A Note on Explicit Particular Solutions for Third and Fourth order Generalized Leonardo-Type Recurrences with Polynomial-Exponential Input

Abstract. Sequences, both classical and modern in scope, can be analyzed through a versatile framework that remains central to mathematics, namely recurrence relations. Previous investigations established explicit iterative procedures for constructing polynomial–exponential particular solutions of generalized Leonardo-type sequences. Building upon that framework, this article develops illustrative examples for the cases

$$m = 3, 4,$$

where the forcing term is given by $C(n) = p(n)d^n$, with $p(n) = \sum_{i=0}^{s} c_i n^i$ a polynomial in n. For such recurrences, we derive particular solutions of the form

$$W_n^{(C)} = n^r \left(\sum_{i=0}^{s} A_i n^i \right) d^n,$$

and demonstrate the computation of the coefficients A_i via the established iterative scheme. These examples reveal how the multiplicity r of the root d in the characteristic polynomial governs the structure of the solution, while also exposing resonance phenomena in non-homogeneous contexts. By presenting explicit cases, the paper offers a transparent and accessible illustration of the general theory, reinforcing the connection between abstract recurrence analysis and concrete symbolic computation.

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1. Introduction

Recurrence sequences, generated through recurrence relations, have long been recognized as fundamental objects in mathematics, with influence extending well beyond the discipline into physics, engineering, biology, computer science, architecture, and even artistic analysis. Despite their elementary formulation, they encapsulate profound structures: modeling growth dynamics, oscillatory behavior, and intricate symbolic identities. Classical second-order families such as the Fibonacci, Lucas, Pell, and Jacobsthal sequences exemplify this richness and continue to serve as paradigmatic cases.

The scope of recurrence theory, however, is not confined to second-order constructions. Higher-order families occupy an equally prominent position, both in abstract theory and in applied modeling. These generalizations broaden the classical framework and uncover deeper algebraic and analytic phenomena. Third-order examples such as the Tribonacci sequence, fourth-order examples such as the Tetranacci sequence, and fifth-order examples such as the Pentanacci sequence extend the paradigm, each governed by characteristic polynomials whose root structures dictate closed-form representations. Homogeneous recurrences highlight the decisive role of characteristic polynomials and root multiplicities, whereas non-homogeneous recurrences introduce external symbolic inputs whose interaction with the root configuration gives rise to resonance phenomena. Taken together, these families establish a coherent framework that unites classical recurrence identities with modern developments in symbolic recurrence theory.

We first recall the definition of m-order homogeneous linear recurrence relations.

Definition 1.1. *A sequence $\{V_n\}_{n\geq 0}$ is called a homogeneous (linear) recurrence relation order $m\in\mathbb{N}$ if it satisfies*

$$V_n=\sum_{k=1}^{m}a_kV_{n-k}=a_1V_{n-1}+a_2V_{n-2}...+a_mV_{n-m} \tag{1.1}$$

for

$$m\geq 1$$

with the initial conditions $V_0,V_1,...,V_{m-1}$

and

$$V_n=a_0, \tag{1.2}$$

for

$$m=0.$$

The recurrence coefficients $a_1,a_2,\ldots,a_m$ and the initial conditions $V_0,V_1,...,V_{m-1}$ are complex scalars. We allow each coefficient a_i, for $1\leq i\leq m$, to be identically zero.

The integer m is called the order of the linear recurrence.

The characteristic polynomial of the sequence $(V_n)_{n\geq 0}$ is given by

$$A(z)=z^m-\sum_{k=1}^{m}a_kz^{m-k}=z^m-a_1z^{m-1}-a_2z^{m-2}-...-a_{m-1}z-a_m=(z-\theta_1)^{u_1}(z-\theta_2)^{u_2}...(z-\theta_v)^{u_v}$$

with distinct $\theta_1,\theta_2,...,\theta_v$ and $u_1+u_2+...+u_v=m$. $\theta_1,\theta_2,...,\theta_v$ are called the (characteristic) root of characteristic equation

$$A(z)=z^m-a_1z^{m-1}-a_2z^{m-2}-...-a_{m-1}z-a_m=(z-\theta_1)^{u_1}(z-\theta_2)^{u_2}...(z-\theta_v)^{u_v}=0. \tag{1.3}$$

For $m \geq 1$, consider the sequence (W_n) defined by the recurrence relation (a **generalized Leonardo-type sequence**)

$$W_n = \sum_{k=1}^{m} a_k W_{n-k} + p(n)bd^n = \sum_{k=1}^{m} a_k W_{n-k} + C(n) \tag{1.4}$$

with initial conditions $W_0, W_1, \ldots, W_{m-1}$ and the recurrence coefficients $a_1, a_2, \ldots, a_m$ are complex scalars or polynomials in $\mathbb{C}[x]$ and with the input function

$$C(n) = p(n)bd^n$$

where

$$p(n) := p(n,x) = \sum_{i=0}^{s} c_i n^i$$

denotes a polynomial in n of order s, with coefficients belonging to $\mathbb{C}[x]$ or $\mathbb{C}$ and $b \in \mathbb{C}[x]$ or $\mathbb{C}$, and $d \in \mathbb{C}$ or $\mathbb{R}$. For more information on generalized Leonardo-type sequences, see Soykan [2] and [1].

We consider the homogeneous recurrence relation (1.1) and its characteristic equation (1.3), corresponding to the sequence (W_n) defined by (1.4).

The particular solution $W_n^{(C)}$ of (1.4) is of the form

$$W_n^{(C)} = n^r (\sum_{i=0}^{s} A_i n^i) d^n = n^r (A_0 + \sum_{i=1}^{s} A_i n^i) d^n, \tag{1.5}$$

where the coefficients $A_i \in \mathbb{C}[x]$ or $\mathbb{C}$ and r is the multiplicity of d as a root of the characteristic equation (1.3), (if d is not a root of characteristic equation (1.3) then $r = 0$).

We proceed to formulate a theorem that provides explicit iterative expressions for the coefficients appearing in the particular solution $W_n^{(C)}$ of (1.4). The derivation is governed by the relationship between the parameter d and the characteristic roots of (1.3). When d coincides with a root of multiplicity r, the iterative procedure requires precise adjustments that reflect this multiplicity, ensuring the correct construction of the solution.

THEOREM 1.2. *[2, p.100, Theorem 5.1] For each $0 \leq i \leq s$, A_i given in (1.5) can be calculated with the iteration as follows:*

- *If $r = 0$, i.e., none of the roots of the characteristic equation (1.3) equals d, then*

$$A_s = -\frac{c_s bd^m}{a_1 d^{m-1} + a_2 d^{m-2} + a_2 d^{m-3} + \ldots + a_{m-2} d^2 + a_{m-1} d + a_m - d^m} = -\frac{c_s bd^m}{-d^m + \sum_{j=1}^{m} a_j d^{m-j}}, \text{ for } n = s$$

  and

$$A_n = -\frac{1}{-d^m + \sum_{j=1}^{m} a_j d^{m-j}} (c_n bd^m - \sum_{k=n+1}^{s} (-1)^{k-n+1} \binom{k}{n} (\sum_{j=1}^{m} j^{k-n} a_j d^{m-j}) A_k)$$

  for $n = s-1, s-2, \ldots, 2, 1, 0$.

- *If $r > 0$ then*

$$A_s = (-1)^{r+1} \frac{c_s bd^m}{(\sum_{j=1}^{m} j^r a_j \times d^{m-j}) \binom{s+r}{r}}, \text{ for } n = s$$

  and

$$A_n = (-1)^{r+1} \frac{1}{(\sum_{j=1}^{m} j^r a_j \times d^{m-j}) \binom{n+r}{r}} (c_n bd^m - \sum_{k=n+1}^{s} (-1)^{k+r-n+1} \binom{k+r}{n} (\sum_{j=1}^{m} j^{k+r-n} a_j \times d^{m-j}) A_k)$$

  for $n = s-1, s-2, \ldots, 2, 1, 0$.

In the following sections, we present explicit particular solutions to (1.4) for $m = 1, 2, 3, 4$, where

$$C(n) = p(n)bd^n, \quad p(n) \text{ is a polynomial in } n.$$

We seek solutions of the form

$$W_n^{(C)} = P(n)d^n,$$

where $P(n)$ is itself a polynomial in n.

2. Special Cases

2.1. The Case $m = 3$. Consider the homogeneous relation

$$V_n = a_1V_{n-1} + a_2V_{n-2} + a_3V_{n-3} \tag{2.1}$$

with the initial conditions V_0, V_1, V_2. Suppose that $\theta_1, \theta_2, \theta_3$ are the roots of characteristic equation

$$z^3 - a_1z^2 - a_2z - a_3 = 0 \tag{2.2}$$

associated with (2.1). Note that if all the roots of (2.2) are equal to d then

$$z^3 - a_1z^2 - a_2z - a_3 = (z-d)^3 = z^3 - 3dz^2 + 3d^2z - d^3 = 0$$

so that $a_1 = 3d$, $a_2 = -3d^2$, $a_3 = d^3$ and (2.1) reduces to

$$V_n = 3dV_{n-1} - 3d^2V_{n-2} + d^3V_{n-3}.$$

We now turn to an example that demonstrates the results derived above.

Example 2.1. *Consider the sequence (W_n) defined by the recurrence relation*

$$W_n = a_1W_{n-1} + a_2W_{n-2} + a_3W_{n-3} + p(n)bd^n$$

where $p(n) := p(n,x)$ is a polynomial in n of order s, with coefficients belonging to $\mathbb{C}[x]$ or $\mathbb{C}$ and $b \in \mathbb{C}[x]$ or $\mathbb{C}$, and $d \in \mathbb{C}$ or $\mathbb{R}$. We seek a particular solution

$$W_n^C = P(n)d^n$$

for the cases $s = 0, 1, 2, 3$ where $P(n)$ is itself a polynomial in n. The order (degree) and coffecients of $P(n)$ depend on the multiplicity r of d as a root of the characteristic equation (2.2) and W_n^C satisfy

$$W_n^C = a_1W_{n-1}^C + a_2W_{n-2}^C + a_3W_{n-3}^C + p(n)bd^n$$

i.e.,

$$P(n)d^n = a_1P(n-1)d^{n-1} + a_2P(n-2)d^{n-2} + a_3P(n-3)d^{n-3} + p(n)bd^n.$$

In each case of s, we consider the homogeneous relation (2.1) and its characteristic equation (2.2), corresponding to the sequence (W_n) with the same initial conditions as W_n, i.e.,

$$V_0 = W_0, V_1 = W_1, V_2 = W_2.$$

We investigate all cases of multiplicity r of of d as a root of the characteristic equation (2.2):

(a): $m = 3$, $s = 0$. *Consider the sequence* (W_n) *defined by*

$$W_n = a_1 W_{n-1} + a_2 W_{n-2} + a_3 W_{n-3} + c_0 b d^n.$$

(i): *Case* $r = 0$*, i.e., none of the roots of the characteristic equation equals* d*:*

$$W_n^C = A_0 d^n, \quad A_0 = -\frac{c_0 b d^3}{a_1 d^2 + a_2 d + a_3 - d^3}.$$

(ii): *Case* $r = 1$*, i.e., exactly one root of the characteristic equation equals* d*:*

$$W_n^C = n A_0 d^n, \quad A_0 = \frac{c_0 b d^3}{(a_1 d^2 + 2a_2 d + 3a_3)}.$$

(iii): *Case* $r = 2$*, i.e., exactly two roots of the characteristic equation equal* d*:*

$$W_n^C = n^2 A_0 d^n, \quad A_0 = -\frac{c_0 b d^3}{(a_1 d^2 + 4a_2 d + 9a_3)}.$$

(iv): *Case* $r = 3$*, i.e., all three roots of the characteristic equation equal* d*:*

$$W_n^C = n^3 A_0 d^n, \quad A_0 = \frac{1}{6} b c_0.$$

(b): $m = 3$, $s = 1$. *Consider the sequence* (W_n) *defined by*

$$W_n = a_1 W_{n-1} + a_2 W_{n-2} + a_3 W_{n-3} + (c_1 n + c_0) b d^n.$$

(i): *Case* $r = 0$ *(no root equal to* d*):*

$$W_n^C = (A_1 n + A_0) d^n$$

where

$$\begin{aligned} A_1 &= -\frac{c_1 b d^3}{(a_1 d^2 + a_2 d + a_3 - d^3)}, \\ A_0 &= -\frac{1}{(a_1 d^2 + a_2 d + a_3 - d^3)} (c_0 b d^3 - (a_1 d^2 + 2a_2 d + 3a_3) A_1), \end{aligned}$$

i.e.,

$$\begin{aligned} A_1 &= -\frac{c_1 b d^3}{(a_1 d^2 + a_2 d + a_3 - d^3)}, \\ A_0 &= -\frac{b d^3}{(a_1 d^2 + a_2 d + a_3 - d^3)^2} (-c_0 d^3 + a_1 (c_0 + c_1) d^2 + a_2 (c_0 + 2c_1) d + a_3 (c_0 + 3c_1)). \end{aligned}$$

(ii): *Case* $r = 1$ *(exactly one root equal to* d*):*

$$W_n^C = n (A_1 n + A_0) d^n$$

where

$$\begin{aligned} A_1 &= \frac{c_1 b d^3}{2\,(a_1 d^2 + 2a_2 d + 3a_3)}, \\ A_0 &= \frac{1}{(a_1 d^2 + 2a_2 d + 3a_3)} (c_0 b d^3 + \left(a_1 d^2 + 4a_2 d + 9a_3\right) A_1), \end{aligned}$$

i.e.,

$$A_1 = \frac{c_1 b d^3}{2\left(a_1 d^2 + 2a_2 d + 3a_3\right)},$$
$$A_0 = \frac{b d^3}{2\left(a_1 d^2 + 2a_2 d + 3a_3\right)^2}(a_1(2c_0 + c_1)d^2 + 4a_2(c_0 + c_1)d + 3a_3(2c_0 + 3c_1)).$$

(iii): *Case* $r = 2$ *(exactly two roots equal to* d*):*

$$W_n^C = n^2(A_1 n + A_0)d^n$$

where

$$A_1 = -\frac{c_1 b d^3}{3\left(a_1 d^2 + 4a_2 d + 9a_3\right)},$$
$$A_0 = -\frac{1}{\left(a_1 d^2 + 4a_2 d + 9a_3\right)}(c_0 b d^3 - \left(a_1 d^2 + 8a_2 d + 27a_3\right) A_1),$$

i.e.,

$$A_1 = -\frac{c_1 b d^3}{3\left(a_1 d^2 + 4a_2 d + 9a_3\right)},$$
$$A_0 = -\frac{b d^3}{3\left(a_1 d^2 + 4a_2 d + 9a_3\right)^2}(a_1(3c_0 + c_1)d^2 + 4a_2(3c_0 + 2c_1)d + 27a_3(c_0 + c_1)).$$

(iv): *Case* $r = 3$ *(all three roots equal to* d*):*

$$W_n^C = n^3(A_1 n + A_0)d^n$$

where

$$A_1 = \frac{1}{24} b c_1,$$
$$A_0 = \frac{1}{12} b(2c_0 + 3c_1).$$

(c): $m = 3$, $s = 2$. *Consider the sequence* (W_n) *defined by*

$$W_n = a_1 W_{n-1} + a_2 W_{n-2} + a_3 W_{n-3} + (c_2 n^2 + c_1 n + c_0) b d^n.$$

(i): *Case* $r = 0$ *(no root equal to* d*):*

$$W_n^C = (A_2 n^2 + A_1 n + A_0)d^n$$

where

$$A_2 = -\frac{c_2 b d^3}{(a_1 d^2 + a_2 d + a_3 - d^3)},$$
$$A_1 = -\frac{1}{(a_1 d^2 + a_2 d + a_3 - d^3)}(c_1 b d^3 - 2(a_1 d^2 + 2a_2 d + 3a_3)A_2),$$
$$A_0 = -\frac{1}{(a_1 d^2 + a_2 d + a_3 - d^3)}(c_0 b d^3 - (a_1 d^2 + 2a_2 d + 3a_3)A_1 + (a_1 d^2 + 4a_2 d + 9a_3)A_2),$$

i.e.,

$$A_2 = -\frac{c_2 b d^3}{(a_1 d^2 + a_2 d + a_3 - d^3)},$$
$$A_1 = -\frac{b d^3}{(a_1 d^2 + a_2 d + a_3 - d^3)^2}(-c_1 d^3 + a_1\left(c_1 + 2c_2\right) d^2 + a_2\left(c_1 + 4c_2\right) d + a_3\left(c_1 + 6c_2\right)),$$

$A_0 = -\frac{bd^3}{(a_1d^2+a_2d+a_3-d^3)^3}(c_0d^6+a_1^2(c_0+c_1+c_2)d^4+a_2^2(c_0+2c_1+4c_2)d^2+a_3^2(c_0+3c_1+9c_2)-a_1(2c_0+c_1-c_2)d^5-2a_2(c_0+c_1-2c_2)d^4-a_3(2c_0+3c_1-9c_2)d^3+a_1a_2(2c_0+3c_1+3c_2)d^3+2a_1a_3(c_0+2c_1+c_2)d^2+a_2a_3(2c_0+5c_1+11c_2)d).$

(ii): *Case $r=1$ (exactly one root equal to d):*

$$W_n^C = n(A_2n^2+A_1n+A_0)d^n$$

where

$$\begin{aligned}
A_2 &= \frac{c_2bd^3}{3\left(a_1d^2+2a_2d+3a_3\right)}, \\
A_1 &= \frac{1}{2\left(a_1d^2+2a_2d+3a_3\right)}(c_1bd^3+3\left(a_1d^2+4a_2d+9a_3\right)A_2), \\
A_0 &= \frac{1}{\left(a_1d^2+2a_2d+3a_3\right)}(c_0bd^3+\left(a_1d^2+4a_2d+9a_3\right)A_1-\left(a_1d^2+8a_2d+27a_3\right)A_2),
\end{aligned}$$

i.e.,

$A_2 = \frac{c_2bd^3}{3\left(a_1d^2+2a_2d+3a_3\right)},$

$A_1 = \frac{bd^3}{2\left(a_1d^2+2a_2d+3a_3\right)^2}(a_1(c_1+c_2)d^2+2a_2(c_1+2c_2)d+3a_3(c_1+3c_2)),$

$A_0 = \frac{bd^3}{6\left(a_1d^2+2a_2d+3a_3\right)^3}(a_1^2(6c_0+3c_1+c_2)d^4+8a_2^2(3c_0+3c_1+2c_2)d^2+27a_3^2(2c_0+3c_1+3c_2)+2a_1a_2(12c_0+9c_1+2c_2)d^3+6a_1a_3(6c_0+6c_1-c_2)d^2+6a_2a_3(12c_0+15c_1+10c_2)d).$

(iii): *Case $r=2$ (exactly two roots equal to d):*

$$W_n^C = n^2(A_2n^2+A_1n+A_0)d^n$$

where

$$\begin{aligned}
A_2 &= -\frac{c_2bd^3}{6\left(a_1d^2+4a_2d+9a_3\right)}, \\
A_1 &= -\frac{1}{3\left(a_1d^2+4a_2d+9a_3\right)}(c_1bd^3-4\left(a_1d^2+8a_2d+27a_3\right)A_2), \\
A_0 &= -\frac{1}{\left(a_1d^2+4a_2d+9a_3\right)}(c_0bd^3-\left(a_1d^2+8a_2d+27a_3\right)A_1+\left(a_1d^2+16a_2d+81a_3\right)A_2),
\end{aligned}$$

i.e.,

$A_2 = -\frac{c_2bd^3}{6\left(a_1d^2+4a_2d+9a_3\right)},$

$A_1 = -\frac{bd^3}{9\left(a_1d^2+4a_2d+9a_3\right)^2}(a_1(3c_1+2c_2)d^2+4a_2(3c_1+4c_2)d+27a_3(c_1+2c_2)),$

$A_0 = -\frac{bd^3}{18\left(a_1d^2+4a_2d+9a_3\right)^3}(a_1^2(18c_0+6c_1+c_2)d^4+32a_2^2(9c_0+6c_1+2c_2)d^2+729a_3^2(2c_0+2c_1+c_2)+4a_1a_2(36c_0+18c_1+c_2)d^3+54a_1a_3(6c_0+4c_1-c_2)d^2+108a_2a_3(12c_0+10c_1+3c_2)d).$

(iv): *Case $r=3$ (all three roots equal to d):*

$$W_n^C = n^3(A_2n^2+A_1n+A_0)d^n$$

where

$$\begin{aligned} A_2 &= \frac{1}{60} b c_2, \\ A_1 &= \frac{1}{24} b (c_1 + 3c_2), \\ A_0 &= \frac{1}{12} b (2c_0 + 3c_1 + 4c_2). \end{aligned}$$

(d): $m = 3$, $s = 3$. *Consider the sequence* (W_n) *defined by*

$$W_n = a_1 W_{n-1} + a_2 W_{n-2} + a_3 W_{n-3} + (c_3 n^3 + c_2 n^2 + c_1 n + c_0) b d^n.$$

(i): *Case* $r = 0$ *(no root equal to* d*):*

$$W_n^C = (A_3 n^3 + A_2 n^2 + A_1 n + A_0) d^n$$

where

$A_3 = -\dfrac{c_3 b d^3}{(a_1 d^2 + a_2 d + a_3 - d^3)}$,

$A_2 = -\dfrac{1}{(a_1 d^2 + a_2 d + a_3 - d^3)} (c_2 b d^3 - 3(a_1 d^2 + 2a_2 d + 3a_3) A_3)$,

$A_1 = -\dfrac{1}{(a_1 d^2 + a_2 d + a_3 - d^3)} (c_1 b d^3 - 2(a_1 d^2 + 2a_2 d + 3a_3) A_2 + 3(a_1 d^2 + 4a_2 d + 9a_3) A_3)$,

$A_0 = -\dfrac{1}{(a_1 d^2 + a_2 d + a_3 - d^3)} (c_0 b d^3 - (a_1 d^2 + 2a_2 d + 3a_3) A_1 + (a_1 d^2 + 4a_2 d + 9a_3) A_2 - (a_1 d^2 +$
$8a_2 d + 27a_3) A_3)$,

i.e.,

$A_3 = -\dfrac{c_3 b d^3}{(a_1 d^2 + a_2 d + a_3 - d^3)}$,

$A_2 = -\dfrac{b d^3}{(a_1 d^2 + a_2 d + a_3 - d^3)^2} (-c_2 d^3 + a_1 (c_2 + 3c_3) d^2 + a_2 (c_2 + 6c_3) d + a_3 (c_2 + 9c_3))$,

$A_1 = -\dfrac{b d^3}{(a_1 d^2 + a_2 d + a_3 - d^3)^3} (c_1 d^6 + a_1^2 (c_1 + 2c_2 + 3c_3) d^4 + a_2^2 (c_1 + 4c_2 + 12c_3) d^2 + a_3^2 (c_1 + 6c_2 + 27c_3)$
$- a_1 (2c_1 + 2c_2 - 3c_3) d^5 - 2a_2 (c_1 + 2c_2 - 6c_3) d^4 - a_3 (2c_1 + 6c_2 - 27c_3) d^3 + a_1 a_2 (2c_1 + 6c_2 + 9c_3) d^3 +$
$2a_1 a_3 (c_1 + 4c_2 + 3c_3) d^2 + a_2 a_3 (2c_1 + 10c_2 + 33c_3) d)$,

$A_0 = -\dfrac{b d^3}{(a_1 d^2 + a_2 d + a_3 - d^3)^4} (-c_0 d^9 + a_1^3 (c_0 + c_1 + c_2 + c_3) d^6 + a_2^3 (c_0 + 2c_1 + 4c_2 + 8c_3) d^3 +$
$a_3^3 (c_0 + 3c_1 + 9c_2 + 27c_3) - a_1^2 (3c_0 + 2c_1 - 4c_3) d^7 - a_2^2 (3c_0 + 4c_1 - 32c_3) d^5 - 3a_3^2 (c_0 + 2c_1 - 36c_3) d^3 +$
$a_1 (3c_0 + c_1 - c_2 + c_3) d^8 + a_2 (3c_0 + 2c_1 - 4c_2 + 8c_3) d^7 + 3a_3 (c_0 + c_1 - 3c_2 + 9c_3) d^6 - 2a_1 a_2 (3c_0 + 3c_1 - c_2 -$
$9c_3) d^6 - 2a_1 a_3 (3c_0 + 4c_1 - 4c_2 - 8c_3) d^5 - 2a_2 a_3 (3c_0 + 5c_1 - c_2 - 55c_3) d^4 + a_1 a_2^2 (3c_0 + 5c_1 + 7c_2 + 5c_3)$
$d^4 + a_1^2 a_2 (3c_0 + 4c_1 + 4c_2 + 4c_3) d^5 + a_1 a_3^2 (3c_0 + 7c_1 + 11c_2 - 17c_3) d^2 + a_1^2 a_3 (3c_0 + 5c_1 + 3c_2 + 5c_3)$
$d^4 + a_2 a_3^2 (3c_0 + 8c_1 + 20c_2 + 44c_3) d + a_2^2 a_3 (3c_0 + 7c_1 + 15c_2 + 31c_3) d^2 + 2a_1 a_2 a_3 (3c_0 + 6c_1 + 8c_2) d^3)$.

(ii): *Case* $r = 1$ *(exactly one root equal to* d*):*

$$W_n^C = n(A_3 n^3 + A_2 n^2 + A_1 n + A_0) d^n$$

where

$A_3 = \dfrac{c_3 b d^3}{4 (a_1 d^2 + 2a_2 d + 3a_3)}$,

$A_2 = \dfrac{1}{3 (a_1 d^2 + 2a_2 d + 3a_3)} (c_2 b d^3 + 6(a_1 d^2 + 4a_2 d + 9a_3) A_3)$,

$A_1 = \frac{1}{2\,(a_1d^2+2a_2d+3a_3)}(c_1bd^3+3(a_1d^2+4a_2d+9a_3)A_2-4(a_1d^2+8a_2d+27a_3)A_3),$
$A_0 = \frac{1}{(a_1d^2+2a_2d+3a_3)}(c_0bd^3+(a_1d^2+4a_2d+9a_3)A_1-(a_1d^2+8a_2d+27a_3)A_2+(a_1d^2+16a_2d+81a_3)A_3),$

i.e.,

$A_3 = \frac{c_3bd^3}{4\,(a_1d^2+2a_2d+3a_3)},$
$A_2 = \frac{bd^3}{6\,(a_1d^2+2a_2d+3a_3)^2}(a_1(2c_2+3c_3)d^2+4a_2(c_2+3c_3)d+3a_3(2c_2+9c_3)),$
$A_1 = \frac{bd^3}{4\,(a_1d^2+2a_2d+3a_3)^3}(a_1^2(2c_1+2c_2+c_3)d^4+8a_2^2(c_1+2c_2+2c_3)d^2+9a_3^2(2c_1+6c_2+9c_3)+4a_1a_2(2c_1+3c_2+c_3)d^3+6a_1a_3(2c_1+4c_2-c_3)d^2+12a_2a_3(2c_1+5c_2+5c_3)d),$
$A_0 = \frac{bd^3}{6\,(a_1d^2+2a_2d+3a_3)^4}(a_1^3(6c_0+3c_1+c_2)d^6+16a_2^3(3c_0+3c_1+2c_2)d^3+81a_3^3(2c_0+3c_1+3c_2)+6a_1a_2^2(12c_0+10c_1+4c_2-3c_3)d^4+6a_1^2a_2(6c_0+4c_1+c_2)d^5+9a_1a_3^2(18c_0+21c_1+7c_2-30c_3)d^2+3a_1^2a_3(18c_0+15c_1-c_2+6c_3)d^4+18a_2a_3^2(18c_0+24c_1+19c_2-6c_3)d+6a_2^2a_3(36c_0+42c_1+28c_2-3c_3)d^2+12a_1a_2a_3(18c_0+18c_1+5c_2-9c_3)d^3).$

(iii): *Case $r=2$ (exactly two roots equal to d):*

$$W_n^C = n^2(A_3n^3+A_2n^2+A_1n+A_0)d^n$$

where

$A_3 = -\frac{c_3bd^3}{10\,(a_1d^2+4a_2d+9a_3)},$
$A_2 = -\frac{1}{6\,(a_1d^2+4a_2d+9a_3)}(c_2bd^3-10(a_1d^2+8a_2d+27a_3)A_3),$
$A_1 = -\frac{1}{3\,(a_1d^2+4a_2d+9a_3)}(c_1bd^3-4(a_1d^2+8a_2d+27a_3)A_2+5(a_1d^2+16a_2d+81a_3)A_3),$
$A_0 = -\frac{1}{(a_1d^2+4a_2d+9a_3)}(c_0bd^3-(a_1d^2+8a_2d+27a_3)A_1+(a_1d^2+16a_2d+81a_3)A_2-(a_1d^2+32a_2d+243a_3)A_3),$

i.e.,

$A_3 = -\frac{c_3bd^3}{10\,(a_1d^2+4a_2d+9a_3)},$
$A_2 = -\frac{bd^3}{6\,(a_1d^2+4a_2d+9a_3)^2}(a_1(c_2+c_3)d^2+4a_2(c_2+2c_3)d+9a_3(c_2+3c_3)),$

$A_1 = -\frac{bd^3}{18\,(a_1d^2+4a_2d+9a_3)^3}(a_1^2(6c_1+4c_2+c_3)d^4+32a_2^2(3c_1+4c_2+2c_3)d^2+243a_3^2(2c_1+4c_2+3c_3)+4a_1a_2(12c_1+12c_2+c_3)d^3+18a_1a_3(6c_1+8c_2-3c_3)d^2+36a_2a_3(12c_1+20c_2+9c_3)d),$
$A_0 = -\frac{bd^3}{90\,(a_1d^2+4a_2d+9a_3)^4}(a_1^3(90c_0+30c_1+5c_2-c_3)d^6+128a_2^3(45c_0+30c_1+10c_2-4c_3)d^3+6561a_3^3(10c_0+10c_1+5c_2-3c_3)+16a_1a_2^2(270c_0+150c_1+25c_2-27c_3)d^4+40a_1^2a_2(27c_0+12c_1+c_2)d^5+1215a_1a_3^2(18c_0+14c_1+c_2-9c_3)d^2+9a_1^2a_3(270c_0+150c_1-25c_2+51c_3)d^4+1944a_2a_3^2(45c_0+40c_1+15c_2-12c_3)d+144a_2^2a_3(270c_0+210c_1+65c_2-33c_3)d^2+144a_1a_2a_3(135c_0+90c_1+5c_2-18c_3)d^3).$

(iv): *Case $r=3$ (all three roots equal to d):*

$$W_n^C = n^3(A_3n^3+A_2n^2+A_1n+A_0)d^n$$

where

$$\begin{aligned} A_3 &= \frac{1}{120} bc_3, \\ A_2 &= \frac{1}{120} b(2c_2 + 9c_3), \\ A_1 &= \frac{1}{24} b(c_1 + 3c_2 + 6c_3), \\ A_0 &= \frac{1}{24} b(4c_0 + 6c_1 + 8c_2 + 9c_3). \end{aligned}$$

2.2. The Case $m = 4$. Consider the homogeneous relation

$$V_n = a_1 V_{n-1} + a_2 V_{n-2} + a_3 V_{n-3} + a_4 V_{n-4} \tag{2.3}$$

with the initial conditions V_0, V_1, V_2, V_3. Suppose that $\theta_1, \theta_2, \theta_3, \theta_4$ are the roots of characteristic equation

$$z^4 - a_1 z^3 - a_2 z^2 - a_3 z - a_4 = 0 \tag{2.4}$$

associated with (2.3). Note that if all the roots of (2.4) are equal to d then

$$z^4 - a_1 z^3 - a_2 z^2 - a_3 z - a_4 = (z - d)^4 = z^4 - 4dz^3 + 6d^2 z^2 - 4d^3 z + d^4 = 0$$

so that $a_1 = 4d$, $a_2 = -6d^2$, $a_3 = 4d^3$, $a_4 = -d^4$ and (2.3) reduces to

$$V_n = 4dV_{n-1} - 6d^2 V_{n-2} + 4d^3 V_{n-3} - d^4 V_{n-4}.$$

EXAMPLE 2.2. *Consider the sequence (W_n) defined by the recurrence relation*

$$W_n = a_1 W_{n-1} + a_2 W_{n-2} + a_3 W_{n-3} + a_4 W_{n-4} + p(n) bd^n$$

where $p(n) := p(n, x)$ is a polynomial in n of order s, with coefficients belonging to $\mathbb{C}[x]$ or $\mathbb{C}$ and $b \in \mathbb{C}[x]$ or $\mathbb{C}$, and $d \in \mathbb{C}$ or $\mathbb{R}$. We seek a particular solution

$$W_n^C = P(n) d^n$$

for the cases $s = 0, 1, 2, 3$ where $P(n)$ is itself a polynomial in n. The order (degree) and coffecients of $P(n)$ depend on the multiplicity r of d as a root of the characteristic equation (2.4) and W_n^C satisfy

$$W_n^C = a_1 W_{n-1}^C + a_2 W_{n-2}^C + a_3 W_{n-3}^C + a_4 W_{n-4}^C + p(n) bd^n$$

i.e.,

$$P(n) d^n = a_1 P(n-1) d^{n-1} + a_2 P(n-2) d^{n-2} + a_3 P(n-3) d^{n-3} + a_4 P(n-4) d^{n-4} + p(n) bd^n.$$

In each case of s, we consider the homogeneous relation (2.3) and its characteristic equation (2.4), corresponding to the sequence (W_n) with the same initial conditions as W_n, i.e.,

$$V_0 = W_0, V_1 = W_1, V_2 = W_2, V_3 = W_3.$$

We investigate all cases of multiplicity r of of d as a root of the characteristic equation (2.4):

(a): $m = 4$, $s = 0$. *Consider the sequence (W_n) defined by*

$$W_n = a_1 W_{n-1} + a_2 W_{n-2} + a_3 W_{n-3} + a_4 W_{n-4} + c_0 bd^n.$$

(i): *Case $r = 0$, i.e., none of the roots of the characteristic equation equals d:*

$$W_n^C = A_0 d^n, \quad A_0 = -\frac{c_0 b d^4}{a_1 d^3 + a_2 d^2 + a_3 d + a_4 - d^4}.$$

(ii): *Case $r = 1$, i.e., exactly one root of the characteristic equation equals d:*

$$W_n^C = n A_0 d^n, \quad A_0 = \frac{c_0 b d^4}{(a_1 d^3 + 2a_2 d^2 + 3a_3 d + 4a_4)}.$$

(iii): *Case $r = 2$, i.e., exactly two roots of the characteristic equation equal d:*

$$W_n^C = n^2 A_0 d^n, \quad A_0 = -\frac{b c_0 d^4}{(a_1 d^3 + 4a_2 d^2 + 9a_3 d + 16a_4)}.$$

(iv): *Case $r = 3$, i.e., exactly three roots of the characteristic equation equal d:*

$$W_n^C = n^3 A_0 d^n, \quad A_0 = \frac{c_0 b d^4}{(a_1 d^3 + 8a_2 d^2 + 27a_3 d + 64a_4)}.$$

(v): *Case $r = 4$, i.e., all four roots of the characteristic equation equal d:*

$$W_n^C = n^4 A_0 d^n, \quad A_0 = \frac{1}{24} b c_0.$$

(b): $m = 4$, $s = 1$. *Consider the sequence (W_n) defined by*

$$W_n = a_1 W_{n-1} + a_2 W_{n-2} + a_3 W_{n-3} + a_4 W_{n-4} + (c_1 n + c_0) b d^n.$$

(i): *Case $r = 0$ (no root equal to d):*

$$W_n^C = (A_1 n + A_0) d^n$$

where

$$\begin{aligned} A_1 &= -\frac{c_1 b d^4}{(a_1 d^3 + a_2 d^2 + a_3 d + a_4 - d^4)}, \\ A_0 &= -\frac{1}{(a_1 d^3 + a_2 d^2 + a_3 d + a_4 - d^4)} (c_0 b d^4 - (a_1 d^3 + 2a_2 d^2 + 3a_3 d + 4a_4) A_1), \end{aligned}$$

i.e.,

$$A_1 = -\frac{c_1 b d^4}{(a_1 d^3 + a_2 d^2 + a_3 d + a_4 - d^4)},$$

$$A_0 = -\frac{b d^4}{(a_1 d^3 + a_2 d^2 + a_3 d + a_4 - d^4)^2} (-c_0 d^4 + a_1 (c_0 + c_1) d^3 + a_2 (c_0 + 2c_1) d^2 + a_3 (c_0 + 3c_1) d + a_4 (c_0 + 4c_1)).$$

(ii): *Case $r = 1$ (exactly one root equal to d):*

$$W_n^C = n(A_1 n + A_0) d^n$$

where

$$\begin{aligned} A_1 &= \frac{c_1 b d^4}{2\left(d^3 a_1 + 2a_2 d^2 + 3d a_3 + 4a_4\right)}, \\ A_0 &= \frac{1}{\left(d^3 a_1 + 2a_2 d^2 + 3d a_3 + 4a_4\right)} (c_0 b d^4 + \left(d^3 a_1 + 4d^2 a_2 + 9d a_3 + 16a_4\right) A_1), \end{aligned}$$

i.e.,

$$A_1 = \frac{c_1 b d^4}{2\left(d^3 a_1 + 2a_2 d^2 + 3d a_3 + 4a_4\right)},$$

$$A_0 = \frac{b d^4}{2\left(d^3 a_1 + 2a_2 d^2 + 3d a_3 + 4a_4\right)^2} (a_1 (2c_0 + c_1) d^3 + 4a_2 (c_0 + c_1) d^2 + 3a_3 (2c_0 + 3c_1) d + 8a_4 (c_0 + 2c_1)).$$

(iii): *Case* $r = 2$ *(exactly two roots equal to* d*):*

$$W_n^C = n^2(A_1 n + A_0)d^n$$

where

$$A_1 = -\frac{c_1 bd^4}{3\left(a_1 d^3 + 4a_2 d^2 + 9a_3 d + 16a_4\right)},$$
$$A_0 = -\frac{1}{\left(a_1 d^3 + 4a_2 d^2 + 9a_3 d + 16a_4\right)}\left(c_0 bd^4 - \left(a_1 d^3 + 8a_2 d^2 + 27a_3 d + 64a_4\right) A_1\right),$$

i.e.,

$$A_1 = -\frac{c_1 bd^4}{3\left(a_1 d^3 + 4a_2 d^2 + 9a_3 d + 16a_4\right)},$$
$$A_0 = -\frac{bd^4}{3\left(a_1 d^3 + 4a_2 d^2 + 9a_3 d + 16a_4\right)^2}\left(a_1(3c_0 + c_1)d^3 + 4a_2(3c_0 + 2c_1)d^2 + 27a_3(c_0 + c_1)d + 16a_4(3c_0 + 4c_1)\right).$$

(iv): *Case* $r = 3$ *(exactly three roots equal to* d*):*

$$W_n^C = n^3(A_1 n + A_0)d^n$$

where

$$A_1 = \frac{c_1 bd^4}{4\left(a_1 d^3 + 8a_2 d^2 + 27a_3 d + 64a_4\right)},$$
$$A_0 = \frac{1}{\left(a_1 d^3 + 8a_2 d^2 + 27a_3 d + 64a_4\right)}\left(c_0 bd^4 + (a_1 d^3 + 16a_2 d^2 + 81a_3 d + 256a_4)A_1\right),$$

i.e.,

$$A_1 = \frac{c_1 bd^4}{4(a_1 d^3 + 8a_2 d^2 + 27a_3 d + 64a_4)},$$
$$A_0 = \frac{bd^4}{4(a_1 d^3 + 8a_2 d^2 + 27a_3 d + 64a_4)^2}\left(a_1(4c_0 + c_1)d^3 + 16a_2(2c_0 + c_1)d^2 + 27a_3(4c_0 + 3c_1)d + 256a_4(c_0 + c_1)\right).$$

(v): *Case* $r = 4$ *(all four roots equal to* d*):*

$$W_n^C = n^4(A_1 n + A_0)d^n$$

where

$$A_1 = \frac{1}{120} bc_1$$
$$A_0 = \frac{1}{24} b(c_0 + 2c_1)$$

(c): $m = 4,\ s = 2$. *Consider the sequence* (W_n) *defined by*

$$W_n = a_1 W_{n-1} + a_2 W_{n-2} + a_3 W_{n-3} + a_4 W_{n-4} + (c_2 n^2 + c_1 n + c_0)bd^n.$$

(i): *Case* $r = 0$ *(no root equal to* d*):*

$$W_n^C = (A_2 n^2 + A_1 n + A_0)d^n$$

where

$$A_2 = -\frac{c_2 bd^4}{(a_1 d^3 + a_2 d^2 + a_3 d + a_4 - d^4)},$$
$$A_1 = -\frac{1}{(a_1 d^3 + a_2 d^2 + a_3 d + a_4 - d^4)}(c_1 bd^4 - 2(a_1 d^3 + 2a_2 d^2 + 3a_3 d + 4a_4)A_2),$$

$A_0 = -\frac{1}{(a_1d^3+a_2d^2+a_3d+a_4-d^4)}(c_0bd^4-(a_1d^3+2a_2d^2+3a_3d+4a_4)A_1+(a_1d^3+4a_2d^2+9a_3d+16a_4)A_2),$

i.e.,

$A_2 = -\frac{c_2bd^4}{(a_1d^3+a_2d^2+a_3d+a_4-d^4)},$

$A_1 = -\frac{bd^4}{(a_1d^3+a_2d^2+a_3d+a_4-d^4)^2}(-c_1d^4+a_1(c_1+2c_2)d^3+a_2(c_1+4c_2)d^2+a_3(c_1+6c_2)d+a_4(c_1+8c_2)),$

$A_0 = -\frac{bd^4}{(a_1d^3+a_2d^2+a_3d+a_4-d^4)^3}(c_0d^8+a_1^2(c_0+c_1+c_2)d^6+a_2^2(c_0+2c_1+4c_2)d^4+a_3^2(c_0+3c_1+9c_2)d^2+a_4^2(c_0+4c_1+16c_2)-a_1(2c_0+c_1-c_2)d^7-2a_2(c_0+c_1-2c_2)d^6-a_3(2c_0+3c_1-9c_2)d^5-2a_4(c_0+2c_1-8c_2)d^4+a_1a_2(2c_0+3c_1+3c_2)d^5+2a_1a_3(c_0+2c_1+c_2)d^4+a_1a_4(2c_0+5c_1-c_2)d^3+a_2a_3(2c_0+5c_1+11c_2)d^3+2a_2a_4(c_0+3c_1+6c_2)d^2+a_3a_4(2c_0+7c_1+23c_2)d).$

(ii): *Case $r=1$ (exactly one root equal to d):*

$$W_n^C = n(A_2n^2+A_1n+A_0)d^n$$

where

$$A_2 = \frac{c_2bd^4}{3(a_1d^3+2a_2d^2+3a_3d+4a_4)},$$
$$A_1 = \frac{1}{2(a_1d^3+2a_2d^2+3a_3d+4a_4)}(c_1bd^4+3\left(d^3a_1+4d^2a_2+9da_3+16a_4\right)A_2),$$
$$A_0 = \frac{1}{(a_1d^3+2a_2d^2+3a_3d+4a_4)}(c_0bd^4+(d^3a_1+4d^2a_2+9da_3+16a_4)A_1-(d^3a_1+8d^2a_2+27da_3+64a_4)A_2),$$

i.e.,

$A_2 = \frac{c_2bd^4}{3(a_1d^3+2a_2d^2+3a_3d+4a_4)},$

$A_1 = \frac{bd^4}{2(a_1d^3+2a_2d^2+3a_3d+4a_4)^2}(a_1(c_1+c_2)d^3+2a_2(c_1+2c_2)d^2+3a_3(c_1+3c_2)d+4a_4(c_1+4c_2)),$

$A_0 = \frac{bd^4}{6(a_1d^3+2a_2d^2+3a_3d+4a_4)^3}(a_1^2(6c_0+3c_1+c_2)d^6+8a_2^2(3c_0+3c_1+2c_2)d^4+27a_3^2(2c_0+3c_1+3c_2)d^2+32a_4^2(3c_0+6c_1+8c_2)+2a_1a_2(12c_0+9c_1+2c_2)d^5+6a_1a_3(6c_0+6c_1-c_2)d^4+4a_1a_4(12c_0+15c_1-10c_2)d^3+6a_2a_3(12c_0+15c_1+10c_2)d^3+16a_2a_4(6c_0+9c_1+4c_2)d^2+12a_3a_4(12c_0+21c_1+22c_2)d).$

(iii): *Case $r=2$ (exactly two roots equal to d):*

$$W_n^C = n^2(A_2n^2+A_1n+A_0)d^n$$

where

$A_2 = -\frac{c_2bd^4}{6(a_1d^3+4a_2d^2+9a_3d+16a_4)},$

$A_1 = -\frac{1}{3(a_1d^3+4a_2d^2+9a_3d+16a_4)}(c_1bd^4-4\left(a_1d^3+8a_2d^2+27a_3d+64a_4\right)A_2),$

$A_0 = -\frac{1}{(a_1d^3+4a_2d^2+9a_3d+16a_4)}(c_0bd^4-(a_1d^3+8a_2d^2+27a_3d+64a_4)A_1+(a_1d^3+16a_2d^2+81a_3d+256a_4)A_2),$

i.e.,

$A_2 = -\frac{c_2bd^4}{6(a_1d^3+4a_2d^2+9a_3d+16a_4)},$

$A_1 = -\frac{bd^4}{9\left(16a_4+9da_3+4d^2a_2+d^3a_1\right)^2}(a_1(3c_1+2c_2)d^3+4a_2(3c_1+4c_2)d^2+27a_3(c_1+2c_2)d+16a_4(3c_1+8c_2)),$

$A_0 = -\frac{bd^4}{18\left(16a_4+9da_3+4d^2a_2+d^3a_1\right)^3}(a_1^2(18c_0+6c_1+c_2)d^6+32a_2^2(9c_0+6c_1+2c_2)d^4+729a_3^2(2c_0+2c_1+c_2)d^2+512a_4^2(9c_0+12c_1+8c_2)+4a_1a_2(36c_0+18c_1+c_2)d^5+54a_1a_3(6c_0+4c_1-c_2)d^4+16a_1a_4(36c_0+30c_1-19c_2)d^3+108a_2a_3(12c_0+10c_1+3c_2)d^3+256a_2a_4(9c_0+9c_1+c_2)d^2+432a_3a_4(12c_0+14c_1+7c_2)d).$

(iv): *Case* $r=3$ *(exactly three roots equal to* d*):*

$$W_n^C = n^3(A_2n^2+A_1n+A_0)d^n$$

where

$A_2 = \frac{c_2bd^4}{10\left(a_1d^3+8a_2d^2+27a_3d+64a_4\right)},$

$A_1 = \frac{1}{4\left(a_1d^3+8a_2d^2+27a_3d+64a_4\right)}(c_1bd^4+5(a_1d^3+16a_2d^2+81a_3d+256a_4)A_2),$

$A_0 = \frac{1}{\left(a_1d^3+8a_2d^2+27a_3d+64a_4\right)}(c_0bd^4+(a_1d^3+16a_2d^2+81a_3d+256a_4)A_1-(a_1d^3+32a_2d^2+243a_3d+1024a_4)A_2),$

i.e.,

$A_2 = \frac{c_2bd^4}{10\left(a_1d^3+8a_2d^2+27a_3d+64a_4\right)},$

$A_1 = \frac{bd^4}{8\left(a_1d^3+8a_2d^2+27a_3d+64a_4\right)^2}(a_1(2c_1+c_2)d^3+16a_2(c_1+c_2)d^2+27a_3(2c_1+3c_2)d+128a_4(c_1+2c_2)),$

$A_0 = \frac{bd^4}{40\left(a_1d^3+8a_2d^2+27a_3d+64a_4\right)^3}(a_1^2(40c_0+10c_1+c_2)d^6+256a_2^2(10c_0+5c_1+c_2)d^4+729a_3^2(40c_0+30c_1+9c_2)d^2+32768a_4^2(5c_0+5c_1+2c_2)+80a_1a_2(8c_0+3c_1)d^5+270a_1a_3(8c_0+4c_1-c_2)d^4+128a_1a_4(40c_0+25c_1-14c_2)d^3+432a_2a_3(40c_0+25c_1+4c_2)d^3+10240a_2a_4(4c_0+3c_1)d^2+17280a_3a_4(8c_0+7c_1+2c_2)d).$

(v): *Case* $r=4$ *(all four roots equal to* d*):*

$$W_n^C = n^4(A_2n^2+A_1n+A_0)d^n$$

where

$$A_2 = \frac{1}{360}bc_2,$$
$$A_1 = \frac{1}{120}b(c_1+4c_2),$$
$$A_0 = \frac{1}{72}b(3c_0+6c_1+11c_2).$$

(d): $m=4$, $s=3$. *Consider the sequence* (W_n) *defined by*

$$W_n = a_1W_{n-1}+a_2W_{n-2}+a_3W_{n-3}+a_4W_{n-4}+(c_3n^3+c_2n^2+c_1n+c_0)bd^n.$$

(i): *Case* $r=0$ *(no root equal to* d*):*

$$W_n^C = (A_3n^3+A_2n^2+A_1n+A_0)d^n$$

where

$A_3 = -\frac{c_3bd^4}{(a_1d^3+a_2d^2+a_3d+a_4-d^4)},$

$A_2 = -\frac{1}{(a_1d^3+a_2d^2+a_3d+a_4-d^4)}(c_2bd^4 - 3(a_1d^3+2a_2d^2+3a_3d+4a_4)A_3),$

$A_1 = -\frac{1}{(a_1d^3+a_2d^2+a_3d+a_4-d^4)}(c_1bd^4 - 2(a_1d^3+2a_2d^2+3a_3d+4a_4)A_2 + 3(a_1d^3+4a_2d^2+9a_3d+16a_4)A_3),$

$A_0 = -\frac{1}{(a_1d^3+a_2d^2+a_3d+a_4-d^4)}(c_0bd^4 - (a_1d^3+2a_2d^2+3a_3d+4a_4)A_1 + (a_1d^3+4a_2d^2+9a_3d+16a_4)A_2 - (a_1d^3+8a_2d^2+27a_3d+64a_4)A_3),$

i.e.,

$A_3 = -\frac{c_3bd^4}{(a_1d^3+a_2d^2+a_3d+a_4-d^4)},$

$A_2 = -\frac{bd^4}{(a_1d^3+a_2d^2+a_3d+a_4-d^4)^2}(-c_2d^4 + a_1(c_2+3c_3)d^3 + a_2(c_2+6c_3)d^2 + a_3(c_2+9c_3)d + (c_2+12c_3)a_4),$

$A_1 = -\frac{bd^4}{(a_1d^3+a_2d^2+a_3d+a_4-d^4)^3}(c_1d^8 + a_1^2(c_1+2c_2+3c_3)d^6 + a_2^2(c_1+4c_2+12c_3)d^4 + a_3^2(c_1+6c_2+27c_3)d^2 + a_4^2(c_1+8c_2+48c_3) - a_1(2c_1+2c_2-3c_3)d^7 - 2a_2(c_1+2c_2-6c_3)d^6 - a_3(2c_1+6c_2-27c_3)d^5 - 2a_4(c_1+4c_2-24c_3)d^4 + a_1a_2(2c_1+6c_2+9c_3)d^5 + 2a_1a_3(c_1+4c_2+3c_3)d^4 + a_1a_4(2c_1+10c_2-3c_3)d^3 + a_2a_3(2c_1+10c_2+33c_3)d^3 + 2a_2a_4(c_1+6c_2+18c_3)d^2 + a_3a_4(2c_1+14c_2+69c_3)d),$

$A_0 = -\frac{bd^4}{(a_1d^3+a_2d^2+a_3d+a_4-d^4)^4}(-c_0d^{12} + a_1^3(c_0+c_1+c_2+c_3)d^9 + a_2^3(c_0+2c_1+4c_2+8c_3)d^6 + a_3^3(c_0+3c_1+9c_2+27c_3)d^3 + a_4^3(c_0+4c_1+16c_2+64c_3) - a_1^2(3c_0+2c_1-4c_3)d^{10} - a_2^2(3c_0+4c_1-32c_3)d^8 - 3a_3^2(c_0+2c_1-36c_3)d^6 - a_4^2(3c_0+8c_1-256c_3)d^4 + a_1(3c_0+c_1-c_2+c_3)d^{11} + a_2(3c_0+2c_1-4c_2+8c_3)d^{10} + 3a_3(c_0+c_1-3c_2+9c_3)d^9 + a_4(3c_0+4c_1-16c_2+64c_3)d^8 - 2a_1a_2(3c_0+3c_1-c_2-9c_3)d^9 - 2a_1a_3(3c_0+4c_1-4c_2-8c_3)d^8 - 2a_1a_4(3c_0+5c_1-9c_2+5c_3)d^7 - 2a_2a_3(3c_0+5c_1-c_2-55c_3)d^7 - 2a_2a_4(3c_0+6c_1-4c_2-72c_3)d^6 - 2a_3a_4(3c_0+7c_1-c_2-161c_3)d^5 + a_1a_2^2(3c_0+5c_1+7c_2+5c_3)d^7 + a_1^2a_2(3c_0+4c_1+4c_2+4c_3)d^8 + a_1a_3^2(3c_0+7c_1+11c_2-17c_3)d^5 + a_1^2a_3(3c_0+5c_1+3c_2+5c_3)d^7 + 3a_1a_4^2(c_0+3c_1+5c_2-29c_3)d^3 + 3a_1^2a_4(c_0+2c_1+4c_3)d^6 + a_2a_3^2(3c_0+8c_1+20c_2+44c_3)d^4 + a_2^2a_3(3c_0+7c_1+15c_2+31c_3)d^5 + a_2a_4^2(3c_0+10c_1+28c_2+40c_3)d^2 + a_2^2a_4(3c_0+8c_1+16c_2+32c_3)d^4 + a_3a_4^2(3c_0+11c_1+39c_2+131c_3)d + a_3^2a_4(3c_0+10c_1+32c_2+100c_3)d^2 + 2a_1a_2a_3(3c_0+6c_1+8c_2)d^6 + 2a_1a_2a_4(3c_0+7c_1+7c_2-5c_3)d^5 + 2a_1a_3a_4(3c_0+8c_1+12c_2-40c_3)d^4 + 2a_2a_3a_4(3c_0+9c_1+23c_2+45c_3)d^3),$

(ii): *Case $r = 1$ (exactly one root equal to d):*

$$W_n^C = n(A_3n^3 + A_2n^2 + A_1n + A_0)d^n$$

where

$A_3 = \frac{c_3bd^4}{4\,(a_1d^3+2a_2d^2+3a_3d+4a_4)},$

$A_2 = \frac{1}{3\,(a_1d^3+2a_2d^2+3a_3d+4a_4)}(c_2bd^4 + 6(d^3a_1+4d^2a_2+9da_3+16a_4)A_3),$

$A_1 = \frac{1}{2\,(a_1d^3+2a_2d^2+3a_3d+4a_4)}(c_1bd^4 + 3(d^3a_1+4d^2a_2+9da_3+16a_4)A_2 - 4(a_1d^3+8a_2d^2+27a_3d+64a_4)A_3),$

$$A_0 = \frac{1}{(a_1d^3 + 2a_2d^2 + 3a_3d + 4a_4)}(c_0bd^4 + (d^3a_1 + 4d^2a_2 + 9da_3 + 16a_4)A_1 - (a_1d^3 + 8a_2d^2 + 27a_3d + 64a_4)A_2 + (a_1d^3 + 16a_2d^2 + 81a_3d + 256a_4)A_3),$$

i.e.,

$$A_3 = \frac{c_3bd^4}{4\,(a_1d^3 + 2a_2d^2 + 3a_3d + 4a_4)},$$

$$A_2 = \frac{bd^4}{6\,(a_1d^3 + 2a_2d^2 + 3a_3d + 4a_4)^2}(a_1(2c_2 + 3c_3)d^3 + 4a_2(c_2 + 3c_3)d^2 + 3a_3(2c_2 + 9c_3)d + 8a_4(c_2 + 6c_3)),$$

$$A_1 = \frac{bd^4}{4\,(a_1d^3 + 2a_2d^2 + 3a_3d + 4a_4)^3}(a_1^2(2c_1 + 2c_2 + c_3)d^6 + 8a_2^2(c_1 + 2c_2 + 2c_3)d^4 + 9a_3^2(2c_1 + 6c_2 + 9c_3)d^2 + 32a_4^2(c_1 + 4c_2 + 8c_3) + 4a_1a_2(2c_1 + 3c_2 + c_3)d^5 + 6a_1a_3(2c_1 + 4c_2 - c_3)d^4 + 8a_1a_4(2c_1 + 5c_2 - 5c_3)d^3 + 12a_2a_3(2c_1 + 5c_2 + 5c_3)d^3 + 32a_2a_4(c_1 + 3c_2 + 2c_3)d^2 + 24a_3a_4(2c_1 + 7c_2 + 11c_3)d),$$

$$A_0 = \frac{bd^4}{6\,(a_1d^3 + 2a_2d^2 + 3a_3d + 4a_4)^4}(a_1^3(6c_0 + 3c_1 + c_2)d^9 + 16a_2^3(3c_0 + 3c_1 + 2c_2)d^6 + 81a_3^3(2c_0 + 3c_1 + 3c_2)d^3 + 128a_4^3(3c_0 + 6c_1 + 8c_2) + 6a_1a_2^2(12c_0 + 10c_1 + 4c_2 - 3c_3)d^7 + 6a_1^2a_2(6c_0 + 4c_1 + c_2)d^8 + 9a_1a_3^2(18c_0 + 21c_1 + 7c_2 - 30c_3)d^5 + 3a_1^2a_3(18c_0 + 15c_1 - c_2 + 6c_3)d^7 + 24a_1a_4^2(12c_0 + 18c_1 + 4c_2 - 63c_3)d^3 + 36a_1^2a_4(2c_0 + 2c_1 - c_2 + 3c_3)d^6 + 18a_2a_3^2(18c_0 + 24c_1 + 19c_2 - 6c_3)d^4 + 6a_2^2a_3(36c_0 + 42c_1 + 28c_2 - 3c_3)d^5 + 192a_2a_4^2(3c_0 + 5c_1 + 4c_2 - 6c_3)d^2 + 96a_2^2a_4(3c_0 + 4c_1 + 2c_2)d^4 + 24a_3a_4^2(36c_0 + 66c_1 + 76c_2 - 15c_3)d + 36a_3^2a_4(18c_0 + 30c_1 + 31c_2 - 3c_3)d^2 + 12a_1a_2a_3(18c_0 + 18c_1 + 5c_2 - 9c_3)d^6 + 24a_1a_2a_4(12c_0 + 14c_1 - 3c_3)d^5 + 24a_1a_3a_4(18c_0 + 24c_1 + 5c_2 - 48c_3)d^4 + 24a_2a_3a_4(36c_0 + 54c_1 + 40c_2 - 27c_3)d^3).$$

(iii): *Case $r = 2$ (exactly two roots equal to d):*

$$W_n^C = n^2(A_3n^3 + A_2n^2 + A_1n + A_0)d^n$$

where

$$A_3 = -\frac{c_3bd^4}{10\,(a_1d^3 + 4a_2d^2 + 9a_3d + 16a_4)},$$

$$A_2 = -\frac{1}{6\,(a_1d^3 + 4a_2d^2 + 9a_3d + 16a_4)}(c_2bd^4 - 10(a_1d^3 + 8a_2d^2 + 27a_3d + 64a_4)A_3),$$

$$A_1 = -\frac{1}{3\,(a_1d^3 + 4a_2d^2 + 9a_3d + 16a_4)}(c_1bd^4 - 4(a_1d^3 + 8a_2d^2 + 27a_3d + 64a_4)A_2 + 5(a_1d^3 + 16a_2d^2 + 81a_3d + 256a_4)A_3),$$

$$A_0 = -\frac{1}{(a_1d^3 + 4a_2d^2 + 9a_3d + 16a_4)}(c_0bd^4 - (a_1d^3 + 8a_2d^2 + 27a_3d + 64a_4)A_1 + (a_1d^3 + 16a_2d^2 + 81a_3d + 256a_4)A_2 - (a_1d^3 + 32a_2d^2 + 243a_3d + 1024a_4)A_3),$$

i.e.,

$$A_3 = -\frac{c_3bd^4}{10\,(a_1d^3 + 4a_2d^2 + 9a_3d + 16a_4)},$$

$$A_2 = -\frac{bd^4}{6\,(a_1d^3 + 4a_2d^2 + 9a_3d + 16a_4)^2}(a_1(c_2 + c_3)d^3 + 4a_2(c_2 + 2c_3)d^2 + 9a_3(c_2 + 3c_3)d + 16a_4(c_2 + 4c_3)),$$

$$A_1 = -\frac{bd^4}{18\,(a_1d^3 + 4a_2d^2 + 9a_3d + 16a_4)^3}(a_1^2(6c_1 + 4c_2 + c_3)d^6 + 32a_2^2(3c_1 + 4c_2 + 2c_3)d^4 + 243a_3^2(2c_1 + 4c_2 + 3c_3)d^2 + 512a_4^2(3c_1 + 8c_2 + 8c_3) + 4a_1a_2(12c_1 + 12c_2 + c_3)d^5 + 18a_1a_3(6c_1 + 8c_2 - 3c_3)d^4 + 16a_1a_4(12c_1 + 20c_2 - 19c_3)d^3 + 36a_2a_3(12c_1 + 20c_2 + 9c_3)d^3 + 256a_2a_4(3c_1 + 6c_2 + c_3)d^2 + 144a_3a_4(12c_1 + 28c_2 + 21c_3)d),$$

$$A_0 = -\frac{bd^4}{90\,(a_1d^3 + 4a_2d^2 + 9a_3d + 16a_4)^4}(a_1^3(90c_0 + 30c_1 + 5c_2 - c_3)d^9 + 128a_2^3(45c_0 + 30c_1 + 10c_2 - 4c_3)d^6 + 6561a_3^3(10c_0 + 10c_1 + 5c_2 - 3c_3)d^3 + 8192a_4^3(45c_0 + 60c_1 + 40c_2 - 32c_3) + 16a_1a_2^2(270c_0 + 150c_1 +$$

$25c_2-27c_3)d^7+40a_1^2a_2(27c_0+12c_1+c_2)d^8+1215a_1a_3^2(18c_0+14c_1+c_2-9c_3)d^5+9a_1^2a_3(270c_0+150c_1-25c_2+51c_3)d^7+768a_1a_4^2(90c_0+90c_1-5c_2-133c_3)d^3+96a_1^2a_4(45c_0+30c_1-15c_2+34c_3)d^6+1944a_2a_3^2(45c_0+40c_1+15c_2-12c_3)d^4+144a_2^2a_3(270c_0+210c_1+65c_2-33c_3)d^5+2048a_2a_4^2(135c_0+150c_1+50c_2-108c_3)d^2+2560a_2^2a_4(27c_0+24c_1+4c_2)d^4+11520a_3a_4^2(54c_0+66c_1+37c_2-33c_3)d+7776a_3^2a_4(45c_0+50c_1+25c_2-18c_3)d^2+144a_1a_2a_3(135c_0+90c_1+5c_2-18c_3)d^6+128a_1a_2a_4(270c_0+210c_1-35c_2+27c_3)d^5+576a_1a_3a_4(135c_0+120c_1-5c_2-96c_3)d^4+1152a_2a_3a_4(270c_0+270c_1+85c_2-99c_3)d^3)$.

(iv): *Case $r=3$ (exactly three roots equal to d):*

$$W_n^C=n^3(A_3n^3+A_2n^2+A_1n+A_0)d^n$$

where

$A_3=\dfrac{c_3bd^4}{20\left(a_1d^3+8a_2d^2+27a_3d+64a_4\right)}$,

$A_2=\dfrac{1}{10\left(a_1d^3+8a_2d^2+27a_3d+64a_4\right)}(c_2bd^4+15(a_1d^3+16a_2d^2+81a_3d+256a_4)A_3)$,

$A_1=\dfrac{1}{4\left(a_1d^3+8a_2d^2+27a_3d+64a_4\right)}(c_1bd^4+5A_2(a_1d^3+16a_2d^2+81a_3d+256a_4)-6A_3(a_1d^3+32a_2d^2+243a_3d+1024a_4))$,

$A_0=\dfrac{1}{\left(a_1d^3+8a_2d^2+27a_3d+64a_4\right)}(c_0bd^4+(a_1d^3+16a_2d^2+81a_3d+256a_4)A_1-(a_1d^3+32a_2d^2+243a_3d+1024a_4)A_2+(a_1d^3+64a_2d^2+729a_3d+4096a_4)A_3)$,

i.e.,

$A_3=\dfrac{c_3bd^4}{20\left(a_1d^3+8a_2d^2+27a_3d+64a_4\right)}$,

$A_2=\dfrac{bd^4}{40\left(a_1d^3+8a_2d^2+27a_3d+64a_4\right)^2}(a_1(4c_2+3c_3)d^3+16a_2(2c_2+3c_3)d^2+27a_3(4c_2+9c_3)d+256a_4(c_2+3c_3))$,

$A_1=\dfrac{bd^4}{160\left(a_1d^3+8a_2d^2+27a_3d+64a_4\right)^3}(a_1^2(40c_1+20c_2+3c_3)d^6+256a_2^2(10c_1+10c_2+3c_3)d^4+729a_3^2(40c_1+60c_2+27c_3)d^2+32768a_4^2(5c_1+10c_2+6c_3)+160a_1a_2(4c_1+3c_2)d^5+270a_1a_3(8c_1+8c_2-3c_3)d^4+256a_1a_4(20c_1+25c_2-21c_3)d^3+864a_2a_3(20c_1+25c_2+6c_3)d^3+20480a_2a_4(2c_1+3c_2)d^2+34560a_3a_4(4c_1+7c_2+3c_3)d)$,

$A_0=\dfrac{bd^4}{160\left(a_1d^3+8a_2d^2+27a_3d+64a_4\right)^4}(a_1^3(160c_0+40c_1+4c_2-c_3)d^9+4096a_2^3(20c_0+10c_1+2c_2-c_3)d^6+19683a_3^3(160c_0+120c_1+36c_2-27c_3)d^3+8388608a_4^3(5c_0+5c_1+2c_2-2c_3)+256a_1a_2^2(120c_0+50c_1+4c_2-5c_3)d^7+16a_1^2a_2(240c_0+80c_1+2c_2+c_3)d^8+729a_1a_3^2(480c_0+280c_1-4c_2-91c_3)d^5+27a_1^2a_3(480c_0+200c_1-36c_2+55c_3)d^7+98304a_1a_4^2(20c_0+15c_1-2c_2-11c_3)d^3+768a_1^2a_4(40c_0+20c_1-9c_2+17c_3)d^6+11664a_2a_3^2(240c_0+160c_1+34c_2-31c_3)d^4+6912a_2^2a_3(120c_0+70c_1+12c_2-7c_3)d^5+524288a_2a_4^2\left(30c_0+25c_1+4c_2-10c_3\right)d^2+65536a_2^2a_4\left(30c_0+20c_1+c_2+c_3\right)d^4+884736a_3a_4^2(60c_0+55c_1+18c_2-19c_3)d+186624a_3^2a_4(120c_0+100c_1+29c_2-25c_3)d^2+864a_1a_2a_3(240c_0+120c_1-2c_2-9c_3)d^6+8192a_1a_2a_4(60c_0+35c_1-7c_2+7c_3)d^5+13824a_1a_3a_4(120c_0+80c_1-9c_2-29c_3)d^4+221184a_2a_3a_4(60c_0+45c_1+7c_2-9c_3)d^3)$.

(v): *Case $r=4$ (all four roots equal to d):*

$$W_n^C=n^4(A_3n^3+A_2n^2+A_1n+A_0)d^n$$

where

$$\begin{aligned} A_3 &= \frac{1}{840} b c_3, \\ A_2 &= \frac{1}{360} b(c_2 + 6c_3), \\ A_1 &= \frac{1}{120} b(c_1 + 4c_2 + 11c_3), \\ A_0 &= \frac{1}{72} b(3c_0 + 6c_1 + 11c_2 + 18c_3). \end{aligned}$$

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