Analysis of a two-server queue with consultation by main server having protected phases of service

Abstract

In a queueing system where service is carried out in multiple phases and is subjected to interruptions, the idea of protecting certain critical phases from interruption becomes highly significant. This paper investigates a two-server queueing system in which a primary (main) server provides consultation to a secondary (regular) server. The main server performs dual functions: servicing customers and offering consultation to the regular server, with consultations being assigned preemptive priority over customer service. Customer service at the main server is subjected to stochastic interruptions. But interruptions are prohibited when the service process is in designated protected phases. Some constraints are imposed through upper bounds on the number of allowable interruptions experienced by a customer at the main server and the number of consultations provided to the regular server during an individual service period. There is a supervisory timing mechanism super clock which governs the admissibility of additional interruptions to an ongoing service at the main server. A threshold clock regulates the restart or resumption of service at both servers following each

consultation event. Customer arrivals and consultation requests are modelled as mutually independent Poisson processes. It is assumed that service times at the main and regular servers are independent and follow phase-type distributions. The system's stability condition is established, and key performance measures are evaluated through numerical analysis.

Key words : main server, regular server, consultation, interruption, protected phases

1 Introduction

Multi-server queueing systems incorporating consultation mechanisms, wherein a designated primary server provides assistance to subordinate servers, are widely encountered in practical settings. For instance, in healthcare systems, a chief physician offers expert guidance to attending doctors regarding patient diagnosis and treatment decisions. The inclusion of consultation mechanisms plays a significant role in enhancing the overall quality and reliability of service.

Chakravarthy [1] introduce a c-server queueing system with consultations, in which one of the servers is designated as the main server and the remaining as regular servers. The main server gives preemptive priority to consultation requests from the regular servers over its own customers. Whenever a regular server requests consultation, the main server immediately attends to it, even if it is currently serving a customer. Consequently, the service of the customer at the main server is interrupted and remains suspended until the consultation is completed. In contrast, the service at the regular server is not considered interrupted, since consultation is treated as an integral part of its service process. If multiple regular servers require consultation simultaneously, they form a queue, and consultations are provided on a first-come, first-served (FIFO) basis. The interrupted service at the main server resumes only after all pending consultations are completed. Notably, there is no restriction on the number of interruptions a customer at the main server may experience; similarly, a regular server may request any number of consultations during the service of a customer.

Klimenock et al. [2] analyze a multi-server queueing model with heterogeneous customers, where two types of customers with different priority levels are considered. Service durations at the servers are assumed to be

independently distributed according to phase-type distributions.

Samouylov et. al. [3] consider a multi-server queueing model that serves two correlated streams of requests. A non-preemptive priority mechanism is implemented by introducing a preliminary delay for one of the streams through intervening buffers, from which requests are retrieved at different rates.

Krishnamoorthy et. al. [4] investigate a two-server queueing model where both servers have service times represented by phase-type distributions of identical order, with one server having a comparatively lower service rate.

Ayyappan G. and Archana G. [5] elaborate a classical queueing system by considering two types of heterogeneous servers. Optional services are provided to customers who are not satisfied with the initial service. The service at server 2 is subject to interruptions due to breakdowns during any stage of service; however, despite such breakdowns, server 2 continues and completes the ongoing service at a reduced rate.

T. Resmi and K Ravikumar [6] analyze a three-server queueing system in which a main server provides consultations to two regular servers. A finite buffer is maintained at the main server, and preemptive priority is given to consultation requests over customer service.

Resmi et. al. [7] consider a two-server queueing model with mutual consultation between servers, where each server provides timely assistance to the other, and the service times are governed by independent phase-type distributions.

Krishnamoorthy et.al [8] study a queueing model with a single server experiencing interruptions, with the number of such interruptions being controlled through a super clock mechanism and a prescribed finite upper bound.

Early work on queueing systems with service interruptions can be traced back to White and Christie in their paper [9]. The customer's service is resumed immediately upon completion of the interruption. Avi-Izhak and Naor [13], Ibe and Trivedi [12], Gaver [10] , Keilson [11], and Fiems et. al [14] analyze queueing systems with generally distributed service processes and interruption durations.

Some phases of service may involve costly or delicate operations, where any interruption could lead to substantial loss of time, resources, or service quality. By designating such phases as "protected," interruptions are either delayed or temporarily disallowed during their execution. This ensures smoother service completion for sensitive stages, reduces inefficiencies caused by repeated restarts, and improves overall system performance. Con-

sequently, incorporating protected phases into interrupted service models provides a more realistic and practical framework for analyzing complex service systems.

Klimenok et. al. [15] examine a multi-server queueing model with finite buffer and negative customers, where arrivals follow a BMAP and services are PH-type. Negative customers eliminate customers in service only during unprotected phases, while protected phases remain immune.

Klimenok and Dudin [16] study admission control policies of complete acceptance and complete rejection, and assumes an infinite buffer capacity.

Krishnamoorthy et. al. [17] consider the concept of protection in a queueing system with service interruptions. If the service process is in any one of the first n phases of an Erlang service process with m phases, it is subjected to interruption, whereas the remaining $m-n$ phases are protected. Following an interruption, the service is either resumed or restarted. Interruptions do not result in any loss of customers from the system.

In the present study, we analyze a two-server queueing model in which a primary (main) server provides consultation to a secondary (regular) server. The service times at both servers are assumed to follow mutually independent phase-type distributions. The service process at the main server is subjected to interruptions upon the occurrence of consultation requests from the regular server.

Although consultation improves the overall quality of service, excessive interruptions to customers at the main server may lead to undesirable delays. To address this, restrictions are imposed on the provision of consultation while the main server is actively serving a customer. Specifically, interruptions are regulated through upper bounds on the number of allowable interruptions and consultations. An additional layer of control can be introduced through a supervisory timing mechanism, often referred to as a super clock. If the super clock has not expired, then an interruption is allowed to the customer at the main server, otherwise, the regular server has to wait until the completion of the service of the customer at the main server.

We assume that some phases of the service at the main server are too costly, so these phases are protected from interruption. They are called proteceted phases while the other pahses where interruptions are allowed, are said to be non-protected pahses. Consultation to the regular server will be denied if the main server is at the protected phase. Under this situation, the regular server can receive assistance only after the main server finishes the current service.

This threefold mechanism − an upper bound on the number of interruptions, phase-based protection, and time-based supervision − significantly enhances the flexibility and realism of the model. It enables the system to achieve an effective balance between efficiency and quality of service by preventing disruptions during critical phases, while still permitting necessary interruptions in a controlled and limited manner. Such a framework is particularly valuable in practical applications such as maintenance systems, communication networks, and healthcare services, where both precise timing and continuity of service play a crucial role in overall system performance.

In many real-world systems, the effect of service interruption depends critically on its duration. This is naturally modeled through a threshold mechanism that determines whether an interrupted service should be resumed or restarted. The threshold clock governs the restart or resumption of services at both the main and regular servers following each consultation. This clock is initiated whenever the regular server temporarily suspends its service to seek consultation. If the regular server receives consultation immediately, the consultation process and the threshold clock commence simultaneously. However, if the regular server must wait for consultation, the threshold clock starts ticking during the waiting period and continues throughout the consultation phase until the main server becomes available. If the threshold clock expires before the consultation process is completed, the services at both servers are restarted from the beginning. Otherwise, they resume from the exact phases at which they were interrupted. For instance, in healthcare systems, if interruption is completed within a clinically acceptable time, treatment can resume from the point of interruption; otherwise, the patient's condition may require a complete reassessment, leading to a restart of service. Similarly, in communication networks, short delays allow ongoing processes to resume, whereas prolonged interruptions result in session timeouts and necessitate restarting the service. In manufacturing systems, brief interruptions may not affect the production process, but longer delays can degrade material quality, forcing a restart. Thus, the threshold clock provides a realistic and unified framework to capture time-sensitive interruption effects across diverse application domains.

2 Description of the model

In this model, We study a queueing system comprising a main server and a regular server, with customer arrivals governed by a Poisson process of rate λ. An arriving customer receives service immediately if at least one server is available; otherwise, the customer joins the queue. The service times at the main and regular servers are independently distributed and follow phase-type distributions, characterized by$(\boldsymbol{\alpha}, U)$ and $(\boldsymbol{\beta}, V)$ with p and r, phases, respectively. Write $\mathbf{U}^0 = -U\mathbf{e}$ and $\mathbf{V}^0 = -V\mathbf{e}$. Here, $\mathbf{e}$ denotes a column vector of ones of appropriate dimension. It is assumed that, among the p phases of service at the main server, $m \leq p$ phases are designated as protected.

Consultation requests occur according to a Poisson process with rate θ. Let L and M denote the maximum allowable numbers of interruptions to a customer at the main server and consultations for a customer at the regular server, respectively. These bounds are imposed to prevent excessive delays and to ensure that customers receiving service do not become impatient and leave the system.

We assume that the durations of the consultation, threshold, and super, clocks are mutually independent, each following a phase-type distribution with representation $(\boldsymbol{\delta}, D)$, $(\boldsymbol{\eta}, E)$, and $(\boldsymbol{\gamma}, G)$. The corresponding phase-type distributions have phases f, d and c, respectively. The exit rate vectors are given by $\mathbf{D}^0 = -D\mathbf{e}$, $\mathbf{E}^0 = -E\mathbf{e}$, $\mathbf{G}^0 = -G\mathbf{e}$, and respectively.

Notations :- For the analysis of the model, we use the following notations. These notations are introduced to facilitate a clear and concise description of the system under study.

- $L_0 = L(1 + c), L_1 = 1 + L_0$
- $\tilde{\boldsymbol{\alpha}} = \mathbf{e}_{L_1}^{'}(1) \otimes \boldsymbol{\alpha}$, $\tilde{\boldsymbol{\eta}} = (\boldsymbol{\eta}, 0)$, $\tilde{\boldsymbol{\gamma}} = (\boldsymbol{\gamma}, 0)$
- $\tilde{G} = \begin{bmatrix} G & G^0 \\ \underline{\mathbf{0}} & 0 \end{bmatrix}$, $\tilde{E} = \begin{bmatrix} E & E^0 \\ \underline{\mathbf{0}} & 0 \end{bmatrix}$, $D^* = D \oplus \tilde{E}$, $G^* = \tilde{G} \oplus D^*$
- $\boldsymbol{\gamma}^* = \tilde{\boldsymbol{\gamma}} \otimes \boldsymbol{\delta}^*$ where $\boldsymbol{\delta}^* = \boldsymbol{\delta} \otimes \tilde{\boldsymbol{\eta}}$
- $\dot{I} = \begin{bmatrix} \mathbf{0} & I_{L_0} \end{bmatrix}_{L_0 \times L_1}$, $\bar{I}_m = \tilde{\boldsymbol{\eta}} \otimes \begin{bmatrix} O & O \\ O & I_{p-m} \end{bmatrix}_{p \times p}$

- $\tilde{\mathbf{e}}_c = \begin{bmatrix} \mathbf{e}_c \otimes \bar{I}_m \\ \tilde{\boldsymbol{\eta}} \otimes I_p \end{bmatrix}$, $I_m^* = \begin{bmatrix} I_m \\ O \end{bmatrix}_{p \times m}$

Consider the queueing model $P = \{P(\tau), \tau \geq 0\}$, where
$P(\tau) = \{N(\tau), S(\tau), K_1(\tau), K_2(\tau), \sigma_1(\tau), \sigma_2(\tau), \sigma_3(\tau), J_1(\tau), J_2(\tau)\}$.
Here $N(\tau)$ denotes the number of customers in the system, $K_1(\tau)$ denotes the number of consultations already availed by the regular server during the service of a specific customer, $K_2(\tau)$ denotes the number of interruptions experienced by a customer at the main server. Further $\sigma_1(\tau)$, $\sigma_2(\tau)$ and $\sigma_3(\tau)$ indicate the phases of the super clock, the consultation process and the threshold clock, respectively, while $J_1(\tau)$ and $J_2(\tau)$ refer to the phases of the main and the regular servers, respectively.

Here $S(\tau)$ denotes the status of the servers at time τ such that

$$S(\tau) = \begin{cases} \tilde{0}, & \text{if the regular server alone is busy} \\ 0, & \text{if the main server is busy,} \\ & \text{irrespective of whether the regular server is busy or not} \\ 1, & \text{if the main server is giving consultation only} \\ 2, & \text{if the main server is engaged in consultation} \\ & \text{while one customer at the main server is interrupted} \\ 3, & \text{if the regular server is awaiting consultation} \\ & \text{upon completion of the current service at the main server} \end{cases}$$

Note that $K_2(\tau) = 0$ implies that no interruption has occurred to the customer at the main server up to time τ and hence the super clock has not been initiated. So when $K_2(\tau) = 0$ the super clock is irrelevant and the variable $\sigma_1(\tau)$ is not defined. Moreover, as the super clock is tied to interruptions of a customer at the main server, it is not present in the "consultation-only" mode where no such customer exists.

As interruptions do not occur beyond the $(m+1)^{th}$ phase, these phases are excluded when $S(\tau) = 2$ or 3.

$\{P(\tau), \tau \geq 0\}$ is a CTMC defined on the state space

$$\{0\} \cup \bigcup_{n=1}^{\infty} \epsilon(n).$$

The terms $\epsilon(n)$'s are defined as
$\epsilon(1) = \epsilon(1,0) \cup \epsilon(1,\tilde{0}) \cup \epsilon(1,1)$,
$\epsilon(n) = \epsilon(n,0) \cup \epsilon(n,1) \cup \epsilon(n,2) \cup \epsilon(n,3), n \geq 2$,

where

$$\begin{aligned}
\epsilon(1,0) &= \{(1,0,0,\tau)\} \cup \{(1,0,k,j_1,\tau) : 1 \le k \le L, 1 \le \tau \le p\} \\
\epsilon(1,\tilde{0}) &= \{(1,\tilde{0},i,t) : 0 \le i \le M\} \\
\epsilon(1,1) &= \{(1,1,i,j_2,j_3,t) : 0 \le i \le M-1\} \\
\epsilon(n,0) &= \{(n,0,i,0,\tau,t) \cup (n,0,i,k,j_1,\tau,t) : 0 \le i \le M, \\
&\quad 1 \le k \le L, 1 \le \tau \le p\}, \\
\epsilon(n,1) &= \{(n,1,i,j_2,j_3,t) : 0 \le i \le M-1\} \\
\epsilon(n,2) &= \{(n,2,i,k,j_1,j_2,j_3,\tau,t) : 0 \le i \le M-1, 0 \le k \le L-1, \\
&\quad 1 \le \tau \le m\} \\
\epsilon(n,3) &= \{(n,3,i,j_3,\tau,t) : 0 \le i \le M-1, 1 \le \tau \le m\}, \\
&\quad \text{for } 0 \le j_1 \le c, 1 \le j_2 \le d, 0 \le j_3 \le f, 1 \le t \le r.
\end{aligned}$$

The infinitesimal generator Z is given by

$$Z = \begin{bmatrix}
-\lambda & A_1 & & & & \\
A_2 & A_3 & A_4 & & & \\
 & A_5 & B_1 & B_0 & & \\
 & & B_2 & B_1 & B_0 & \\
 & & & \ddots & \ddots & \ddots \\
 & & & & & \\
\end{bmatrix} \tag{1}$$

Here B_0, B_1 and B_2 are square matrices of orders C_0. Further A_3 is a square matrix of order C_1 and A_1, A_2, A_4, A_5 are matrices of orders $1 \times C_1$, $C_1 \times 1$, $C_1 \times C_0$ and $C_0 \times C_1$, respectively, where
$C_0 = L_1(M+1)pr + Md(f+1)r + L_0Md(f+1)mr + M(f+1)mr$ and
$C_1 = L_1p + (M+1)r + Mrd(f+1)$ with the matrices given by the following expressions:

$$A_1 = \begin{bmatrix} \boldsymbol{\alpha} & \underline{\mathbf{0}} \end{bmatrix}, A_2 = \begin{bmatrix} \mathbf{e}_{L_1} \otimes U^0 \\ \mathbf{e}_{M+1} \otimes V^0 \\ \mathbf{0} \end{bmatrix}, A_3 = \begin{bmatrix} I_{L_1} \otimes U & O & O \\ O & A_{31} & A_{32} \\ O & A_{33} & I_M \otimes D^* \otimes I_r \end{bmatrix},$$

$$A_4 = \begin{bmatrix} A_{41} & A_{42} & O \end{bmatrix}, A_5 = \begin{bmatrix} A_{51} & A_{52} & A_{53} \end{bmatrix},$$

$$B_0 = \lambda I,\ B_1 = \begin{bmatrix} B_{11} & O & B_{12} & B_{13} \\ B_{14} & B_{15} & O & O \\ B_{16} & O & B_{17} & O \\ O & O & O & B_{18} \end{bmatrix},\ B_2 = \begin{bmatrix} B_{21} & A_{53} & O \end{bmatrix}.$$

Here the block matrices are

$$A_{31} = I_{M+1} \otimes V - \theta \left[\begin{array}{cc} I_M & \mathbf{0} \\ \underline{\mathbf{0}} & 0 \end{array} \right]_{M+1} \otimes I_r,\ A_{32} = \theta \left[\begin{array}{c} I_M \\ \underline{\mathbf{0}} \end{array} \right]_{(M+1)\times M} \otimes \boldsymbol{\delta} \otimes \tilde{\boldsymbol{\eta}} \otimes I_r,$$

$$A_{33} = \left[\begin{array}{cc} O & I_M \otimes D^0 \otimes \tilde{\Delta}_r \end{array} \right]_{Md(f+1)r\times(M+1)r},$$

$$A_{41} = \left[\begin{array}{c} \mathbf{e}'_{M+1}(1) \otimes I_{L_1} \otimes I_p \otimes \boldsymbol{\beta} \\ I_{M+1} \otimes \tilde{\boldsymbol{\alpha}} \otimes I_r \\ O \end{array} \right]_{C_1\times(M+1)L_1pr},\ A_{42} = \left[\begin{array}{c} O \\ I_{Md(f+1)r} \end{array} \right]_{C_1\times Md(f+1)r},$$

$$A_{51} = \left[\begin{array}{c} \mathbf{e}_{M+1} \otimes I_{L_1} \otimes I_p \otimes V^0 \\ O \end{array} \right]_{C_0\times L_0p},\ A_{52} = \left[\begin{array}{c} I_{M+1} \otimes \mathbf{e}_{L_1} \otimes U^0 \otimes I_r \\ O \end{array} \right]_{C_0\times(M+1)r},$$

$$A_{53} = \left[\begin{array}{c} O \\ I_M \otimes \boldsymbol{\delta} \otimes I_{f+1} \otimes U^0 \otimes I_r \end{array} \right]_{C_0\times Md(f+1)r},$$

$$B_{11} = I_{M+1} \otimes I_{L_1} \otimes (U \oplus V) - \theta \left[\begin{array}{cc} I_M & \mathbf{0} \\ \underline{\mathbf{0}} & 0 \end{array} \right] \otimes I_{L_1} \otimes I_{pr},$$

$$B_{12} = \theta \left[\begin{array}{c} I_M \\ \underline{\mathbf{0}} \end{array} \right]_{(M+1)\times M} \otimes Q \otimes I_m^* \otimes I_r,\ B_{13} = \theta \left[\begin{array}{c} I_M \\ \underline{\mathbf{0}} \end{array} \right]_{(M+1)\times M} \otimes Q^* \otimes I_r,$$

$$B_{14} = \left[\begin{array}{cc} O & I_M \otimes D^0 \otimes \Delta^0 \end{array} \right]_{Md(f+1)r\times L_1(M+1)pr},\ B_{15} = I_M \otimes D^* \otimes I_r,$$

$$B_{16} = \left[\begin{array}{cc} O & I_M \otimes \dot{I} \otimes D^0 \otimes \tilde{\Delta} \end{array} \right]_{L_0Md(f+1)mr\times L_1(M+1)pr},$$

$$B_{17} = I_M \otimes I_L \otimes G^* \otimes I_{mr},\ B_{18} = I_{Md} \otimes (\tilde{E} \oplus U) \otimes I_r,$$

$$B_{21} = \left[\begin{array}{c} \tilde{U}^0 + \tilde{V}^0 \\ O \end{array} \right]_{C_0\times L_1(M+1)pr}.$$

Here

$$Q = \left[\begin{array}{c} diag(\tilde{\boldsymbol{\gamma}}, I_{L-1} \otimes \hat{I}_c) \\ O \end{array} \right]_{L_1\times L_0} \otimes \boldsymbol{\delta} \otimes \tilde{\boldsymbol{\eta}},\ Q^* = \left[\begin{array}{c} \bar{I}_m \\ \mathbf{e}_{L-1} \otimes \tilde{\mathbf{e}}_c \\ \mathbf{e}_{c+1} \otimes \tilde{\boldsymbol{\eta}} \otimes I_p \end{array} \right]_{L_1p\times(f+1)p}$$

,

$\tilde{U}^0 = I_{M+1} \otimes \mathbf{e}_{L_1} \otimes U^0 \otimes \boldsymbol{\alpha} \otimes I_r$, $\tilde{V}^0 = \mathbf{e}_{M+1} \otimes I_{L_1} \otimes I_p \otimes V^0 \otimes \boldsymbol{\beta}$,

$$\tilde{\Delta}_q = \left[\begin{array}{c} \mathbf{e}_f \otimes I_r \\ \mathbf{e}_r \otimes \boldsymbol{\beta} \end{array} \right],\ \Delta^0 = \left[\begin{array}{c} \mathbf{e}_f \otimes \tilde{\boldsymbol{\alpha}} \otimes I_r \\ \mathbf{e}_r \otimes \tilde{\boldsymbol{\alpha}} \otimes \boldsymbol{\beta} \end{array} \right],\ \tilde{\Delta} = \left[\begin{array}{c} \mathbf{e}_f \otimes (I_m^*)' \otimes I_r \\ \mathbf{e}_{mr} \otimes \boldsymbol{\alpha} \otimes \boldsymbol{\beta} \end{array} \right].$$

3 Steady state analysis

In this section, we analyze the steady-state behavior of the queueing model under consideration. We begin by establishing the stability condition necessary for the system to operate in equilibrium.

3.1 Stability condition

Consider the generator matrix $B_0 + B_1 + B_2$. Let $\boldsymbol{\nu}$ be its the steady-state probability vector. Then ν and the generator satisfy the equations $\boldsymbol{\nu}(B_0 + B_1 + B_2) = 0$ and $\boldsymbol{\nu}\mathbf{e} = 1$.

Theorem 3.1: The Markov chain $\{P(\tau), \tau \geq 0\}$ is stable if and only if

$$\boldsymbol{\nu}B_0\mathbf{e} < \boldsymbol{\nu}B_2\mathbf{e}. \tag{2}$$

Proof: The system is said to be stable if and only if the mean downward drift from any given level exceeds the mean upward drift to the next higher level. This condition ensures that the underlying process does not drift to infinity and admits a steady-state distribution. See Neuts [18].

The stability condition in equation (2) can be equivalently expressed in terms of the traffic intensity ρ. Specifically, the system is stable if $\rho < 1$, where ρ is is defined as

$$\rho = \frac{\boldsymbol{\nu}B_0\mathbf{e}}{\boldsymbol{\nu}B_2\mathbf{e}}. \tag{3}$$

The validity of this stability criterion is further supported through numerical experiments presented in Section 4.

3.2 Steady state probability vector

Let $\boldsymbol{q} = (\boldsymbol{q}_0, \boldsymbol{q}_1, \boldsymbol{q}_2, ...)$ denotes the steady state probability vector of the Markov chain $\{P(\tau), \tau \geq 0\}$, where $\boldsymbol{q}_n$ represents the probability vector corresponding to level n.

Note that $\boldsymbol{q}_0$ is a scalar, $\boldsymbol{q}_1 = (\boldsymbol{q}_{10}, \boldsymbol{q}_{1\tilde{0}}, \boldsymbol{q}_{11})$ and $\boldsymbol{q}_n = (\boldsymbol{q}_{n0}, \boldsymbol{q}_{n1}, \boldsymbol{q}_{n2}, \boldsymbol{q}_{n3})$, for $n \geq 2$. The condition $\boldsymbol{q}Z = 0$ with $\boldsymbol{q}\mathbf{e} = 1$. Under the stability condition,

the sub vectors of $\boldsymbol{q}$ can be determined by solving the following system of balance equations that characterize the steady-state behavior of the system.

$$\boldsymbol{q}_n = \boldsymbol{q}_2 R^{n-2}, n \geq 3 \tag{4}$$

where R is the minimal non-negative matrix satisfying the equation

$$R^2 B_2 + R B_1 + B_0 = 0. \tag{5}$$

Once the matrix R is determined, the sub vectors $\boldsymbol{q}_0, \boldsymbol{q}_1$ and $\boldsymbol{q}_2$ can be obtained using the matrix-geometric equation

$$\begin{bmatrix} \boldsymbol{q}_0 & \boldsymbol{q}_1 & \boldsymbol{q}_2 \end{bmatrix} \begin{bmatrix} -\lambda & A_1 & \\ A_2 & A_3 & A_4 \\ & A_5 & B_1 + RB_2 \end{bmatrix} = 0, \tag{6}$$

together with the normalizing condition

$$\boldsymbol{q}_0 + \boldsymbol{q}_1 \mathbf{e} + \boldsymbol{q}_2 (I - R)^{-1} \mathbf{e} = 1. \tag{7}$$

3.3 Performance measures

Some important system performance measures are listed below, together with their respective computational formulae, in order to elucidate the qualitative behavior of the model. To facilitate this, the vectors $\boldsymbol{q}_n$, $n \geq 1$ are partitioned as

$$\boldsymbol{q}_1 = (\boldsymbol{q}_{10}, \boldsymbol{q}_{1\tilde{0}}, \boldsymbol{q}_{11})$$

and

$$\boldsymbol{q}_n = (\boldsymbol{q}_{n0}, \boldsymbol{q}_{n1}, \boldsymbol{q}_{n2}, \boldsymbol{q}_{n3}), n \geq 2.$$

Observe that $\boldsymbol{q}_0$ is a scalar, $\boldsymbol{q}_{10}$, $\boldsymbol{q}_{1\tilde{0}}$, $\boldsymbol{q}_{11}$, $\boldsymbol{q}_{n0}$, $\boldsymbol{q}_{n1}$, $\boldsymbol{q}_{n2}$ and $\boldsymbol{q}_{n3}$ are vectors of dimensions $L_1 p$, $(1+M)r$, $Mrd(1+f)$, $(1+M)L_1 pr$, $Md(1+f)r$, $L_0 Md(1+f)mr$ and $M(1+f)mr$, respectively.

(1) Mean number of customers in the system

$$N_S = \sum_{n=1}^{\infty} n \boldsymbol{q}_n \mathbf{e}. \tag{8}$$

(2) Mean number of customers in the queue

$$N_Q = \sum_{n=2}^{\infty}(n-1)\boldsymbol{q}_{n1}\mathbf{e} + \sum_{n=3}^{\infty}(n-2)(\boldsymbol{q}_{n0}\mathbf{e} + \boldsymbol{q}_{n2}\mathbf{e} + \boldsymbol{q}_{n3}\mathbf{e}). \tag{9}$$

(3) Interruption's effective rate

$$\rho_i = \theta\sum_{n=2}^{\infty}\sum_{j=0}^{M-1}\sum_{\tau=1}^{m}\boldsymbol{q}_{n0j0\tau}\mathbf{e} + \theta\sum_{n=2}^{\infty}\sum_{j=0}^{M-1}\sum_{k=1}^{L-1}\sum_{j_1=1}^{c}\sum_{\tau=1}^{m}\boldsymbol{q}_{n0jkj_1\tau}\mathbf{e}. \tag{10}$$

(4) Consultation's effective rate

$$\rho_c = \theta\sum_{j=0}^{M-1}\boldsymbol{q}_{1\tilde{0}j}\mathbf{e} + \theta\sum_{n=2}^{\infty}\sum_{j=0}^{M-1}\boldsymbol{q}_{n0j}\mathbf{e}. \tag{11}$$

(5) Probability that the main server is not engaged in service

$$\psi_m = \boldsymbol{q}_0\mathbf{e} + \boldsymbol{q}_{1\tilde{0}}\mathbf{e}. \tag{12}$$

(6) Probability that the regular server is not engaged in service

$$\psi_r = \boldsymbol{q}_0\mathbf{e} + \boldsymbol{q}_{10}\mathbf{e}. \tag{13}$$

(7) Probability that the main server is engaged in service

$$\zeta_m = \boldsymbol{q}_{10}\mathbf{e} + \sum_{n=2}^{\infty}\boldsymbol{q}_{n0}\mathbf{e} + \sum_{n=2}^{\infty}\boldsymbol{q}_{n3}\mathbf{e}. \tag{14}$$

(8) Probability that the regular server is engaged in service

$$\zeta_r = \boldsymbol{q}_{1\tilde{0}}\mathbf{e} + \sum_{n=2}^{\infty}\boldsymbol{q}_{n0}\mathbf{e}. \tag{15}$$

(9) Probability that the main server is experiencing interruption

$$\mu = \sum_{n=2}^{\infty}\boldsymbol{q}_{n2}\mathbf{e}. \tag{16}$$

(10) Probability that the regular server is receiving consultation

$$\nu_c = \sum_{n=1}^{\infty} \boldsymbol{q}_{n1}\mathbf{e} + \sum_{n=2}^{\infty} \boldsymbol{q}_{n2}\mathbf{e}. \tag{17}$$

(11) Probability that the regular server is awaiting consultation from the main server

$$\nu_w = \sum_{n=2}^{\infty} \boldsymbol{q}_{n3}\mathbf{e}. \tag{18}$$

4 Numerical results

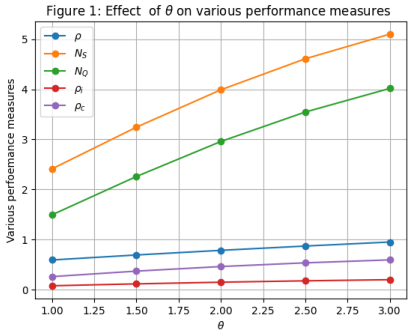
Let us assume
$M = 3$, $L = 3$

$$\boldsymbol{\alpha} = \begin{bmatrix} 0.4 & 0.3 & 0.1 & 0.2 \end{bmatrix}, U = \begin{bmatrix} -12 & 3 & 1 & 2 \\ 3 & -15 & 1 & 2 \\ 0 & 0 & -5 & 1 \\ 0 & 0 & 2 & -7 \end{bmatrix},$$

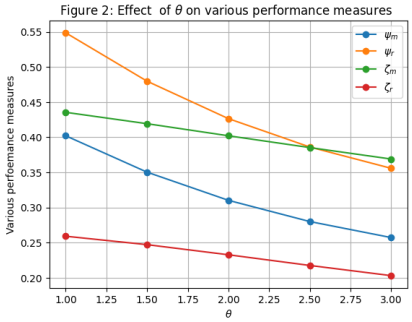
$$\boldsymbol{\beta} = \begin{bmatrix} 0.4 & 0.6 \end{bmatrix}, V = \begin{bmatrix} -12 & 6 \\ 5 & -10 \end{bmatrix}, \boldsymbol{\delta} = \begin{bmatrix} 0.4 & 0.6 \end{bmatrix}, D = \begin{bmatrix} -6 & 4 \\ 3 & -4 \end{bmatrix},$$

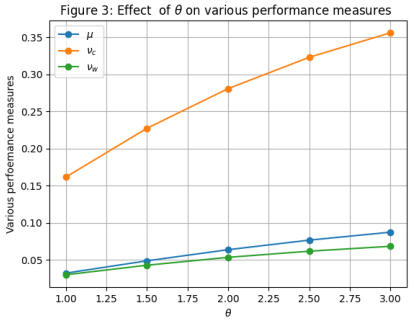
$$\boldsymbol{\eta} = \begin{bmatrix} 0.5 & 0.5 \end{bmatrix}, E = \begin{bmatrix} -12 & 3 \\ 3 & -12 \end{bmatrix}, \boldsymbol{\gamma} = \begin{bmatrix} 0.6 & 0.4 \end{bmatrix}, G = \begin{bmatrix} -12 & 8 \\ 8 & -12 \end{bmatrix}.$$

The parameter values, vectors, and matrices are specified to ensure that ρ remains less than 1.



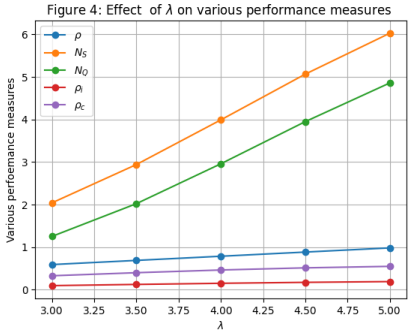
Referring to Figures 1–3, as the consultation rate θ increases, the traffic intensity ρ also increases, leading to a rise in ρ_i and ρ_c. Consequently, μ and

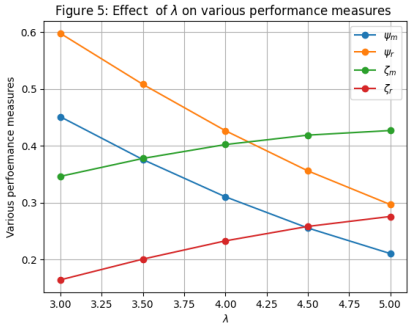




ν_c increase. With higher values of θ, consultations occur more frequently, causing the upper bound on the number of interruptions to be reached more rapidly, or the super clock to be triggered more often. As a result, the main server is compelled to complete the service of the current customer before engaging in further consultations. This, in turn, increases the waiting time for the regular server to receive consultation, leading to an increase in ν_w. Since μ, ν_c and ν_w increase, customers spend more time both in the system and in the queue, resulting in higher values of N_S and N_Q. Consequently, ψ_m and ψ_r decrease. Furthermore, as the main server allocates more time to consultations, less time is available for serving customers, which leads to a decrease in ζ_m and ζ_r.

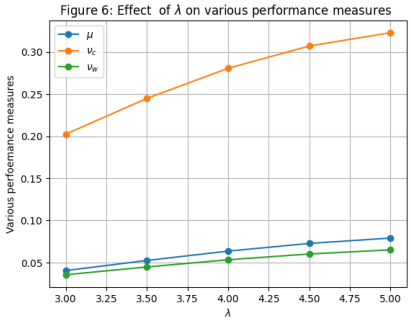
From Figures 4–6, as the arrival rate λ increases, the traffic intensity ρ increases, leading to a buildup of customers in the system. Consequently, N_S and N_Q increase, along with ρ_i and ρ_c, resulting in higher values of μ, ν_c and ν_w. Moreover, with more customers in the queue, the servers spend more time in service, leading to an increase in ζ_m and ζ_r, and a corresponding decrease in ψ_m and ψ_r.





Concluding remarks and suggestions for further study

In this paper, we analyze a two-server queueing model in which the main server provides consultation to the regular server. The service of a customer at the main server is subject to interruption upon consultation requests; however, certain phases are protected from interruption. Although consultations improve service quality, excessive interruptions are undesirable. To address this, restrictions are imposed on permitting consultations while the main server is busy. Interruptions are regulated through upper bounds on interruptions and consultations, along with a super clock mechanism. The stability condition of the system is established, and numerical illustrations are provided. Service times at both servers are assumed to be independent and follow phase-type distributions. Even though phase-type distributions provide analytical tractability and flexibility, their use in complex systems such as the present multi-server queue with consultations leads to certain limitations. In particular, the resulting state space grows rapidly with the number of phases, making numerical computation challenging. Moreover, phase-type distributions may not adequately capture heavy-tailed behavior



and high variability observed in real-world service and interruption processes. In practical systems, consultation and interruption mechanisms may also exhibit non-Markovian and state-dependent characteristics, which are not fully represented under the PH assumption. Therefore, future research may focus on incorporating heavy-tailed distributions, Markov-modulated processes, or simulation-based approaches to enhance the realism of the model. It will also be interesting to study models without protected phases of service at the main server. Then interruption may be permitted at any phase.

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