# Anti-Frobenius Algebras and anti-Bialgebras

**Abstract**

This work investigates $q$-generalized associative algebras, with a focus on their bimodule and matched pair structures, establishing a cohesive framework that bridges associative and antiassociative systems. A detailed analysis of the double constructions of quadratic antiassociative algebras, referred to as the double construction of anti-Frobenius algebras, is presented, highlighting their structural properties and their role in advancing the theory of antiassociative algebras. Additionally, we derive the corresponding antisymmetric infinitesimal antiassociative bialgebra structures. Finally, a comparative study of these was conducted with the Mock biagebras.

**Keywords. Antiassociative Algebras, Anti-Frobenius Algebras, Anti-Bialgebras.**
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## 1 Introduction

The concept of antiassociative algebras was first introduced in the literature by Okubo and Kamiya in [9]. They outlined the essential properties of these algebras and introduced the framework of Jordan-Lie (super) algebras, which are closely related to both Jordan-super and antiassociative algebras. Subsequently, in 2014, M. Markl and E. Remm [8] explored the Koszulness of operads for $n$-ary algebras, focusing specifically on the antiassociative operation. This operation, defined as $(ab)c+a(bc) = 0$ for all $a, b, c$, revealed that the corresponding operad is not Koszul. Consequently, while standard cohomology lacks meaningful interpretation, deformation cohomology coincides with triple cohomology [5,6], which governs the deformations of antiassociative algebras.

More recently, P. Zummanovich [10] introduced the notion of Mock-Lie algebras, a class of commutative algebras satisfying the Jacobi identity. These algebras had previously appeared in the literature under various names (see [10] and [4] for details). Zummanovich highlighted two notable properties of Mock-Lie algebras. First, algebras over the operad Koszul dual to the Mock-Lie operad can be characterized equivalently as:

- anticommutative antiassociative algebras,
- anticommutative 2-Engel algebras (satisfying $(xy)y = 0$),
- anticommutative alternative algebras.

Second, Mock-Lie algebras can be constructed from antiassociative algebras in much the same way as they are from associative ones, establishing a strong connection between these two classes.

The study of Frobenius algebras and antisymmetric infinitesimal bialgebras by C. Bai in [1] further enriches this field. A symmetric Frobenius algebra is an associative algebra $\mathcal{A}$ equipped with a non-degenerate symmetric invariant bilinear form. In contrast, an infinitesimal bialgebra is a triple $(\mathcal{A}, m, \Delta)$ where $(\mathcal{A}, m)$ is an associative algebra, $(\mathcal{A}, \Delta)$ is a coassociative coalgebra, and the compatibility condition

$$\Delta(ab) = \sum ab_1 \otimes b_2 + a_1 \otimes a_2 b$$

holds for all $a, b \in \mathcal{A}$. These bialgebras were initially introduced by Joni and Rota [7] to provide a foundation for the calculus of divided differences, and subsequent work has identified additional examples and applications.

The importance of antiassociative algebras extends beyond their role in the construction of Mock-Lie algebras. Their deformations, governed by triple cohomology, offer insights into their structure and applications [5, 6]. Establishing antiassociative bialgebra structures, particularly in low dimensions, may provide a more robust framework for understanding related mathematical constructs.The cohomology and deformation theories of Mock Lie algebras were studied in [2]. In this work, a cohomology theory based on two operators, called zigzag cohomology, was constructed, and low-degree cohomology spaces were detailed. Further, bialgebras, the Yang-Baxter equation, and Manin triples for Mock-Lie algebras were investigated in [3]. A Manin triple of Mock-Lie algebras is precisely equivalent to the structure of a Mock-Lie bialgebra, establishing a fundamental correspondence between the two frameworks [3].

In this paper, we define $q$-generalized associative algebras, along with their related algebraic structures such as bimodules and matched pairs. Additionally, we explore the double constructions of quadratic antiassociative algebras, referred to as the double construction of anti-Frobenius algebras, drawing inspiration from Bai's pioneering methodology on Frobenius algebras. We demonstrate that the anticommutator derived from the double construction of an anti-Frobenius algebra naturally forms a Manin triple, thereby establishing a Mock-Lie bialgebra. Moreover, this double construction exhibits a range of properties analogous to those of a Mock-Lie bialgebra, and it is fundamentally equivalent to an antisymmetric infinitesimal anti-bialgebra.

The paper begins in Section 2 with a review of the foundational ideas and properties of $q$-generalized associative algebras, including an exploration of their bimodules and matched pairs. Section 3 then examines the double constructions of anti-Frobenius algebras, providing a deeper understanding of their unique structural aspects. In Section 4, the focus shifts to antisymmetric infinitesimal anti-bialgebras, with a detailed presentation of their defining features. Section 5 offers a comparative analysis of antiassociative and Mock-Lie bialgebras, highlighting their connections.

Throughout this paper, $\mathcal{K}$ is a field of characteristic 0.

# 2 $q$-generalized associative algebras

## 2.1 Preliminaries

**Definition 2.1** *Let "$\cdot$" be a bilinear product in a vector space $\mathcal{A}$. Suppose that it satisfies the following law:*

$$(x \cdot y) \cdot z = -x \cdot (y \cdot z). \tag{2.1}$$

*Then, we call the pair $(\mathcal{A}, \cdot)$ an* ***antiassociative algebra****. Combining both associative (q=1) and antiassociative (q=-1) cases, any algebra $\mathcal{A}$ satisfying*

$$(x \cdot y) \cdot z = qx \cdot (y \cdot z), q = 1, -1$$

*is called a* ***$q$-associative algebra****.*

**Definition 2.2** *[10] An algebra $(\mathcal{A}, \diamond)$ over $\mathcal{K}$ is called mock Lie if it is commutative:*

$$x \diamond y = y \diamond x, \tag{2.2}$$

*and satisfies the Jacobi identity:*

$$(x \diamond y) \diamond z + (z \diamond x) \diamond y + (y \diamond z) \diamond x = 0 \tag{2.3}$$

*for any $x$,$y$,$z \in \mathcal{A}$.*

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**Theorem 2.3** *[10] Given an antiassociative algebra $(\mathcal{A}, \cdot)$, the new algebra $\mathcal{A}^\dagger$ with multiplication give by the "anticommutator"*

$$a \diamond b = (a \cdot b + b \cdot a),$$

*is a mock-Lie algebra.*

So, it follows from Theorem2.3 that an antiassociative algbera is an admissible Mock Lie algebra.
Now, let us give a generalized definition.

**Definition 2.4** *Let $(\mathcal{A}, \cdot)$ be an algebra over field $\mathcal{K}$. $(\mathcal{A}, \cdot)$ is called $q$-**generalized associative algebra** when it satisfies the following law:*

$$(x \cdot y) \cdot z = qx \cdot (y \cdot z), q \in \mathcal{K} - \{0\}. \tag{2.4}$$

**Example 2.5** *Let $(\mathcal{A}, \circ)$ be a Zinbiel algebra, that is for all $x, y, z \in \mathcal{A}$, we have*

$$(x \circ y) \circ z = x \circ (y \circ z) + x \circ (z \circ y).$$

*If "$\circ$" is commutative, we have*

$$(x \circ y) \circ z = x \circ (y \circ z) + x \circ (y \circ z) = 2x \circ (y \circ z)$$

*Hence, a commutative Zinbiel algebra is a 2-generalized associative algebra.*

## 2.2 Bimodules and matched pairs of $q$-generalized associative algebras

**Definition 2.6** *Let $\mathcal{A}$ be a q-generalized associative algebra and let $V$ be a vector space. Let $l, r, \mathcal{A} \to gl(V)$ be two linear maps. $V$ (or the pair $(l, r)$, or $(l, r, V)$) is called a bimodule of $\mathcal{A}$ if*

$$l(xy)v = ql(x)l(y)v, r(xy)v = q^{-1}r(y)r(x)v, l(x)r(y)v = q^{-1}r(y)l(x)v$$

*for all $x, y \in \mathcal{A}, v \in V$.*

**Remark 2.7** *Let $\mathcal{A}$ be a q-generalized associative algebra and $(l, r, V)$ the bimodule of $\mathcal{A}$*

- *For the particular case of $q = 1$, $(l, r, V))$ is a bimodule of associative algebra $\mathcal{A}$ ie*

$$l(xy)v = l(x)l(y)v, r(xy)v = r(y)r(x)v, l(x)r(y)v = r(y)l(x)v, \quad \forall x, y \in \mathcal{A}, v \in V, \tag{2.5}$$

  *which is well known in the litterature.*

- *When $q = -1$, $(l, r, V))$ is a bimodule of antiassociative algebra $\mathcal{A}$ ie*

$$l(xy)v = -l(x)l(y)v, r(xy)v = -r(y)r(x)v, l(x)r(y)v = -r(y)l(x)v \tag{2.6}$$

  *for all $x, y \in \mathcal{A}, v \in V$.*

- *When $q = 2$, $(l, r, V))$ is a bimodule of a commutative Zinbiel algebra of $\mathcal{A}$ ie*

$$l(xy)v = 2l(x)l(y)v, r(xy)v = \frac{1}{2}r(y)r(x)v, l(x)r(y)v = \frac{1}{2}r(y)l(x)v, \quad \forall x, y \in \mathcal{A}, v \in V. \tag{2.7}$$

**Proposition 2.8** *$(l, r, V)$ is a bimodule of a q-generalized associative algebra $\mathcal{A}$ if and only if the direct sum $\mathcal{A} \oplus V$ of vectors spaces is turned into a q-generalized associative algebra by defining multiplication in $\mathcal{A} \oplus V$ by*

$$(x + a) * (y + b) = x \cdot y + (l(x)b + r(y)a)$$

*for all $x, y \in \mathcal{A}, a, b \in V$.*

**Proof:** We have:

$$\begin{aligned}[(x_1+v_1)*(x_2+v_2)]*(x_3+v_3) &= (x_1\cdot x_2)\cdot x_3 + l(x_1\cdot x_2)v_3 \\ &\quad +r(x_3)(l(x_1)v_2) + r(x_3)(r(x_2)v_1) \\ &= qx_1\cdot(x_2\cdot x_3) + ql(x_1)l(x_2)v_3 \\ &\quad +ql(x_1)r(x_3)v_2 + qr(x_2\cdot x_3)v_1 \\ &= q(x_1+v_1)*[(x_2+v_2)*(x_3+v_3)]\end{aligned}$$

for all $x_1, x_2, x_3 \in \mathcal{A},\ v_1, v_2, v_3 \in V$.

□

We denote such $q$-generalized associative algebra $(\mathcal{A}\oplus V, *)$ by $\mathcal{A}\ltimes_{l,r} V$ or simply $\mathcal{A}\ltimes V$.

**Lemma 2.9** *Let $(l, r, V)$ be a bimodule of a $q$-generalized associative algebra $\mathcal{A}$.*

*(i) Let $l^*, r^* : \mathcal{A}\to gl(V^*)$ be the linear maps given by*

$$\langle l^*(x)u^*, v\rangle = \langle l(x)v, u^*\rangle, \langle r^*(x)u^*, v\rangle = \langle r(x)v, u^*\rangle \tag{2.8}$$

*for all $x\in\mathcal{A}$, $u^*\in V^*, v\in V$. Then, $(q^{-2}r^*, q^2l^*, V^*)$ is a bimodule of $\mathcal{A}$.*

*(ii) $(l, 0, V), (0, r, V), (q^{-2}r^*, 0, V^*)$ and $(0, q^2l^*, V^*)$ are bimodules.*

**Proof:** Let $(l, r, V)$ be a bimodule of a $q$-generalized associative algebra $\mathcal{A}$. Show that $(q^{-2}r^*, q^2l^*, V^*)$ is a bimodule of $\mathcal{A}$. Let $x, y\in\mathcal{A}, u^*\in V^*, v\in V$, we have

(i)

$$\langle q^{-2}r^*(xy)u^*, v\rangle = \langle q^{-2}r(xy)v, u^*\rangle = \langle q^{-3}r(y)r(x)v, u^*\rangle = \langle q(q^{-2}r^*)(x)(q^{-2}r^*)(y)u^*, v\rangle$$

leading to $q^{-2}r^*(xy)u^* = q(q^{-2}r^*)(x)(q^{-2}r^*)(y)u^*$;

(ii)

$$\langle q^2l^*(xy)u^*, v\rangle = \langle q^2l(xy)v, u^*\rangle = \langle q^3l(x)l(y)v, u^*\rangle = \langle q^{-1}(q^2l^*)(y)(q^2l^*)(x)u^*, v\rangle$$

giving $q^2l^*(xy)u^* = q^{-1}(q^2l^*)(y)(q^2l^*)(x)u^*$;

(iii)

$$\langle (q^{-2}r^*)(x)(q^2l^*)(y)u^*, v\rangle = \langle l(y)r(x)v, u^*\rangle = \langle q^{-1}r(x)l(y)v, u^*\rangle = \langle q^{-1}(q^2l^*)(y)(q^{-2}r^*)(x)u^*, v\rangle$$

providing that $(q^{-2}r^*)(x)(q^2l^*)(y)u^* = q^{-1}(q^2l^*)(y)(q^{-2}r^*)(x)u^*$. Hence, $(q^{-2}r^*, q^2l^*, V^*)$ is a bimodule of $\mathcal{A}$.

Similarly, we can show also that $(l, 0, V), (0, r, V), (q^{-2}r^*, 0, V^*)$ and $(0, q^2l^*, V^*)$ are well bimodules of $\mathcal{A}$.

□

**Remark 2.10**
- *For $q = 1$ we obtain a bimodule dual of an associative algebra which is well known in [1].*
- *For $q = \pm 1$ the dual bimodule of a bimodule of an antiassociative algebra $\mathcal{A}$ and of a bimodule of an associative algebra $\mathcal{A}$ are equal ie $(r^*, l^*, V^*)$.*

**Example 2.11** *Let $(\mathcal{A}, \cdot)$ be a $q$-generalized associative algebra. Let $L.(x)$ and $R.(x)$ denote the left and right multiplication operators, respectively, that is, $L.(x)(y) = x\cdot y, R.(x)(y) = y\cdot x$. For any $x, y\in\mathcal{A}$. Let $L. : \mathcal{A}\to gl(\mathcal{A})$ with $x\mapsto L.(x)$ and $R. : \mathcal{A}\to gl(\mathcal{A})$ with $x\mapsto R.(x)$ (for every $x\in\mathcal{A}$) be two linear maps. Then $(L., 0), (0, R.)$ and $(L., R.)$ are bimodules of $\mathcal{A}$ too.*

**Theorem 2.12** *Let $(\mathcal{A},\cdot)$ and $(\mathcal{B},\circ)$ be two q-generalized associative algebras. Suppose that there are linear maps $l_{\mathcal{A}}, r_{\mathcal{A}} : \mathcal{A} \to gl(\mathcal{B})$ and $l_{\mathcal{B}}, r_{\mathcal{B}} : \mathcal{B} \to gl(\mathcal{A})$ such that $(l_{\mathcal{A}}, r_{\mathcal{A}})$ is a bimodule of $\mathcal{A}$ and $(l_{\mathcal{B}}, r_{\mathcal{B}})$ is a bimodule of $\mathcal{B}$, satisfying the following conditions:*

$$l_{\mathcal{A}}(x)(a\circ b) = q^{-1}l_{\mathcal{A}}(r_{\mathcal{B}}(a)x)b + q^{-1}(l_{\mathcal{A}}(x)a)\circ b, \tag{2.9}$$

$$r_{\mathcal{A}}(x)(a\circ b) = qr_{\mathcal{A}}(l_{\mathcal{B}}(b)x)a + qa\circ(r_{\mathcal{A}}(x)b), \tag{2.10}$$

$$l_{\mathcal{B}}(a)(x\cdot y) = q^{-1}l_{\mathcal{B}}(r_{\mathcal{A}}(x)a)y + q^{-1}(l_{\mathcal{B}}(a)x)\cdot y, \tag{2.11}$$

$$r_{\mathcal{B}}(a)(x\cdot y) = qr_{\mathcal{B}}(l_{\mathcal{A}}(y)a)x + qx\cdot(r_{\mathcal{B}}(a)y), \tag{2.12}$$

$$l_{\mathcal{A}}(l_{\mathcal{B}}(a)x)b + (r_{\mathcal{A}}(x)a)\circ b - qr_{\mathcal{A}}(r_{\mathcal{B}}(b)x)a - qa\circ(l_{\mathcal{A}}(x)b) = 0, \tag{2.13}$$

$$l_{\mathcal{B}}(l_{\mathcal{A}}(x)a)y + (r_{\mathcal{B}}(a)x)\cdot y - qr_{\mathcal{B}}(r_{\mathcal{A}}(y)a)x - qx\cdot(l_{\mathcal{B}}(a)y) = 0 \tag{2.14}$$

*for any $x, y \in \mathcal{A}, a, b \in \mathcal{B}$. Then, there is a q-generalized associative algebra structure on the direct sum $\mathcal{A}\oplus\mathcal{B}$ of the underlying vector spaces of $\mathcal{A}$ and $\mathcal{B}$ given by*

$$(x+a)*(y+b) = (x\cdot y + l_{\mathcal{B}}(a)y + r_{\mathcal{B}}(b)x) + (a\circ b + l_{\mathcal{A}}(x)b + r_{\mathcal{A}}(y)a) \tag{2.15}$$

*for all $x, y \in \mathcal{A}, a, b \in \mathcal{B}$. We denote this q-generalized associative algebra by $\mathcal{A}\bowtie^{l_{\mathcal{A}},r_{\mathcal{A}}}_{l_{\mathcal{B}},r_{\mathcal{B}}}\mathcal{B}$ or simply $\mathcal{A}\bowtie\mathcal{B}$.*

**Proof:** We have:

$$\begin{aligned}
((x+a)*(y+b))*(z+c) &= [(x\cdot y + l_{\mathcal{B}}(a)y + r_{\mathcal{B}}(b)x)\\
&\quad +(a\circ b + l_{\mathcal{A}}(x)b + r_{\mathcal{A}}(y)a)]*(z+c)\\
&= (x\cdot y)\cdot z + (l_{\mathcal{B}}(a)y)\cdot z + (r_{\mathcal{B}}(b)x)\cdot z\\
&\quad +l_{\mathcal{B}}(a\circ b)z + l_{\mathcal{B}}(l_{\mathcal{A}}(x)b)z + l_{\mathcal{B}}(r_{\mathcal{A}}(y)a)z\\
&\quad +r_{\mathcal{B}}(c)(x\cdot y) + r_{\mathcal{B}}(c)(l_{\mathcal{B}}(a)y) + r_{\mathcal{B}}(c)(r_{\mathcal{B}}(b)x)\\
&\quad +a\circ(b\circ c) + (l_{\mathcal{A}}(x)b)\circ c\\
&\quad +(l_{\mathcal{A}}(x)b)\circ c + (r_{\mathcal{A}}(y)a)\circ c\\
&\quad +l_{\mathcal{A}}(x\cdot y)c + l_{\mathcal{A}}(l_{\mathcal{B}}(a)y)c\\
&\quad +l_{\mathcal{A}}(r_{\mathcal{B}}(b)x)c + r_{\mathcal{A}}(z)(a\circ b)\\
&\quad +r_{\mathcal{A}}(z)(l_{\mathcal{A}}(x)b) + r_{\mathcal{A}}(z)(r_{\mathcal{A}}(y)a)
\end{aligned}$$

and

$$\begin{aligned}
q(x+a)*[(y+b)*(z+c)] &= q(x+a)*[(y\cdot z + l_{\mathcal{B}}(b)z + r_{\mathcal{B}}(c)y)\\
&\quad +(b\circ c + l_{\mathcal{A}}(y)c + r_{\mathcal{A}}(z)b)]\\
&= qx\cdot(y\cdot z) + qx\cdot(l_{\mathcal{B}}(b)z) + qx\cdot(r_{\mathcal{B}}(c)y)\\
&\quad ql_{\mathcal{B}}(a)(y\cdot z) + ql_{\mathcal{B}}(a)(l_{\mathcal{B}}(b)z) + ql_{\mathcal{B}}(a)(r_{\mathcal{B}}(c)y)\\
&\quad qr_{\mathcal{B}}(b\circ c)x + qr_{\mathcal{B}}(l_{\mathcal{A}}(y)c)x + qr_{\mathcal{B}}(r_{\mathcal{A}}(z)b)x\\
&\quad qa\circ b\circ c + qa\circ(l_{\mathcal{A}}(y)c) + qa\circ(r_{\mathcal{A}}(z)b)\\
&\quad ql_{\mathcal{A}}(x)(b\circ c) + ql_{\mathcal{A}}(x)(l_{\mathcal{A}}(y)c)ql_{\mathcal{A}}(x)(r_{\mathcal{A}}(z)b)\\
&\quad +qr_{\mathcal{A}}(y\cdot z)a + qr_{\mathcal{A}}(l_{\mathcal{B}}(b)z)a + qr_{\mathcal{A}}(r_{\mathcal{B}}(c)y)\circ a.
\end{aligned}$$

Then $((x+a)*(y+b))*(z+c) = q(x+a)*((y+b)*(z+c))$.

□

**Definition 2.13** *Let $(\mathcal{A},\cdot)$ and $(\mathcal{B},\circ)$ be two q-generalized associative algebras. Suppose that there are linear maps $l_{\mathcal{A}}, r_{\mathcal{A}} : \mathcal{A} \to gl(\mathcal{B})$ and $l_{\mathcal{B}}, r_{\mathcal{B}} : \mathcal{B} \to gl(\mathcal{A})$ such that $(l_{\mathcal{A}}, r_{\mathcal{A}})$ is a bimodule of $\mathcal{A}$ and $(l_{\mathcal{B}}, r_{\mathcal{B}})$ is a bimodule of $\mathcal{B}$. If the equations (2.9) - (2.14) are satisfied, then $(\mathcal{A},\mathcal{B},l_{\mathcal{A}},r_{\mathcal{A}},l_{\mathcal{B}},r_{\mathcal{B}})$ is called a* ***matched pair of $q$-generalized associative algebras****.*

**Remark 2.14** *In the previous definition*

- *for* $q = 1$, $(\mathcal{A}, \mathcal{B}, l_{\mathcal{A}}, r_{\mathcal{A}}, l_{\mathcal{B}}, r_{\mathcal{B}})$ *is called a matched pair of associative algebras which is well known in [1];*
- *for* $q = -1$, $(\mathcal{A}, \mathcal{B}, l_{\mathcal{A}}, r_{\mathcal{A}}, l_{\mathcal{B}}, r_{\mathcal{B}})$ *is called a matched pair of antiassociative algebras;*
- *when* $q = 2$, $(\mathcal{A}, \mathcal{B}, l_{\mathcal{A}}, r_{\mathcal{A}}, l_{\mathcal{B}}, r_{\mathcal{B}})$ *is called a matched pair of a commutative Zinbiel algebras.*

It would be interesting to establish a double construction of $q$-generalized Frobenius algebras if it is possible to find a compatible natural non-degenerate invariant symmetric bilinear form. Thus, in the following, we take $q = -1$ because the case $q = 1$ is made in [1].

# 3 Double constructions of anti-Frobenius algebras

**Definition 3.1** *We call* $(\mathcal{A}, B)$ *a double construction of an anti-Frobenius algebra associated to* $\mathcal{A}_1$ *and* $\mathcal{A}_1^*$ *if it satisfies the conditions*

*(1)* $\mathcal{A} = \mathcal{A}_1 \oplus \mathcal{A}_1^*$ *as the direct sum of vector spaces;*

*(2)* $\mathcal{A}_1$ *and* $\mathcal{A}_1^*$ *are antiassociative subalgebras of* $\mathcal{A}$*;*

*(3)* $B$ *is the natural non-degenerate invariant symmetric bilinear form on* $\mathcal{A}_1 \oplus \mathcal{A}_1^*$ *given by*

$$B(x + a^*, y + b^*) = \langle x, b^*\rangle + \langle a^*, y\rangle \tag{3.1}$$

*for all* $x, y \in \mathcal{A}_1, a^*, b^* \in \mathcal{A}_1^*$ *where* $\langle , \rangle$ *is the natural pair between the vector space* $\mathcal{A}_1$ *and its dual space* $\mathcal{A}_1^*$.

Let $(\mathcal{A}, \cdot)$ be an antiassociative algebra. Suppose that there is an antiassociative algebra structure " $\circ$ " on its dual space $\mathcal{A}^*$. We construct an antiassociative algebra structure on the direct sum $\mathcal{A} \oplus \mathcal{A}^*$ of the underlying vector spaces of $\mathcal{A}$ and $\mathcal{A}^*$ such that $(\mathcal{A}, \cdot)$ and $(\mathcal{A}^*, \circ)$ are subalgebras and the symmetric bilinear form on $\mathcal{A} \oplus \mathcal{A}^*$ given by (3.1) is invariant. That is, $(\mathcal{A} \oplus \mathcal{A}^*, \mathcal{B})$ is an anti-Frobenius algebra. Such a construction is called a double construction of a quadratic antiassociative algebra associated to $(\mathcal{A}, \cdot)$ and $(\mathcal{A}^*, \circ)$ and we denote it by $(\mathcal{A} \oplus \mathcal{A}^*, B)$.

**Theorem 3.2** *Let* $(\mathcal{A}, \cdot)$ *be an antiassociative algebra. Suppose that there is an antiassociative algebra structure " $\circ$ " on its dual space* $\mathcal{A}^*$*. Then, there is a double construction of an anti-Frobenius associated to* $(\mathcal{A}, \cdot)$ *and* $(\mathcal{A}^*, \circ)$ *if and only if* $(\mathcal{A}, \mathcal{A}^*, R_\cdot^*, L_\cdot^*, R_\circ^*, L_\circ^*)$ *is a matched pair of antiassociative algebras.*

**Proof:** Let us consider the four maps

$$L_\cdot^* : \mathcal{A} \to gl(\mathcal{A}^*), \langle L_\cdot^*(x)u^*, v\rangle = \langle L_\cdot(x)v, u^*\rangle = \langle xv, u^*\rangle,$$

$$R_\cdot^* : \mathcal{A} \to gl(\mathcal{A}^*), \langle R_\cdot^*(x)u^*, v\rangle = \langle R_\cdot(x)v, u^*\rangle = \langle vx, u^*\rangle,$$

$$R_\circ^* : \mathcal{A}^* \to gl(\mathcal{A}), \langle R_\circ^*(x^*)u, v^*\rangle = \langle R_\circ(x^*)v^*, u\rangle = \langle v^* \circ x^*, u\rangle,$$

$$L_\circ^* : \mathcal{A}^* \to gl(\mathcal{A}), \langle L_\circ^*(x^*)u, v^*\rangle = \langle L_\circ(x^*)v^*, u\rangle = \langle x^* \circ v^*, u\rangle,$$

for all $x, v, u \in \mathcal{A}$, $x^*, v^*, u^* \in \mathcal{A}^*$. If $(\mathcal{A}, \mathcal{A}^*, R_\cdot^*, L_\cdot^*, R_\circ^*, L_\circ^*)$ is a matched pair of antiassociative algebras, then the bilinear form $B(\cdot, \cdot)$ defined by the equation (3.1) is invariant on the antiassociative algebra $\mathcal{A} \bowtie^{R_\cdot^*, L_\cdot^*}_{R_\circ^*, L_\circ^*} \mathcal{A}^*$ with its product $*$ given by the equation (2.15), that is $B[(x+a^*)*(y+b^*), (z+c^*)] = B[(x+a^*), (y+b^*)*(z+c^*)]$, where $B(x+a^*, y+b^*) = \langle x, b^*\rangle + \langle a^*, y\rangle$, for all $x, y \in \mathcal{A}^*, a^*, b^* \in \mathcal{A}^*$ and $(x+a^*)*(y+b^*) = (x\cdot y+l_{\mathcal{B}}(a)y+r_{\mathcal{B}}(b)x)+(a\circ b+l_{\mathcal{A}}(x)b+r_{\mathcal{A}}(y)a)$ with $l_{\mathcal{A}} = R^*, r_{\mathcal{A}} = L^*, l_{\mathcal{B}} = R^*_{\circ}, r_{\mathcal{B}} = L^*_{\circ}$. Indeed, we have

$$\begin{aligned} B[(x+a^*)*(y+b^*),(z+c^*)] &= B[(x\cdot y + l_{\mathcal{A}^*}(a^*)y + r_{\mathcal{A}^*}(b^*)x) + (a^*\circ b^* \\ &\quad + l_{\mathcal{A}}(x)b^* + r_{\mathcal{A}}(y)a^*), z + c^*] \\ &= \langle (x\cdot y + l_{\mathcal{A}^*}(a^*)y + r_{\mathcal{A}^*}(b^*)x), c^*\rangle \\ &\quad + \langle (a^*\circ b^* + l_{\mathcal{A}}(x)b^* + r_{\mathcal{A}}(y)a^*), z\rangle \\ &= \langle x\cdot y, c^*\rangle + \langle l_{\mathcal{A}^*}(a^*)y, c^*\rangle + \langle r_{\mathcal{A}^*}(b^*)x), c^*\rangle \\ &\quad + \langle a^*\circ b^*, z\rangle + \langle l_{\mathcal{A}}(x)b^*, z\rangle + \langle r_{\mathcal{A}}(y)a^*), z\rangle \\ &= \langle x\cdot y, c^*\rangle + \langle c^*\circ a^*, y\rangle + \langle b^*\circ c^*, x\rangle \\ &\quad + \langle a^*\circ b^*, z\rangle + \langle z\cdot x, b^*\rangle + \langle y\cdot z, a^*\rangle \end{aligned}$$

and

$$\begin{aligned} B[x+a^*,(y+b^*)*(z+c^*)] &= B[x+a^*,(y\cdot z + l_{\mathcal{A}^*}(b^*)z + r_{\mathcal{A}^*}(c^*)y) \\ &\quad + (b^*\circ c^* + l_{A}(y)c^* + r_{\mathcal{A}}(z)b^*)] \\ &= \langle x, (b^*\circ c^* + l_{\mathcal{A}}(y)c^* + r_{\mathcal{A}}(z)b^*)\rangle \\ &\quad + \langle (y\cdot z + l_{\mathcal{A}^*}(b^*)z + r_{\mathcal{A}^*}(c^*)y, a^*\rangle \\ &= \langle x, b^*\circ c^*\rangle + \langle x, l_{\mathcal{A}}(y)c^*\rangle + \langle x, r_{\mathcal{A}}(z)b^*)\rangle \\ &\quad + \langle y\cdot z, a^*\rangle + \langle l_{\mathcal{A}^*}(b^*)z, a^*\rangle + \langle r_{\mathcal{A}^*}(c^*)y, a^*\rangle \\ &= \langle x, b^*\circ c^*\rangle + \langle c^*, x\cdot y\rangle + \langle b^*, z\cdot x\rangle \\ &\quad + \langle y\cdot z, a^*\rangle + \langle a^*\circ b^*, z\rangle + \langle c^*\circ a^*, y\rangle. \end{aligned}$$

Thus, $B$ is well invariant. Conversely, set

$$x * a^* = l_{\mathcal{A}}(x)a^* + r_{\mathcal{A}^*}(a^*)x, a^* * x = l_{\mathcal{A}^*}(a^*)x + r_{\mathcal{A}}(x)a^\star,$$

for $x \in \mathcal{A}, a^* \in \mathcal{A}^*$. Then, $(\mathcal{A}, \mathcal{A}^*, R^*_{\cdot}, L^*_{\cdot}, R^*_{\circ}, L^*_{\circ})$ is a matched pair of antiassociative algebras, since the double construction of the anti-Frobenius algebra associated to $(\mathcal{A}, \cdot)$ and $(\mathcal{A}^*, \circ)$ produces the equations (2.9) - (2.14).

□

**Theorem 3.3** *Let $(\mathcal{A}, \cdot)$ be an antiassociative algebra. Suppose that there is an antiassociative algebra structure " $\circ$ " on its dual space $\mathcal{A}^*$. Then, $(\mathcal{A}, \mathcal{A}^*, R^*_{\cdot}, L^*_{\cdot}, R^*_{\circ}, L^*_{\circ})$ is a matched pair of antiassociative algebras if and only if for any $x \in \mathcal{A}$ and $a^*, b^* \in \mathcal{A}^*$,*

$$L^*_{\circ}(a^*)(x\cdot y) = -L^*_{\circ}(R^*(y)a^*)x - x\cdot(L^*_{\circ}(a^*)y), \tag{3.2}$$

$$R^*_{\circ}(R^*(x)a^*)y + (L^*_{\circ}(a^*)x)\cdot y = -L^*_{\circ}(L(y)a^*)x - x\cdot(R^*_{\circ}(a)y) \tag{3.3}$$

**Proof:** Obviously, (3.2) gives (2.9) and (3.3) reduces to (2.13) when $l_{\mathcal{A}} = R^*, r_{\mathcal{A}} = L^*, l_{\mathcal{B}} = l_{\mathcal{A}^*} = R^*_{\circ}, r_{\mathcal{B}} = r_{\mathcal{A}^*} = L^*_{\circ}$. Now, show that

$$\begin{aligned} (2.9) \Longleftrightarrow (2.10) \Longleftrightarrow (2.11) \Longleftrightarrow (2.12) \\ \text{and } (2.13) \Longleftrightarrow (2.14). \end{aligned}$$

Suppose (2.9) and (2.13) are satisfied and show that one has:

$$\begin{aligned} L^*(x)(a^*\circ b^*) = -L^*(R^*_{\circ}(b^*)x)a^* - a^*\circ(L^*(x)b^*) \\ R^*_{\circ}(x\cdot y) = -R^*_{\circ}(L^*(x)a^*)y - (R^*_{\circ}(a)x)\cdot y \\ R^*_{\cdot}(x)(a^*\circ b^*) = -R^*_{\cdot}(L^*_{\circ}(a^*)x)b^* - (R^*_{\cdot}(x)a^*)\circ b^*, \\ R^*_{\cdot}(R^*_{\circ}(a^*)x)b^* + L^*_{\cdot}(x)a^*\circ b^* + L^*_{\cdot}(L^*_{\circ}(b^*)x)a^* + a^*\circ(R^*_{\cdot}(x)b^*) = 0. \end{aligned}$$

We have :

$$\langle R^*(x)a^*, y\rangle = \langle L^*(y)a^*, x\rangle = \langle y\cdot x, a^*\rangle; \langle R^*_{\circ}(b^*)x, a^*\rangle = \langle L^*_{\circ}(a^*)x, b^*\rangle = \langle a^*\circ b^*, x\rangle$$

for all $x, y \in \mathcal{A}, a^*, b^* \in \mathcal{A}^*$.Then

(i)

$$\langle R^*(x)(a^*\circ b^*),y\rangle=\langle y\cdot x,a^*\circ b^*\rangle=\langle L^*(y)(a^*\circ b^*),x\rangle;$$
$$\langle -R^*(L^*_\circ(a^*)x)b^*,y\rangle=\langle -L^*(y)b^*,L^*_\circ(a^*)x\rangle=\langle -a^*\circ(L^*(y)b^*),x\rangle$$
$$\langle -(R^*(x)a^*)\circ b^*,y\rangle=\langle -R^*(x)a^*,R^*_\circ(b^*)y\rangle=\langle -L^*(R^*_\circ(b^*)y)a^*,x\rangle$$

leading to (2.9) $\Longleftrightarrow$ (2.10);

(ii)

$$\begin{aligned}\langle L^*(y)(a^*\circ b^*),x\rangle &= \langle -a^*\circ(L^*(y)b^*),x\rangle+\langle -L^*(R^*_\circ(b^*)y)\cdot x,a^*\rangle\\ &= \langle -R^*_\circ(L^*(y)b^*)x,a^*\rangle+\langle -(R^*_\circ(b^*)y)\cdot x,a^*\rangle\\ &= \langle R^*_\circ(b^*)(y\cdot x),a^*\rangle\end{aligned}$$

giving (2.10) $\Longleftrightarrow$ (2.11);

(iii)

$$\begin{aligned}\langle R^*(x)(a^*\circ b^*),y\rangle &= \langle -R^*(L^*_\circ(a^*)x)b^*,y\rangle+\langle -(R^*(x)a^*)\circ b^*,y\rangle\\ &= \langle -y\cdot L^*_\circ(a^*)x,b^*\rangle+\langle -L^*_\circ(R^*(x)a^*)y,b^*\rangle\\ &= \langle L^*_\circ(a^*)(y\cdot x),b^*\rangle\end{aligned}$$

providing that (2.9) $\Longleftrightarrow$ (2.12);

(iv)

$$\langle L^*(L^*_\circ(b^*)x)a^*,y\rangle=\langle (L^*_\circ(b^*)x)\cdot y,a^*\rangle;\langle a^*\circ(R^*(x)b^*),y\rangle=\langle R^*_\circ(R^*(x)b^*)y,a^*\rangle;$$
$$\langle (L^*(x)a^*)\circ b^*,y\rangle=\langle R_\circ(b^*)y,L^*(x)a^*\rangle=\langle x\cdot(R^*_\circ(b^*)y),a^*\rangle;$$
$$\langle R^*(R^*_\circ(a^*)x)b^*,y\rangle=\langle L^*(y)b^*,R^*_\circ(a^*)x\rangle=\langle L^*_\circ(L^*(y)b^*)x,a^*\rangle$$

implying that (2.13) $\Longleftrightarrow$ (2.14).

□

# 4 Antisymmetric infinitesimal anti-bialgebras

**Definition 4.1** *Let $V_1$, $V_2$ be two vector spaces. For a linear map $\phi : V_1 \to V_2$, we denote the dual (linear) map by $\phi^* : V_2^* \to V_1^*$ given by*

$$\langle v,\phi^*(u^*)\rangle=\langle\phi(v),u^*\rangle$$

*for all $v\in V_1$, $u^*\in V_2^*$.*

Let $\mathcal{A}$ be an antiassociative algebra. Let $\sigma:\mathcal{A}\otimes\mathcal{A}\to\mathcal{A}\otimes\mathcal{A}$ be the exchange operator defined as

$$\sigma(x\otimes y)=y\otimes x,\ \ \forall x,y\in\mathcal{A}.$$

**Theorem 4.2** *Let $(\mathcal{A},\cdot)$ be an antiassociative algebra. Suppose there is an antiassociative algebra structure " $\circ$ " on its dual space $\mathcal{A}^*$ given by a linear map $\Delta^*:\mathcal{A}^*\otimes\mathcal{A}^*\to\mathcal{A}^*$. Then, $(\mathcal{A},\mathcal{A}^*,R^*,L^*,R^*_\circ,L^*_\circ)$ is a matched pair of antiassociative algebras if and only if $\triangle:\mathcal{A}\to\mathcal{A}\otimes\mathcal{A}$ satisfies the following two conditions:*

$$\triangle(x\cdot y)=-(id\otimes L(x))\triangle(y)-(R(y)\otimes id)\triangle(x), \tag{4.1}$$

$$(L(y)\otimes\mathrm{id}+\mathrm{id}\otimes R(y))\Delta(x)+\sigma[(L(x)\otimes\mathrm{id}+\mathrm{id}\otimes R(x))\Delta(y)]=0 \tag{4.2}$$

*for all $x,y\in\mathcal{A}$.*

**Proof:** For any $x, y \in \mathcal{A}$ and any $a, b \in \mathcal{A}^*$, we have

$$\begin{aligned}
&\langle \Delta(x \cdot y), a \otimes b \rangle = \langle x \cdot y, a \circ b \rangle, = \langle L_\circ^*(a)(x \cdot y), b \rangle, \\
&\langle -(R_\cdot(y) \otimes \mathrm{id})\Delta(x), a \otimes b \rangle = \langle x, -(R_\cdot^*(y)a) \circ b \rangle = \langle -L_\circ^*(R_\cdot^*(y)a)x, b \rangle, \\
&\langle -(\mathrm{id} \otimes L_\cdot(x))\Delta(y), a \otimes b \rangle = \langle y, -a \circ (L_\cdot^*(x)b) \rangle = \langle -x \cdot (L_\circ^*(a)y), b \rangle.
\end{aligned}$$

Then Eq. (2.9) is equivalent to Eq. (4.1). Moreover, we have

$$\begin{aligned}
&\langle \sigma(\mathrm{id} \otimes R_\cdot(x))\Delta(y), a \otimes b \rangle = \langle y, b \circ (R_\cdot^*(x)a) \rangle = \langle R_\circ^*(R_\cdot^*(x)a)y, b \rangle, \\
&\langle (\mathrm{id} \otimes R_\cdot(y))\Delta(x), a \otimes b \rangle = \langle x, a \circ (R_\cdot^*(y)b) \rangle = \langle (L_\circ^*(a)x) \cdot y, b \rangle, \\
&\langle \sigma(L_\cdot(x) \otimes \mathrm{id})\Delta(y), a \otimes b \rangle = \langle y, (L_\cdot^*(x)b) \circ a \rangle = \langle x \cdot (R_\circ^*(a)y), b \rangle, \\
&\langle (L_\cdot(y) \otimes \mathrm{id})\Delta(x), a \otimes b \rangle = \langle x, (L_\cdot^*(y)a) \circ b \rangle = \langle L_\circ^*(L_\cdot^*(y)a)x, b \rangle.
\end{aligned}$$

Then Eq. (2.10) is equivalent to Eq. (4.2). Hence the conclusion holds.

□

**Remark 4.3** *From the symmetry of the antiassociative algebra $(\mathcal{A}, \cdot)$ and $(\mathcal{A}^*, *)$ appearing in the double construction, we can also consider the operation $\beta : \mathcal{A}^* \to \mathcal{A}^* \otimes \mathcal{A}^*$ such that $\beta^* : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ gives an antiassociative algebra structure on $\mathcal{A}$. Thus, $\triangle$ satisfies the equations (4.1) and (4.2) if and only if $\beta$ satisfies*

$$\begin{gathered}
\beta(a^* \circ b^*) = -(\mathrm{id} \otimes L_\circ(a^*))\beta(b^*) - (R_\circ(b^*) \otimes \mathrm{id})\beta(a^*), \\
(L_\circ(b^*) \otimes \mathrm{id} + \mathrm{id} \otimes R_\circ(b^*))\beta(a^*) + \sigma[(L_\circ(a^*) \otimes \mathrm{id} + \mathrm{id} \otimes R_\circ(a^*))\beta(b^*)] = 0
\end{gathered}$$

*for all $a^*, b^* \in \mathcal{A}^*$.*

**Definition 4.4** *Let $\mathcal{A}$ be an antiassociative algebra. An* ***antisymmetric infinitesimal anti-bialgebra*** *structure on $\mathcal{A}$ is a linear map $\triangle : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ such that*

*(a) $\triangle^* : \mathcal{A}^* \otimes \mathcal{A}^* \to \mathcal{A}^*$ defines an antiassociative algebra structure on $\mathcal{A}^*$;*

*(b) $\triangle$ satisfies (4.1) and (4.2).*

*We denote it by $(\mathcal{A}, \triangle)$ or $(\mathcal{A}, \mathcal{A}^*)$.*

**Corollary 4.5** *Let $(\mathcal{A}, \cdot)$ and $(\mathcal{A}^*, \circ)$ be two antiassociative algebras. Then, the following conditions are equivalent.*

*(1) There is a double construction of an anti-Frobenius algebra associated to $(\mathcal{A}, \cdot)$ and $(\mathcal{A}^*, \circ)$;*

*(2) $(\mathcal{A}, \mathcal{A}^*, R^*, L^*, R_\circ^*, L_\circ^*)$ is a matched pair of antiassociative algebras;*

*(3) $(\mathcal{A}, \mathcal{A}^*)$ is an antisymmetric infinitesimal anti-bialgebra.*

**Proof:** It follows from Theorems 2.12 and 4.2. □

**Definition 4.6** *Let $(\mathcal{A}, \Delta_\mathcal{A})$ and $(\mathcal{C}, \Delta_\mathcal{C})$ be two antisymmetric infinitesimal anti-bialgebras. A* ***homomorphism of antisymmetric infinitesimal anti-bialgebras*** *$\varphi : \mathcal{A} \to \mathcal{C}$ is a homomorphism of antiassociative algebras such that $(\varphi \otimes \varphi)\Delta_\mathcal{A}(x) = \Delta_\mathcal{C}(\varphi(x))$ for all $x \in \mathcal{A}$.*

*An* ***isomorphism of antisymmetric infinitesimal*** *anti-bialgebras is an invertible homomorphism of antisymmetric infinitesimal anti-bialgebras.*

**Definition 4.7** *Let $(\mathcal{A}_1 \bowtie \mathcal{A}_1^*, \mathcal{B}_1)$ and $(\mathcal{A}_2 \bowtie \mathcal{A}_2^*, \mathcal{B}_2)$ be two double constructions of anti-Frobenius algebras. They are isomorphic if and only if there exists an isomorphism of antiassociative algebras $\varphi : \mathcal{A}_1 \bowtie \mathcal{A}_1^* \to \mathcal{A}_2 \bowtie \mathcal{A}_2^*$ such that*

$$\varphi(\mathcal{A}_1) = \mathcal{A}_2, \varphi(\mathcal{A}_1^*) = \mathcal{A}_2^*, \mathcal{B}_1(x, y) = \mathcal{B}_2(\varphi(x), \varphi(y))$$

*for all $x, y \in (\mathcal{A}_1 \bowtie \mathcal{A}_1^*)$.*

**Proposition 4.8** *Two double constructions of anti-Frobenius algebras are isomorphic if and only if their corresponding antisymmetric infinitesimal anti-bialgebras are isomorphic.*

**Proposition 4.9** *Let $(\mathcal{A}, \Delta)$ be an antisymmetric infinitesimal anti-bialgebra. Then, its dual $(\mathcal{A}^*, \beta)$ given in Remark 4.3 is also an antisymmetric infinitesimal anti-bialgebra.*

**Example 4.10** *Let $\{e_1, ..., e_n\}$ be a basis of $\mathcal{A}$ and $\{e_1^*, ..., e_n^*\}$ be its duals basis. Set $e_i \cdot e_j = \sum_{k=1}^n c_{ij}^k e_k$ and $e_i^* \circ e_j^* = \sum_{k=1}^n f_{ij}^k e_k^*$. Therefore, we have $\Delta(e_k) = \sum_{i,j=1}^n f_{ij}^k e_i \otimes e_j$.*

$$R^*(e_i)e_j^* = \sum_{k=1}^n c_{ij}^k e_k^*, \quad L^*(e_i)e_j^* = \sum_{k=1}^n c_{ij}^k e_k^*,$$
$$R_\circ^*(e_i^*)e_j = \sum_{k=1}^n f_{ki}^j e_k, \quad L_\circ^*(e_i^*)e_j = \sum_{k=1}^n f_{ki}^j e_k.$$

*Notifie that, for $n = 2$, there are two non-isomorphic 2-dimensional antiassociative algebras $\mathcal{A}$ given by the following:*

$$e_i \cdot e_j = 0, \qquad e_1 \cdot e_1 = e_2.$$

*Now, we discuss their antisymetric infinitesimal anti-bialgebras and anti-Frobenius algebras on the direct sum $A \oplus A^*$.*
*Let*

$$\Delta(e_1) = f_{11}^1 e_1 \otimes e_1 + f_{12}^1 e_1 \otimes e_2 + f_{21}^1 e_2 \otimes e_1 + f_{22}^1 e_2 \otimes e_2, \quad f_{ij}^1 \in \mathbb{C},$$

*and*

$$\Delta(e_2) = f_{11}^2 e_1 \otimes e_1 + f_{12}^2 e_1 \otimes e_2 + f_{21}^2 e_2 \otimes e_1 + f_{22}^2 e_2 \otimes e_2, \quad f_{ij}^2 \in \mathbb{C}.$$

*Case(I). $e_i \cdot e_j = 0$: This is the trivial class. By the use of conditions (4.1-4.2) the antisymetric infinitesimal anti-bialgebra $\Delta$ is any.*
*Case(II). $e_1 \cdot e_1 = e_2$. Conditions (4.1-4.2) lead to the following two systems*

$$\begin{cases} -f_{11}^2 = 0, & -f_{12}^2 = f_{11}^1, \\ -f_{11}^1 = f_{21}^2, & f_{21}^1 + f_{12}^1 = -f_{22}^2 \end{cases} \quad \text{and} \quad \begin{cases} 2f_{11}^1 = 0, \\ 2f_{12}^1 + f_{21}^1 = 0, \end{cases}$$

*which implies $f_{11}^1 = f_{11}^2 = f_{12}^2 = f_{21}^2 = f_{22}^2 = 0$ and $f_{12}^1 = -f_{21}^1$. Thus the antisymetric infinitesimal anti-bialgebra $\Delta$ is given by the following relation*

$$\Delta(e_1) = f_{12}^1 e_1 \otimes e_2 - f_{12}^1 e_2 \otimes e_1 + f_{22}^1 e_2 \otimes e_2, \quad \Delta(e_2) = 0. \tag{4.3}$$

*The product on the dual space is the following relations:*

$$e_1^* \circ e_2^* = f_{12}^1 e_1^*, \quad e_2^* \circ e_1^* = -f_{12}^1 e_1^* \quad \text{and} \quad e_2^* \circ e_2^* = f_{22}^1 e_1^*. \tag{4.4}$$

*Using relation (2.15) when $l_{\mathcal{A}} = R^*, r_{\mathcal{A}} = L^*, l_{\mathcal{B}} = l_{\mathcal{A}^*} = R_\circ^*, r_{\mathcal{B}} = r_{\mathcal{A}^*} = L_\circ^*$, we obtain the double construction of quadratic antiassociative algebra $(\mathcal{A} \oplus \mathcal{A}^*, *, \mathcal{B})$ associated to $(\mathcal{A}, \cdot)$ and $(\mathcal{A}^*, \circ)$*

*given explicitly by the following relations:*

$$
\begin{aligned}
(e_1+e_1^*)*(e_1+e_1^*) &= (e_1\cdot e_1+R_\circ^*(e_1^*)e_1+L_\circ^*(e_1^*)e_1)+(e_1^*\circ e_1^*+R_\cdot^*(e_1)e_1^*+L_\cdot^*(e_1)e_1^*),\\
&= (1-2f_{12}^1)e_2,\\
(e_1+e_1^*)*(e_1+e_2^*) &= (e_1\cdot e_1+R_\circ^*(e_1^*)e_1+L_\circ^*(e_2^*)e_1)+(e_1^*\circ e_2^*+R_\cdot^*(e_1)e_2^*+L_\cdot^*(e_1)e_1^*),\\
&= -f_{12}^1e_1+(1-f_{12}^1-f_{22}^1)e_2+2f_{12}^1e_1^*,\\
(e_1+e_1^*)*(e_2+e_1^*) &= (e_1\cdot e_2+R_\circ^*(e_1^*)e_2+L_\circ^*(e_1^*)e_1)+(e_1^*\circ e_1^*+R_\cdot^*(e_1)e_1^*+L_\cdot^*(e_2)e_1^*),\\
&= -f_{12}^1(e_2+e_1^*),\\
(e_1+e_1^*)*(e_2+e_2^*) &= (e_1\cdot e_2+R_\circ^*(e_1^*)e_2+L_\circ^*(e_2^*)e_1)+(e_1^*\circ e_2^*+R_\cdot^*(e_1)e_2^*+L_\cdot^*(e_2)e_1^*),\\
&= f_{12}^1(e_1+e_1^*)+f_{22}^1e_2,\\
(e_1+e_2^*)*(e_1+e_1^*) &= (e_1\cdot e_1+R_\circ^*(e_2^*)e_1+L_\circ^*(e_1^*)e_1)+(e_2^*\circ e_1^*+R_\cdot^*(e_1)e_1^*+L_\cdot^*(e_1)e_1^*),\\
&= (1-f_{22}^1-f_{12}^1)e_2+f_{12}^1e_1,\\
(e_1+e_2^*)*(e_1+e_2^*) &= (e_1\cdot e_1+R_\circ^*(e_2^*)e_1+L_\circ^*(e_2^*)e_1)+(e_2^*\circ e_2^*+R_\cdot^*(e_1)e_2^*+L_\cdot^*(e_1)e_2^*),\\
&= (2f_{12}^1+f_{22}^1)e_1+(1+f_{22}^1)e_2+(2f_{12}^1+f_{22}^1e_1^*,\\
(e_2+e_1^*)*(e_1+e_1^*) &= (e_2\cdot e_1+R_\circ^*(e_1^*)e_1+L_\circ^*(e_2^*)e_1)+(e_1^*\circ e_1^*+R_\cdot^*(e_2)e_1^*+L_\cdot^*(e_1)e_1^*),\\
&= -f_{12}^1e_1^*-f_{12}^1e_2,\\
(e_2+e_1^*)*(e_1+e_2^*) &= (e_2\cdot e_1+R_\circ^*(e_1^*)e_1+L_\circ^*(e_2^*)e_2)+(e_1^*\circ e_2^*+R_\cdot^*(e_2)e_2^*+L_\cdot^*(e_1)e_1^*),\\
&= -f_{12}^1e_2+(f_{12}^1+f_{22}^1)e_1^*,\\
(e_2+e_2^*)*(e_1+e_1^*) &= (e_2\cdot e_1+R_\circ^*(e_2^*)e_1+L_\circ^*(e_1^*)e_2)+(e_2^*\circ e_1^*+R_\cdot^*(e_2)e_1^*+L_\cdot^*(e_1)e_2^*),\\
&= f_{12}^1e_1+f_{22}^1e_2,\\
(e_2+e_2^*)*(e_1+e_2^*) &= (e_2\cdot e_1+R_\circ^*(e_2^*)e_1+L_\circ^*(e_2^*)e_2)+(e_2^*\circ e_2^*+R_\cdot^*(e_2)e_2^*+L_\cdot^*(e_1)e_2^*),\\
&= f_{12}^1(e_1+e_1^*)+f_{22}^1(e_2+2e_1^*),\\
(e_2+e_2^*)*(e_2+e_1^*) &= (e_2\cdot e_2+R_\circ^*(e_2^*)e_2+L_\circ^*(e_1^*)e_2)+(e_2^*\circ e_1^*+R_\cdot^*(e_2)e_1^*+L_\cdot^*(e_2)e_2^*),\\
&= -f_{12}^1e_1^*,\\
(e_2+e_2^*)*(e_2+e_2^*) &= (e_2\cdot e_2+R_\circ^*(e_2^*)e_2+L_\circ^*(e_2^*)e_2)+(e_2^*\circ e_2^*+R_\cdot^*(e_2)e_2^*+L_\cdot^*(e_2)e_2^*),\\
&= 3f_{22}^1e_1^*,\\
(e_1+e_2^*)*(e_2+e_1^*) &= (e_1\cdot e_2+R_\circ^*(e_2^*)e_2+L_\circ^*(e_1^*)e_1)+(e_2^*\circ e_1^*+R_\cdot^*(e_1)e_1^*+L_\cdot^*(e_2)e_2^*),\\
&= -f_{12}^1e_2+f_{22}^1e_1^*,\\
(e_1+e_2^*)*(e_2+e_2^*) &= (e_1\cdot e_2+R_\circ^*(e_2^*)e_2+L_\circ^*(e_2^*)e_1)+(e_2^*\circ e_2^*+R_\cdot^*(e_1)e_2^*+L_\cdot^*(e_2)e_2^*),\\
&= (f_{12}^1+f_{22}^1)e_2+(2f_{22}^1+f_{12}^1)e_1^*,\\
(e_2+e_1^*)*(e_2+e_1^*) &= (e_2\cdot e_2+R_\circ^*(e_1^*)e_2+L_\circ^*(e_1^*)e_2)+(e_1^*\circ e_1^*+R_\cdot^*(e_2)e_1^*+L_\cdot^*(e_2)e_1^*),\\
&= -2f_{12}^1e_1^*,\\
(e_2+e_1^*)*(e_2+e_2^*) &= (e_2\cdot e_2+R_\circ^*(e_1^*)e_2+L_\circ^*(e_2^*)e_2)+(e_1^*\circ e_2^*+R_\cdot^*(e_2)e_2^*+L_\cdot^*(e_2)e_1^*),\\
&= f_{22}^1e_1^*.
\end{aligned}
$$

# 5 A comparative analysis of antiassociative and Mock-Lie bialgebras

**Definition 5.1** *[10] A vector space $V$ is a module over a Mock Lie algebra $\mathcal{G}$, if there is a linear map(a representation) $\rho:\mathcal{G}\to End(V)$ such that*

$$\rho(x\diamond y)(v)=-\rho(x)(\rho(y)v)-\rho(y)(\rho(x)v) \tag{5.1}$$

*for any $x,y\in\mathcal{G}$ and $v\in V$.*

**Theorem 5.2** *[3] Let $\mathcal{G}$ and $\mathcal{H}$ be two Mock Lie algebras and let $\mu : \mathcal{H} \to \mathfrak{gl}(\mathcal{G})$ and $\rho : \mathcal{G} \to \mathfrak{gl}(\mathcal{H})$ be two Mock Lie algebra representations. Then, $(\mathcal{G}, \mathcal{H}, \rho, \mu)$ is called a matched pair of the Mock Lie algebras $\mathcal{G}$ and $\mathcal{H}$, denoted by $\mathcal{H} \bowtie_{\mu}^{-1,\rho} \mathcal{G}$ if and only if $\mu$ and $\rho$ satisfy: for all $x, y \in \mathcal{G}, a, b \in \mathcal{H}$,*

$$\rho(x)[a, b] + [\rho(x)a, b] + [a, \rho(x)b] + \rho(\mu(a)x)b + \rho(\mu(b)x)a = 0, \tag{5.2}$$

$$\mu(a)[x, y] + [\mu(a)x, y] + [x, \mu(a)y] + \mu(\rho(x)a)y + \mu(\rho(y)a)x = 0. \tag{5.3}$$

*In this case, $(\mathcal{G} \oplus \mathcal{H}, [,])$ defines a Mock Lie algebra with respect to the product $*$ satisfying:*

$$[(x + a), (y + b)] = [x, y] + \mu(a)y + \mu(b)x + [a, b] + \rho(x)b + \rho(y)a. \tag{5.4}$$

**Proposition 5.3** *Let $(\mathcal{A}, \cdot)$ be an antiassociative algebra. Define the anticommutator by*

$$[x, y] = x \cdot y + y \cdot x, \quad \forall x, y \in A. \tag{5.5}$$

*Then it is a Mock Lie algebra and we denote it by $(\mathfrak{G}(\mathcal{A}), [\ ,\ ])$ or simply $\mathfrak{G}(\mathcal{A})$, which is called* **the the sub-adjacent Mock Lie algebra of** $(\mathcal{A}, \cdot)$.

**Corollary 5.4** *Let $(\mathcal{A}, \cdot)$ be an antiassociative algebra and $V$ be a vector space over $\mathbb{K}$. Consider two linear maps, $l, r : \mathcal{A} \to \mathfrak{gl}(V)$, such that $(l, r, V)$ is a bimodule of $\mathcal{A}$. Then, the map: $l + r : \mathcal{A} \longrightarrow \mathfrak{gl}(V)$ $x \longmapsto l_x + r_x$, is a linear representation of the sub-adjacent Mock Lie algebra $(\mathfrak{G}(\mathcal{A}), [\ ,\ ])$ of $\mathcal{A}$.*

**Proof:** Let $(l, r, V)$ be a bimodule of the antiassociative algebra $\mathcal{A}$. Then, $\forall x, y \in \mathcal{A}$ $\quad l_x r_y = -r_y l_x; l_{xy} = -l_x l_y; r_{yx} = -r_y r_x$. Besides, it is a matter of straightforward computation to show that $l + r$ is a linear map on $\mathcal{A}$. Then, we have:

$$\begin{aligned} -[(l+r)(x), (l+r)(y)] &= -[l_x + r_x, l_y + r_y] \\ &= -[l_x, l_y] - [l_x, r_y] - [r_x, l_y] - [r_x, r_y] \\ &= -[l_x, l_y] - [r_x, r_y] \\ &= -l_x l_y - l_y l_x - r_x r_y - r_y r_x \\ &= (l+r)_{xy} + (l+r)_{yx} = (l+r)_{[x,y]}. \end{aligned}$$

Therefore, $(l, r, V)$ is a bimodule of $\mathcal{A}$ implies that $l + r$ is a representation of the linear representation of the sub-adjacent Mock Lie algebra of $\mathcal{A}$. □

**Proposition 5.5** *Let $(\mathcal{A}, \mathcal{B}, l_{\mathcal{A}}, r_{\mathcal{A}}, l_{\mathcal{B}}, r_{\mathcal{B}})$ be a matched pair of antiassociative algebras. Then, $(\ \mathfrak{G}(\mathcal{A}), \mathfrak{G}(\mathcal{B}), l_{\mathcal{A}} + r_{\mathcal{A}}, l_{\mathcal{B}} + r_{\mathcal{B}}\ )$ is a matched pair of sub-adjacent Mock Lie algebras $\mathfrak{G}(\mathcal{A})$ and $\mathfrak{G}(\mathcal{B})$.*

**Proof:** By the direct computation and with the corollary 5.5, we obtain the result.

□

So, we have the following result:

**Proposition 5.6** *The anticommutator of a double construction of anti-Frobenius algebra is a Manin triple of Mock Lie algebras.*

Recall a Mock Lie bialgebra structure on a Mock Lie algebra $\mathfrak{G}$ is a linear map $\delta : \mathfrak{G} \to \mathfrak{G} \otimes \mathfrak{G}$ such that $\delta^* : \mathfrak{G}^* \otimes \mathfrak{G}^* \to \mathfrak{G}^*$ defines a Mock Lie algebra structure on $\mathfrak{G}^*$ and $\delta$ satisfies

$$\delta[x, y] = -(\mathrm{ad}(x) \otimes \mathrm{id} + \mathrm{id} \otimes \mathrm{ad}(x))\delta(y) - (\mathrm{ad}(y) \otimes \mathrm{id} + \mathrm{id} \otimes \mathrm{ad}(y))\delta(x), \quad \forall x, y \in \mathfrak{G}, \tag{5.6}$$

where $\mathrm{ad}(x)(y) = [x, y]$ for any $x, y \in \mathfrak{G}$. We denoted it by $(\mathfrak{G}, \delta)$.

**Proposition 5.7** *Let $(\mathcal{A}, \Delta)$ be an antisymmetric infinitesimal anti-bialgebra. Then $(\mathfrak{G}(\mathcal{A}), \delta)$ is a Mock Lie bialgebra, where $\delta = \Delta + \sigma\Delta$.*

**Example 5.8** *From the example4.10, the antisymetric infinitesimal anti-bialgebra* $\Delta$ *on* $\mathcal{A} : e_1 \cdot e_1 = e_2$ *is given by the following relation*

$$\Delta(e_1) = f_{12}^1 e_1 \otimes e_2 - f_{12}^1 e_2 \otimes e_1 + f_{22}^1 e_2 \otimes e_2, \quad \Delta(e_2) = 0.$$

*Then, the Mock Lie bialgebra* $(\mathfrak{G}(\mathcal{A}), \delta)$ *is given by*

$$\delta(e_1) = 2f_{22}^1 e_2 \otimes e_2, \quad \delta(e_2) = 0.$$

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