Original Research Article

A Dependent Risk Model with Constant Dividend Barrier and Stochastic Premium Income

Abstract: This paper considers the Gerber-Shiu discounted penalty function for a class of dependent risk models with a constant dividend barrier and stochastic premium income. The system of integro-differential equations satisfied by the Gerber-Shiu discounted penalty function is derived. In particular, an analytical expression for the Laplace transform of ruin time is derived under exponential claim conditions. Finally, numerical examples are presented.
Keywords: Random premium income; Constant dividend barrier; Gerber-Shiu discount penalty function; Integro-differential equations.

1 Introduction

The classical compound Poisson risk model assumes that premium income is a linear function of time, and that claim amounts and claim inter-arrival times are mutually independent. However, in actual insurance operations, an insurer's premium income often exhibits randomness. Bao (2006) pioneered the introduction of stochastic income into risk models by assuming that the income process follows a Poisson process; building on this, Bao and Ye (2007) further considered a delayed renewal risk model in which premium income follows a Poisson process. Labbé and Sendova (2009) investigated a risk model where premium income follows a compound Poisson process.

To overcome the limitation of independence between claim amounts and claim inter-arrival times, researchers have proposed various dependence structures. A common approach is to use Copula functions to characterize the dependence between claim amounts and claim inter-arrival times, for example, see Cossette et al. (2008) and the references therein. Albrecher and Boxma (2004) constructed a risk model where the claim inter-arrival times determine the distribution of the subsequent claim amounts via a random threshold, and derived the integro-differential equation satisfied by the Gerber-Shiu function. Boudreault et al. (2006) developed a risk model where the distribution of the next claim amounts depends on the length of the previous inter-arrival times, obtaining the integro-differential equation for the Gerber-Shiu function and its Laplace transform. Zhang and Yang (2011) incorporated a perturbation term into the risk model and established a dependence structure between claim amounts and inter-arrival times using the FGM Copula, deriving the integro-differential equation and the defective renewal equation satisfied by the Gerber-Shiu function. Xie and Zou (2013) formulated a risk model where the inter-arrival time determines the distributions of both the subsequent claim amount and the premium amount, utilizing the Dickson-Hipp operator to obtain the Laplace transform of the Gerber-Shiu function and the defective renewal equation.

Since its inception, participating insurance has become a significant product in the insurance market. Under a constant dividend barrier strategy, when the surplus reaches a predetermined level, the excess is distributed to policyholders as dividends. Lin et al. (2003) provided a systematic analysis of the classical compound Poisson risk model with a constant dividend barrier, deriving the integro-differential equation satisfied by the Gerber-Shiu function. Landriault (2008) introduced a constant dividend barrier into the dependent risk model proposed by Boudreault et al. (2006), obtaining the corresponding Gerber-Shiu function. Zou et al. (2014), building upon Xie and Zou (2013), incorporated a constant dividend barrier, derived the integral equations satisfied by the Gerber-Shiu discounted penalty function and the expected dividend payments, and discussed the problem of the optimal dividend barrier.

This paper constructs a dependent risk model incorporating a constant dividend barrier and stochastic premium income. The integro-differential equations satisfied by the Gerber-Shiu discounted penalty function under two types of mechanisms are derived. Under the assumption of exponentially distributed claim sizes, an analytical expression for the Laplace transform of the ruin time is further obtained, and a systematic analysis of the impact of key parameters on the ruin time is conducted using numerical examples.

2 The Model

This paper considers the surplus process of an insurance company at time t, denoted by $\{U_b(t), t \geq 0\}$, which satisfies:

$$U_b(t) = u + \sum_{j=1}^{N(t)} X_j - \sum_{i=1}^{N_1(t)} Y_i, \quad t \geq 0, \tag{1}$$

where $u = U_b(0) \geq 0$ is the initial surplus. When the surplus reaches the dividend barrier $b > 0$, the excess is distributed to policyholders as dividends, and the surplus is maintained at level b. $\{X_j\}_{j\geq 1}$ is a sequence of independent and identically distributed (i.i.d.) positive random variables, representing the amounts of each premium income, with a distribution function G. $N_1(t)$ is a Poisson process with intensity $\lambda_1 > 0$, representing the number of claims up to time t; Y_i is the ith claim amount. $N(t)$ is a Poisson process with intensity $\lambda > 0$, representing the number of premium income occurrences up to time t.

Let $\{V_i\}_{i\geq 1}$ denote the sequence of inter-claim times, which follows an exponential distribution with parameter λ_1; $\{W_i\}_{i\geq 1}$ denotes the sequence of inter-premium income times, following an exponential distribution with parameter λ; $\{M_i\}_{i\geq 1}$ is a sequence of i.i.d. non-negative random variables, representing the random thresholds, following an exponential distribution with parameter λ_2.

Define the time of ruin as $T_b = \inf\{t \geq 0 : U_b(t) < 0\}$ (with $T_b = \infty$ if the set is empty). Let $w(x_1, x_2) \geq 0$ be the penalty function and let $\delta \geq 0$ be the discount factor. The Gerber-Shiu discounted penalty function triggered by the ith type of mechanism is defined as:

$$\Phi_i(u) = \mathbb{E}\left[e^{-\delta T_b} w(U_b(T_b^-), |U_b(T_b)|) I(T_b < \infty, J = i) \,\middle|\, U_b(0) = u\right], \quad u \geq 0,\ i = 1, 2, \tag{2}$$

where $U_b(T_b^-)$ is the instantaneous surplus prior to ruin, $|U_b(T_b)|$ is the deficit at ruin, and J is a random variable indicating the type of ruin mechanism.

Assume the risk process is governed by the following mechanism: If the inter-arrival time V_i exceeds the threshold M_i, then the next claim amount Y_i follows the distribution function F_1, and the corresponding Gerber-Shiu function is denoted as $\Phi_1(u)$; otherwise, the claim amount Y_i follows the distribution function F_2, and the corresponding Gerber-Shiu function is denoted as $\Phi_2(u)$.

This paper assumes that the premium income amounts follow an exponential distribution, given by:

$$G(x) = 1 - e^{-\mu x}, \quad x > 0, \tag{3}$$

where $\mu > 0$ is the parameter of the exponential distribution for premium amounts.

3 Integro-Differential Equations

This section derives the integro-differential equations satisfied by $\Phi_1(u)$ and $\Phi_2(u)$ on the interval $0 \leq u \leq b$.

Theorem: When $0 \leq u \leq b$, $\Phi_1(u)$ and $\Phi_2(u)$ satisfy the following integro-differential equations, respectively:

$$\begin{aligned}
\Phi_1^{(1)}(u) = {} & \frac{(\lambda_1 + \delta)\mu}{\lambda_1 + \lambda_2 + \delta}\Phi_1(u) + \frac{\lambda_1\lambda_2}{(\lambda + \lambda_1 + \delta)(\lambda^* + \delta)}\left(\frac{d}{du} - \mu\right)\left(\int_0^u \Phi_1(u - y) f_1(y)\, dy\right) \\
& + \frac{\lambda_1\lambda_2}{(\lambda + \lambda_1 + \delta)(\lambda^* + \delta)}\left(\omega_1'(u) - \mu\omega_1(u)\right) \\
& + \frac{\lambda_1}{\lambda^* + \delta}\left(\frac{d}{du} - \mu\right)\left(\int_0^u \Phi_2(u - y) f_2(y)\, dy\right) + \frac{\lambda_1}{\lambda^* + \delta}\left(\omega_2'(u) - \mu\omega_2(u)\right),
\end{aligned} \tag{4}$$

$$\Phi_2^{(1)}(u) = \frac{(\lambda_1+\delta)\mu}{\lambda_1+\lambda_2+\delta}\Phi_2(u) + \frac{\lambda_1\lambda_2}{(\lambda+\lambda_1+\delta)(\lambda^*+\delta)}\left(\frac{d}{du}-\mu\right)\left(\int_0^u \Phi_1(u-y)f_1(y)\,dy\right)$$
$$+\frac{\lambda_1\lambda_2}{(\lambda+\lambda_1+\delta)(\lambda^*+\delta)}(\omega_1'(u)-\mu\omega_1(u)) \tag{5}$$
$$+\frac{\lambda_1}{\lambda^*+\delta}\left(\frac{d}{du}-\mu\right)\left(\int_0^u \Phi_2(u-y)f_2(y)\,dy\right)+\frac{\lambda_1}{\lambda^*+\delta}(\omega_2'(u)-\mu\omega_2(u)).$$

where $\lambda^* = \lambda+\lambda_1+\lambda_2$ and $w_i(u) = \int_u^\infty w(u, y-u)\,dF_i(y)$, $i = 1, 2$.

Proof. Let $L = \min\{V_1, W_1\}$ be the time of the first event (premium income or claim). For $0 \le u \le b$, it holds that

$$\Phi_1(u) = \int_0^\infty \Pr(L=t, L=W_1)e^{-\delta t}\left(\int_0^{b-u}\Phi_1(u+x)\,dG(x)+\int_{b-u}^\infty \Phi_1(b)\,dG(x)\right)dt$$
$$+\int_0^\infty \Pr(L=t,L=V_1)e^{-\delta t}\Bigg(\Pr(M_1<t)\left[\int_0^u\Phi_1(u-y)\,dF_1(y)+\int_u^\infty w(u,y-u)\,dF_1(y)\right]$$
$$+\Pr(M_1\ge t)\left[\int_0^u\Phi_2(u-y)\,dF_2(y)+\int_u^\infty w(u,y-u)\,dF_2(y)\right]\Bigg)dt. \tag{4}$$

Note that

$$\Pr(L=W_1)=\frac{\lambda}{\lambda+\lambda_1},\quad \Pr(L=V_1)=\frac{\lambda_1}{\lambda+\lambda_1},$$
$$\Pr(L>t\mid L=W_1)=\Pr(L>t\mid L=V_1)=\exp\big(-(\lambda+\lambda_1)t\big).$$

Equation (4) can be rewritten as

$$\Phi_1(u)=\frac{\lambda}{\lambda+\lambda_1+\delta}\left(\int_0^{b-u}\Phi_1(u+x)\,dG(x)+\int_{b-u}^\infty\Phi_1(b)\,dG(x)\right)$$
$$+\frac{\lambda_1\lambda_2}{(\lambda+\lambda_1+\delta)(\lambda^*+\delta)}\left[\int_0^u\Phi_1(u-y)\,dF_1(y)+w_1(u)\right] \tag{5}$$
$$+\frac{\lambda_1}{\lambda^*+\delta}\left[\int_0^u\Phi_2(u-y)\,dF_2(y)+w_2(u)\right],$$

where $\lambda^*=\lambda+\lambda_1+\lambda_2$, $w_i(u)=\int_u^\infty w(u,y-u)\,dF_i(y)$, $i=1,2$.

Using the exponential distribution of premium amounts (3) and the substitution $u+x=y$, we obtain

$$\Phi_1(u)=\frac{\lambda\mu}{\lambda+\lambda_1+\delta}\left(\int_u^b\Phi_1(y)e^{-\mu(y-u)}\,dy+\Phi_1(b)e^{-\mu(b-u)}\right)$$
$$+\frac{\lambda_1\lambda_2}{(\lambda+\lambda_1+\delta)(\lambda^*+\delta)}\left[\int_0^u\Phi_1(u-y)\,dF_1(y)+w_1(u)\right] \tag{6}$$
$$+\frac{\lambda_1}{\lambda^*+\delta}\left[\int_0^u\Phi_2(u-y)\,dF_2(y)+w_2(u)\right].$$

Differentiating (6) gives

$$\Phi_1^{(1)}(u)=\frac{(\lambda_1+\delta)\mu}{\lambda_1+\lambda_2+\delta}\Phi_1(u)+\frac{\lambda_1\lambda_2}{(\lambda+\lambda_1+\delta)(\lambda^*+\delta)}\left(\frac{d}{du}-\mu\right)\left(\int_0^u\Phi_1(u-y)f_1(y)\,dy\right)$$
$$+\frac{\lambda_1\lambda_2}{(\lambda+\lambda_1+\delta)(\lambda^*+\delta)}(\omega_1'(u)-\mu\omega_1(u)) \tag{7}$$
$$+\frac{\lambda_1}{\lambda^*+\delta}\left(\frac{d}{du}-\mu\right)\left(\int_0^u\Phi_2(u-y)f_2(y)\,dy\right)+\frac{\lambda_1}{\lambda^*+\delta}(\omega_2'(u)-\mu\omega_2(u)).$$

Through a similar argument, we have

$$\begin{aligned}\Phi_2(u) = {} & \frac{\lambda\mu}{\lambda+\lambda_1+\delta}\left(\int_u^b \Phi_2(y)e^{-\mu(y-u)}\,dy + \Phi_2(b)e^{-\mu(b-u)}\right) \\ & + \frac{\lambda_1\lambda_2}{(\lambda+\lambda_1+\delta)(\lambda^*+\delta)}\left[\int_0^u \Phi_1(u-y)\,dF_1(y) + w_1(u)\right] \\ & + \frac{\lambda_1}{\lambda^*+\delta}\left[\int_0^u \Phi_2(u-y)\,dF_2(y) + w_2(u)\right],\end{aligned} \tag{8}$$

and differentiating (8) yields

$$\begin{aligned}\Phi_2^{(1)}(u) = {} & \frac{(\lambda_1+\delta)\mu}{\lambda_1+\lambda_2+\delta}\Phi_2(u) + \frac{\lambda_1\lambda_2}{(\lambda+\lambda_1+\delta)(\lambda^*+\delta)}\left(\frac{d}{du}-\mu\right)\left(\int_0^u \Phi_1(u-y)f_1(y)\,dy\right) \\ & + \frac{\lambda_1\lambda_2}{(\lambda+\lambda_1+\delta)(\lambda^*+\delta)}(\omega_1'(u)-\mu\omega_1(u)) \\ & + \frac{\lambda_1}{\lambda^*+\delta}\left(\frac{d}{du}-\mu\right)\left(\int_0^u \Phi_2(u-y)f_2(y)\,dy\right) + \frac{\lambda_1}{\lambda^*+\delta}(\omega_2'(u)-\mu\omega_2(u)).\end{aligned} \tag{9}$$

Equations (7) and (9) are exactly (**??**) and (**??**), which completes the proof. □

4 Exact results for the Laplace transform of ruin time under exponential claim amounts

In this section, we assume that both types of claim amounts follow exponential distributions, i.e., $F_i(y) = 1 - e^{-\mu_i y}$, $i = 1, 2$, where $\mu_1, \mu_2 > 0$. Let $w(x_1, x_2) \equiv 1$; then $\Phi_1(u)$ and $\Phi_2(u)$ reduce to the Laplace transforms of the ruin time, denoted as $\Psi_1(u)$ and $\Psi_2(u)$, respectively.

To ensure the stable operation of the insurance company, the following safety loading condition is assumed to hold :

$$\lambda \cdot \frac{1}{\mu} > \lambda_1\left(\frac{\lambda_2}{\lambda_1+\lambda_2}\cdot\frac{1}{\mu_1} + \frac{\lambda_1}{\lambda_1+\lambda_2}\cdot\frac{1}{\mu_2}\right). \tag{10}$$

Equations (**??**) and (**??**) become

$$\begin{aligned}\Psi_1^{(1)}(u) = {} & \frac{(\lambda_1+\delta)\mu}{\lambda_1+\lambda_2+\delta}\Psi_1(u) + \frac{\lambda_1\lambda_2}{(\lambda+\lambda_1+\delta)(\lambda^*+\delta)}\left(\frac{d}{du}-\mu\right)\left(\int_0^u \Psi_1(u-y)\mu_1 e^{-\mu_1 y}dy\right) \\ & - \frac{\lambda_1\lambda_2}{(\lambda+\lambda_1+\delta)(\lambda^*+\delta)}(\mu_1 e^{-\mu_1 u} + \mu e^{-\mu_1 u}) \\ & + \frac{\lambda_1}{\lambda^*+\delta}\left(\frac{d}{du}-\mu\right)\left(\int_0^u \Psi_2(u-y)\mu_2 e^{-\mu_2 y}dy\right) \\ & - \frac{\lambda_1}{\lambda^*+\delta}(\mu_2 e^{-\mu_2 u} + \mu e^{-\mu_2 u}),\end{aligned} \tag{11}$$

$$\begin{aligned}\Psi_2^{(1)}(u) = {} & \frac{(\lambda_1+\delta)\mu}{\lambda_1+\lambda_2+\delta}\Psi_2(u) + \frac{\lambda_1\lambda_2}{(\lambda+\lambda_1+\delta)(\lambda^*+\delta)}\left(\frac{d}{du}-\mu\right)\left(\int_0^u \Psi_1(u-y)\mu_1 e^{-\mu_1 y}dy\right) \\ & - \frac{\lambda_1\lambda_2}{(\lambda+\lambda_1+\delta)(\lambda^*+\delta)}(\mu_1 e^{-\mu_1 u} + \mu e^{-\mu_1 u}) \\ & + \frac{\lambda_1}{\lambda^*+\delta}\left(\frac{d}{du}-\mu\right)\left(\int_0^u \Psi_2(u-y)\mu_2 e^{-\mu_2 y}dy\right) \\ & - \frac{\lambda_1}{\lambda^*+\delta}(\mu_2 e^{-\mu_2 u} + \mu e^{-\mu_2 u}).\end{aligned} \tag{12}$$

Define the convolution integral:

$$I_1(u) = \int_0^u \Psi_1(u-y)\mu_1 e^{-\mu_1 y}\,dy, \qquad I_2(u) = \int_0^u \Psi_2(u-y)\mu_2 e^{-\mu_2 y}\,dy. \tag{13}$$

Using properties of the exponential function, we have

$$I_1'(u) = \mu_1\Psi_1(u) - \mu_1 I_1(u), \tag{14}$$

$$I_2'(u) = \mu_2\Psi_2(u) - \mu_2 I_2(u). \tag{15}$$

Furthermore, define:

$$J_1(u) = \left(\frac{d}{du} - \mu\right) I_1(u) = \mu_1\Psi_1(u) - (\mu_1 + \mu)I_1(u), \tag{16}$$

$$J_2(u) = \left(\frac{d}{du} - \mu\right) I_2(u) = \mu_2\Psi_2(u) - (\mu_2 + \mu)I_2(u). \tag{17}$$

Using the above relationships, the differential relationships between J_1, J_2 and Ψ_1, Ψ_2 can be established:

$$J_1'(u) + \mu_1 J_1(u) = \mu_1\Psi_1'(u) - \mu\mu_1\Psi_1(u), \tag{18}$$

$$J_2'(u) + \mu_2 J_2(u) = \mu_2\Psi_2'(u) - \mu\mu_2\Psi_2(u). \tag{19}$$

Define

$$A = \frac{\lambda\mu}{\lambda + \lambda_1 + \delta},$$
$$B = \frac{\lambda_1\lambda_2}{(\lambda + \lambda_1 + \delta)(\lambda^* + \delta)},$$
$$C = \frac{\lambda_1}{\lambda^* + \delta}.$$

Since the exponential parameters for the two claim types differ $(\mu_1 \neq \mu_2)$, a composite operator is defined:

$$\mathcal{L} = \left(\frac{d}{du} + \mu_1\right)\left(\frac{d}{du} + \mu_2\right).$$

Applying $\mathcal{L}$ to (11) and (12), using (18) and (19) to eliminate J_1, J_2, and noting that the non-homogeneous terms $e^{-\mu_1 u}$ and $e^{-\mu_2 u}$ vanish under the action of $\mathcal{L}$, two coupled third-order differential equations are obtained:

$$\Psi_1^{(3)}(u) + \alpha_1\Psi_1^{(2)}(u) + \beta_1\Psi_1^{(1)}(u) + \gamma_1\Psi_1(u) - Q_1(D)\Psi_2(u) = 0, \tag{20}$$

$$\Psi_2^{(3)}(u) + \alpha_2\Psi_2^{(2)}(u) + \beta_2\Psi_2^{(1)}(u) + \gamma_2\Psi_2(u) - Q_2(D)\Psi_1(u) = 0, \tag{21}$$

where

$$\begin{aligned}
\alpha_1 &= \mu_1 + \mu_2 - A - B\mu_1,\\
\beta_1 &= \mu_1\mu_2 - A(\mu_1 + \mu_2) - B\mu_1(\mu_2 - \mu),\\
\gamma_1 &= \mu_1\mu_2(B\mu - A),\\
\alpha_2 &= \mu_1 + \mu_2 - A - C\mu_2,\\
\beta_2 &= \mu_1\mu_2 - A(\mu_1 + \mu_2) - C\mu_2(\mu_1 - \mu),\\
\gamma_2 &= \mu_1\mu_2(C\mu - A).
\end{aligned}$$

The differential operator $Q_1(D)$ and $Q_2(D)$ are

$$Q_1(D) = C\mu_2 D^2 + C\mu_2(\mu_1 - \mu)D - C\mu\mu_1\mu_2,$$

$$Q_2(D) = B\mu_1 D^2 + B\mu_1(\mu_2 - \mu)D - B\mu\mu_1\mu_2.$$

From equation (21), we obtain

$$Q_2(D)\Psi_1(u) = P_2(D)\Psi_2(u), \tag{22}$$

where $P_2(D) = D^3 + \alpha_2 D^2 + \beta_2 D + \gamma_2$.

Similarly, from equation (20), we obtain

$$Q_1(D)\Psi_2(u) = P_1(D)\Psi_1(u), \tag{23}$$

where $P_1(D) = D^3 + \alpha_1 D^2 + \beta_1 D + \gamma_1$.

Using equations (22) and (23) to eliminate $\Psi_1(u)$ yields a sixth-order homogeneous differential equation for $\Psi_2(u)$:

$$\left[P_1(D)P_2(D) - Q_1(D)Q_2(D)\right]\Psi_2(u) \equiv 0,$$

where D denotes the differentiation operator. Since the product of the operators is commutative, $\Psi_1(u)$ also satisfies the same differential equation, thus yielding

$$\Psi_i^{(6)}(u) + p_5\Psi_i^{(5)}(u) + p_4\Psi_i^{(4)}(u) + p_3\Psi_i^{(3)}(u) + p_2\Psi_i^{(2)}(u) + p_1\Psi_i^{(1)}(u) + p_0\Psi_i(u) = 0, \quad i = 1, 2, \tag{24}$$

where

$$\begin{aligned}
p_5 &= 2(\mu_1 + \mu_2) - 2A - B\mu_1 - C\mu_2,\\
p_4 &= \alpha_1\alpha_2 + \beta_1 + \beta_2 - BC\mu_1\mu_2,\\
p_3 &= \alpha_1\beta_2 + \alpha_2\beta_1 + \gamma_1 + \gamma_2 - BC\mu_1\mu_2(\mu_1 + \mu_2 - 2\mu),\\
p_2 &= \alpha_1\gamma_2 + \alpha_2\gamma_1 + \beta_1\beta_2 - BC\mu_1\mu_2\left[\mu_1\mu_2 - 2\mu(\mu_1 + \mu_2) + \mu^2\right],\\
p_1 &= \beta_1\gamma_2 + \beta_2\gamma_1 + BC\mu\mu_1\mu_2\left[2\mu_1\mu_2 - \mu(\mu_1 + \mu_2)\right],\\
p_0 &= A\mu_1^2\mu_2^2(A - B\mu - C\mu).
\end{aligned}$$

The corresponding characteristic equation is

$$r^6 + p_5 r^5 + p_4 r^4 + p_3 r^3 + p_2 r^2 + p_1 r + p_0 = 0. \tag{25}$$

Let the six characteristic roots be $z_1, z_2, \dots, z_6$. The general solutions for $\Psi_1(u)$ and $\Psi_2(u)$ are

$$\Psi_1(u) = \sum_{i=1}^{6} c_{1i} e^{z_i u}, \tag{26}$$

$$\Psi_2(u) = \sum_{i=1}^{6} c_{2i} e^{z_i u}. \tag{27}$$

From equation (22) we obtain

$$c_{2i} = \frac{P_1(z_i)}{Q_1(z_i)} c_{1i} \equiv h_i c_{1i}, \quad i = 1, 2, \dots, 6.$$

Substituting (26) and (27) back into equations (11) and (12) and comparing the coefficients of the exponential terms, the coefficients for the $e^{-\mu_1 u}$ and $e^{-\mu_2 u}$ terms must be zero, leading to two algebraic equations:

$$\sum_{i=1}^{6} \frac{\mu_1}{z_i + \mu_1} c_{1i} = 1, \tag{28}$$

$$\sum_{i=1}^{6} \frac{\mu_2 h_i}{z_i + \mu_2} c_{1i} = 1. \tag{29}$$

Substituting $u = 0$ into (11) and (12) and their derivative forms yields the following four algebraic equations:

$$\sum_{i=1}^{6}(z_i - A)c_{1i} = -M, \tag{30}$$

$$\sum_{i=1}^{6} h_i(z_i - A)c_{1i} = -M, \tag{31}$$

$$\sum_{i=1}^{6} A_i c_{1i} = N, \tag{32}$$

$$\sum_{i=1}^{6} B_i c_{1i} = N, \tag{33}$$

where

$$\begin{aligned}
M &= B(\mu_1 + \mu) + C(\mu_2 + \mu), \\
N &= B\mu_1(\mu_1 + \mu) + C\mu_2(\mu_2 + \mu), \\
A_i &= z_i^2 - (A + B\mu_1)z_i + B\mu_1(\mu_1 + \mu) - \left[C\mu_2 z_i - C\mu_2(\mu_2 + \mu)\right]h_i, \\
B_i &= -B\mu_1 z_i + B\mu_1(\mu_1 + \mu) + \left[z_i^2 - (A + C\mu_2)z_i + C\mu_2(\mu_2 + \mu)\right]h_i.
\end{aligned}$$

Combining equations (28)–(33) gives

$$\begin{pmatrix} \frac{\mu_1}{z_1+\mu_1} & \cdots & \frac{\mu_1}{z_6+\mu_1} \\ \frac{\mu_2 h_1}{z_1+\mu_2} & \cdots & \frac{\mu_2 h_6}{z_6+\mu_2} \\ z_1 - A & \cdots & z_6 - A \\ h_1(z_1 - A) & \cdots & h_6(z_6 - A) \\ A_1 & \cdots & A_6 \\ B_1 & \cdots & B_6 \end{pmatrix} \begin{pmatrix} c_{11} \\ c_{12} \\ c_{13} \\ c_{14} \\ c_{15} \\ c_{16} \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ -M \\ -M \\ N \\ N \end{pmatrix}. \tag{34}$$

By Cramer's rule, we have $c_{1i} = \frac{\det(\mathbf{T}_i)}{\det(\mathbf{T})}$, $i = 1, \ldots, 6$, where **T** is the coefficient matrix and $\mathbf{T}_i$ is the matrix obtained by replacing the ith column with the right-hand side vector.

5 Numerical analysis

This section presents numerical examples to illustrate the behavior of the Laplace transform of the ruin time and to systematically analyze the impact of various model parameters. The baseline parameters are selected as $\lambda = 1.0$, $\lambda_1 = 0.8$, $\lambda_2 = 0.6$, $\mu = 0.3$, $\mu_1 = 1.5$, $\mu_2 = 2.0$, $\delta = 0.05$, $b = 5.0$, where $\lambda^* = \lambda + \lambda_1 + \lambda_2 = 2.4$ is given by definition. Under these parameters, the characteristic equation (25) becomes

$$z^6 + 5.8364z^5 + 12.062z^4 + 9.7480z^3 + 1.3877z^2 - 1.5052z + 0.0730 = 0, \tag{35}$$

whose eigenvalues are $z_1 = 0.1759$, $z_2 = 0.0886$, $z_3 = -0.9401$, $z_4 = -1.5000$, $z_5 = -1.6607$, $z_6 = -2.0000$. Solving the linear system (34) yields the undetermined coefficients. Substituting these into (26) and (27) gives the complete expressions for $\Psi_1(u)$ and $\Psi_2(u)$.

Figure 1 shows the variation curves of $\Psi_1(u)$ and $\Psi_2(u)$ with the initial surplus u under the baseline parameters. Both curves start from positive initial values, decrease monotonically with u, and tend to 0. At $u = 0$, $\Psi_1(0) = 0.54$, $\Psi_2(0) = 0.26$; as u increases, both decay rapidly. At the dividend barrier $b = 5$, both attenuate to near zero. The function value corresponding to the first mechanism is consistently higher than that corresponding to the second.

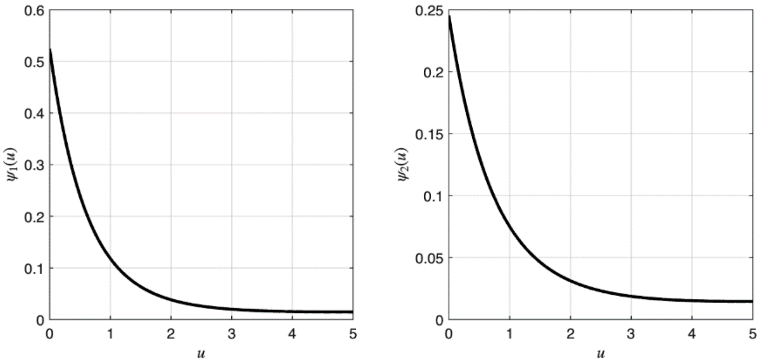

Figure 1: Plot of the Laplace transform of the ruin time

Next, consider the influence of parameters λ and λ_1 on the behavior of the Laplace transform of the ruin time.

Figure 2 shows the changes in the two types of functions when $\lambda = 0.5, 1.0, 1.5$. The influence of λ on the function values is concentrated in the region with smaller initial surplus. A larger λ leads to faster decay of both curves.

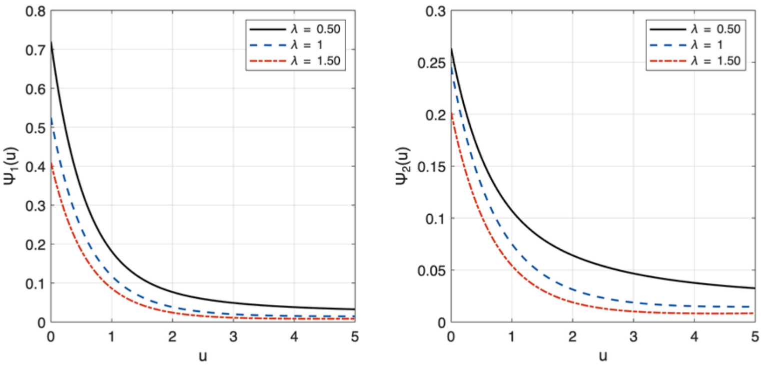

Figure 2: Impact of different λ on $\Psi_1(\mathrm{u})$ and $\Psi_2(\mathrm{u})$

Figure 3 presents the case for $\lambda_1 = 0.5, 1.0, 1.5$. An increase in λ_1 causes a significant upward shift in both types of functions. The larger λ_1 is, the slower the function decays, and a non-negligible ruin risk is retained even at higher initial surplus levels.

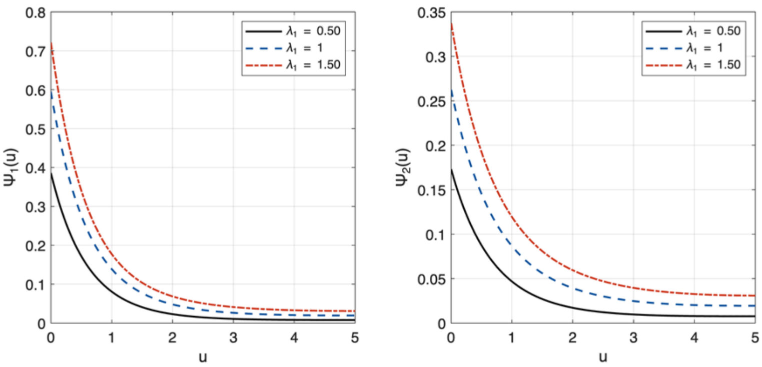

Figure 3: Impact of different λ_1 on $\Psi_1(\mathrm{u})$ and $\Psi_2(\mathrm{u})$

This numerical example validates the effectiveness of the analytical solutions derived in this paper and visually demonstrates the fundamental behavior of the Laplace transform of the ruin time under a constant dividend barrier, decreasing monotonically from the initial surplus towards zero. The rapid decay of the function indicates that a moderate initial surplus can significantly reduce the ruin risk.

6 Conclusion

This paper constructs a dependent risk model with a constant dividend barrier and stochastic premium income, and studies its Gerber-Shiu discounted penalty function. The integro-differential equations satisfied by the Gerber-Shiu function are obtained. Under the assumption of exponential claim amounts, an analytical solution is derived. Finally, the behavior of the Laplace transform of the ruin time is demonstrated through numerical simulations.

This paper enriches and extends the analytical methods for solving the Gerber-Shiu function under complex risk models, providing an effective quantitative tool for insurance company risk assessment and capital management.

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