+i)}\mathrm{e}^{-\frac{b(c-dK+u_1-H_0)}{d(1+i)}}\right].$$

□

The opposite scenario is also possible. The quantity Q_1 may depend linearly on the price P_1 and Q_2 exponentially on P_2. Thus, $Q_1 = a - bP_1 + u_1$ and $Q_2 = c\mathrm{e}^{-dP_2} + u_2$. Since $Q_2 = I_1$,

$$P_2 = -\frac{1}{d}\ln\frac{I_1 - u_2}{c}.$$

With $E_1 u_2 = 0$, this gives

$$E_1 P_2 = -\frac{1}{d}\ln\frac{I_1}{c}.$$

Therefore, the model states

$$\begin{aligned} &Q_1 = a - bP_1 + u_1, \\ &Q_1 + I_1 = H_0, \\ &E_1 P_2 = -\tfrac{1}{d}\ln\tfrac{I_1}{c} \\ &e_1 P_2 = (1+i)P_1 + K. \end{aligned}$$

Since $e_1 P_2 = E_1 P_2$, it transforms to

$$\begin{aligned} &Q_1 = a - bP_1 + u_1, \\ &Q_1 + I_1 = H_0, \\ &I_1 = c\mathrm{e}^{-d((1+i)P_1+K)}. \end{aligned} \tag{2}$$

Theorem 2. *The solution on P_1 of the system (2) given in the form of the Lambert W-function is*

$$P_1 = \frac{a + u_1 - H_0}{b} + \frac{1}{d(1+i)}W\left[\frac{cd(1+i)}{b}\mathrm{e}^{-\frac{a+u_1-H_0+bK}{b}}\right].$$

Proof. We have

$$-bP_1 + c\mathrm{e}^{-d((1+i)P_1+K)} = H_0 - a - u_1.$$

Firstly, multiplying by $\mathrm{e}^{d((1+i)P_1+K)}$ results in

$$\left(P_1 - \frac{a + u_1 - H_0}{b}\right)\mathrm{e}^{d(1+i)P_1} = \frac{c}{b\mathrm{e}^K},$$

and then multiplying by $d(1+i)\mathrm{e}^{-\frac{a+u_1-H_0}{b}}$ gives

$$\left(d(1+i)P_1 - \frac{d(1+i)(a + u_1 - H_0)}{b}\right)\mathrm{e}^{d(1+i)P_1 - \frac{d(1+i)(a+u_1-H_0)}{b}} = \frac{cd(1+i)}{b}\mathrm{e}^{-\frac{a+u_1-H_0+bK}{b}}.$$

So, $d(1+i)P_1 - \frac{d(1+i)(a+u_1-H_0)}{b}$ corresponds exactly to the Lambert W-function

$$W\left[\frac{cd(1+i)}{b}\mathrm{e}^{-\frac{a+u_1-H_0+bK}{b}}\right].$$

Finally,

$$P_1 = \frac{a+u_1-H_0}{b} + \frac{1}{d(1+i)}W\left[\frac{cd(1+i)}{b}\mathrm{e}^{-\frac{a+u_1-H_0+bK}{b}}\right].$$

□

Both semi-exponential models may be simplified if we assume $i = 0$. This assumption is natural: low annual interest rates result in negligible monthly incomes. In this case, the formulas for P_1 are

$$P_1 = \frac{c-dK+u_1-H_0}{d} + \frac{1}{b}W\left[\frac{ab}{d}\mathrm{e}^{-\frac{b(c-dK+u_1-H_0)}{d}}\right] \tag{3}$$

for system (1) and

$$P_1 = \frac{a+u_1-H_0}{b} + \frac{1}{d}W\left[\frac{cd}{b}\mathrm{e}^{-\frac{a+u_1-H_0+bK}{b}}\right]$$

for system (2).

We present a numerical example of the first semi-exponential model to derive prices. Let $Q_1 = 10\mathrm{e}^{-0.1P_1} + u_1$, $Q_2 = 12 - 2P_2 + u_2$, $K = 1$ and $H_0 = 8$, so that the linearisation of the model is equivalent to the Helmberger linear model.

The system (1) becomes

$$\begin{aligned} &Q_1 = 10\mathrm{e}^{-0.1P_1} + u_1 \\ &Q_1 + I_1 = 8 \\ &I_1 = 10 - 2P_1. \end{aligned}$$

The solution for P_1 is, according to (3), given by

$$P_1 = \frac{2+u_1}{2} + 10\,W\left[\frac{1}{2}\,\mathrm{e}^{-0.1\,\frac{2+u_1}{2}}\right].$$

Values of the Lambert W-function for an argument n can, for example, be obtained using the WolframAlpha *ProductLog*[n] function. Different values of the stochastic term u_1 yield different values of P_1, as shown in Table 1.

Table 1: Values of P_1 for different u_1

$\mathbf{u_1}$	-2	-1	0	1	2
$\mathbf{P_1}$	3.51	3.89	4.26	4.64	5.03

Assuming all values of u_1 are equally probable, we obtain $E_1P_1 = 4.27$. Now we use this data for the second period. We have $e_1P_2 = P_1 + K = 5.27$. From $E_1P_2 = 6 - \frac{1}{2}I_1$, we can now deduce $I_1 = 1.46$.

3 Optimal cooling start time

In this section, we present a variation on the original Blackburn and Scudder melon-harvesting and processing model. We consider an agricultural product in two different regimes of mass (and hence value) loss and determine the optimal moment for the regime change.

Let us assume we are dealing with an agricultural product sensitive to mass loss through, for example, evaporation. Let M_0 denote the mass of the product at the time of harvesting. In an uncooled environment, evaporation follows an exponential law, so the mass at a given time $t > 0$ is described by

$$M(t) = M_0\,\mathrm{e}^{-at},$$

where a is a parameter, expressed in reciprocal time units, that describes the evaporation rate. We assume that post-harvest storage in a non-cooled environment incurs no cost. In a cooled environment, evaporation follows a linear law. If cooling begins at time T_c, the mass at time $t > T_c$ is given by

$$M(t) = M(T_c)[1 - b(t - T_c)] = M_0\mathrm{e}^{-aT_c}[1 - b(t - T_c)]. \tag{4}$$

Here, b is a parameter describing the linear evaporation rate in a cooled environment. We assume that mass loss is much smaller in a cooled environment. Hence, it is preferable to start cooling as soon as possible. However, cooled storage incurs certain expenses. The problem is to determine the optimal (i.e., the latest) time to start cooling so that, for a given initial mass M_0 and a given future time T_∞, the remaining mass is at least M_∞, $M_\infty > 0$. Thus, we have to solve the equation

$$M_0\mathrm{e}^{-aT_c}[1 - b(T_\infty - T_c)] = M_\infty$$

for the unknown cooling start time T_c. By introducing a new dimensionless variable $z = 1 - b(T_\infty - T_c)$, the variable of interest can be expressed as

$$T_c = \frac{1 - b\,T_\infty}{b} - \frac{z}{b} = B - \frac{z}{b}.$$

Here, the parameter B arises naturally as the time horizon limiting the applicability of our model. The equation to be solved simplifies to

$$z\,\mathrm{e}^{\frac{a}{b}z} = \frac{M_\infty}{M_0}\mathrm{e}^{a\,B} = \gamma\,\mathrm{e}^{a\,B}.$$

Here, the dimensionless parameter $\gamma = \frac{M_\infty}{M_0}$ quantifies the acceptable loss of the fraction of mass (and hence value) of the product at the final time T_∞. The right-hand side of the equation consists of known quantities; by denoting it by D, the equation reduces to

$$z\,\mathrm{e}^{\frac{a}{b}\,z} = D,$$

which is readily solved in terms of the Lambert W-function. It suffices to denote $\frac{a}{b} = C$, multiply through by C, and write the solution of the resulting equation

$$C\,z\,\mathrm{e}^{C\,z} = C\,D$$

as $z = \frac{1}{C}W[C\,D]$. Now, the optimal starting time for cooling can be neatly expressed in terms of the model parameters as

$$T_c = B - \frac{1}{a}W\left[\frac{a}{b}\gamma\mathrm{e}^{a\,B}\right].$$

Theorem 3. *The solution of the equation (4) for T_c is given in terms of the Lambert W-function as*

$$T_c = \frac{1 - b\,T_\infty}{b} - \frac{1}{a}\,W\left[\frac{a}{b}\gamma \mathrm{e}^{a\,\frac{1-b\,T_\infty}{b}}\right].$$

The obtained formula links, in a compact way, parameters dictated by the product biology (evaporation rates) with those depending on management decisions (terminal time and acceptable losses), and enables theoretical investigation of their interplay.

4 Discussion and further developments

In this short note, we have aimed to highlight the advantages of exactly solvable models for certain problems of agrotechnical and agroeconomic interest. In addition to providing explicit solutions, our models have didactic value, and we hope they will encourage researchers to apply and develop similar models for other relevant problems. In the remainder of this section, we outline some possible directions for further work.

Regarding intra-year pricing and storage, our models could be extended to a third period, as cooling may prolong the expiry dates of perishable agricultural goods beyond two periods. If this is also the final period for selling the goods, the model can, as in Zabihi and Khakzar Bafruei [12], address the optimal timing of discounts to encourage sales rather than product loss.

Similarly, our model for the optimal cooling start time could be further refined by decoupling mass and value, which are currently linked by a simple linear relationship, and by introducing a time-dependent function that penalises decreasing freshness. We invite readers to investigate the effects of both linearly and exponentially decreasing value on management decisions. Another way to extend our model and make it more realistic would be to explicitly account for the cost of cooling. We hope to report on this in the near future.

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