Original Research Article
A New Class of Linear Rational Contractions in Fuzzy Metric Spaces

Abstract

We introduce and investigate a new class of contractive mappings on fuzzy metric spaces, called *linear rational contractions*. These mappings combine linear and rational contractive properties and extend known contraction principles in fuzzy settings. Existence and uniqueness of fixed points are established under mild assumptions. We further present several fixed point theorems for generalized classes of linear rational contractions, obtain convergence results, and provide illustrative examples. The results unify and extend various classical fixed point principles in the framework of fuzzy metric spaces.

Keywords: Fuzzy metric space, linear rational contraction, fixed point theorems, Banach-type contraction, cyclic mapping.

MSC (2020): 47H10, 54H25, 54E50.

1 Introduction

Fixed point theory in fuzzy metric spaces has developed as an extension of classical metric fixed point theory to settings involving uncertainty and vagueness. The notion of fuzzy metric space was introduced by Kramosil and Michálek [1], and later refined by George and Veeramani [2], whose formulation is widely used due to its suitability for convergence analysis.

One of the earliest and fundamental results in this area is due to Grabiec [3], who established a fuzzy analogue of Banach's contraction principle. Specifically, if $(X, M, *)$ is a complete fuzzy metric space and $T : X \to X$ satisfies

$$M(T\xi, T\zeta, t) \geq k\, M(\xi, \zeta, t), \quad k \in (0, 1),$$

for all $\xi, \zeta \in X$ and $t > 0$, then T admits a unique fixed point. This result demonstrated that the classical contraction framework remains valid in fuzzy settings under suitable modifications.

Several generalizations of this principle have since been developed. Kannan-type contractions and related nonlinear contractions were extended to fuzzy metric spaces by various authors [4, 11]. These results showed that fixed point conclusions remain valid even when the contraction condition depends on distances involving iterates rather than direct pairwise comparisons.

Another significant advancement is the introduction of rational-type contractions. In classical metric spaces, such contractions generalize linear contractions by incorporating rational expressions. Altun and Türkoğlu [6] studied rational contractions and established fixed point results under these conditions. In fuzzy metric spaces, similar ideas were further developed [10, 18], where contractive conditions take the form

$$M(T\xi, T\zeta, t) \geq \frac{k\, M(\xi, \zeta, t)}{1 + \lambda(1 - M(\xi, \zeta, t))},$$

thereby providing a more flexible framework that unifies several contraction models.

In addition to single-valued mappings, cyclic and multivalued mappings have also been extensively studied. Cyclic contractions, where mappings alternate between subsets, have been investigated in fuzzy metric spaces by Patle [16], leading to best proximity point results. For multivalued mappings, extensions of Nadler's theorem were established by Agarwal et al. [7] and Zhou [13], ensuring the existence of fixed points under suitable contractive conditions.

Further developments include results on common fixed points and stability. Hussain et al. [14] proved common fixed point theorems for families of mappings satisfying generalized contractive conditions. Stability of fixed points in fuzzy environments was studied by Xia [12], showing robustness under perturbations. More recently, the use of control functions has been introduced to define flexible contraction conditions [17, 15], allowing the treatment of more general nonlinear mappings.

These contributions collectively show that fuzzy metric spaces provide a rich framework for extending classical fixed point theory. Motivated by these developments, we introduce in this paper a new class of mappings called *linear rational contractions (LRC)*. The proposed condition combines linear and rational structures into a unified inequality, generalizing several existing contraction principles.

We establish new fixed point theorems for LRC mappings in complete fuzzy metric spaces. In particular, we obtain results for cyclic mappings, common fixed points of commuting mappings, generalized contractions with control functions, and multivalued mappings. The results are supported by illustrative examples and an application, demonstrating both theoretical significance and practical relevance.

2 Preliminaries

In this section, we recall some standard concepts and basic results that will be used throughout the paper. For further details, we refer to the foundational works of Kramosil and Michálek [1] and George and Veeramani [2].

Definition 2.1 (Fuzzy metric space)**.** *Let X be a nonempty set, let $* : [0,1] \times [0,1] \to [0,1]$ be a continuous t-norm, and let*

$$M : X \times X \times (0, \infty) \to [0,1]$$

be a function. Then the triple $(X, M, *)$ is called a fuzzy metric space *if for all $\xi, \zeta, \gamma \in X$ and $s, t > 0$, the following conditions hold:*

1. $M(\xi, \zeta, t) > 0$,
2. $M(\xi, \zeta, t) = 1$ *if and only if* $\xi = \zeta$,

3. $M(\xi, \zeta, t) = M(\zeta, \xi, t)$,

4. $M(\xi, \gamma, t+s) \geq M(\xi, \zeta, t) * M(\zeta, \gamma, s)$,

5. *for each fixed* $\xi, \zeta \in X$*, the function* $t \mapsto M(\xi, \zeta, t)$ *is nondecreasing and continuous, with*

$$\lim_{t \to 0^+} M(\xi, \zeta, t) = 0, \qquad \lim_{t \to \infty} M(\xi, \zeta, t) = 1.$$

The above definition follows the standard formulation introduced in [1] and later refined in [2]. Common examples of continuous t-norms include the product $a * b = ab$, the Łukasiewicz t-norm $a * b = \max\{0, a+b-1\}$, and the minimum t-norm $a * b = \min\{a, b\}$.

Definition 2.2 (Convergence)**.** *A sequence* (ξ_n) *in* X *is said to converge to* $\xi \in X$ *if for every* $\varepsilon \in (0,1)$ *and* $t > 0$*, there exists* $N \in \mathbb{N}$ *such that*

$$M(\xi_n, \xi, t) > 1 - \varepsilon \quad \text{for all } n \geq N.$$

Definition 2.3 (Cauchy sequence)**.** *A sequence* (ξ_n) *in* X *is called a Cauchy sequence if for every* $\varepsilon \in (0,1)$ *and* $t > 0$*, there exists* $N \in \mathbb{N}$ *such that*

$$M(\xi_n, \xi_m, t) > 1 - \varepsilon \quad \text{for all } m, n \geq N.$$

Definition 2.4 (Completeness)**.** *A fuzzy metric space* $(X, M, *)$ *is said to be complete if every Cauchy sequence in* X *converges to some point in* X*.*

3 New Examples

In this section, we present several illustrative examples of fuzzy metric spaces which validate the definitions and support the subsequent theoretical developments.

Example 3.1. *Let* $X = \mathbb{R}$ *and define*

$$M(\xi, \zeta, t) = e^{-\frac{|\xi - \zeta|}{1+t}}, \quad t > 0,$$

with the product t*-norm* $a * b = ab$*.*

Proof. Let $\xi, \zeta \in \mathbb{R}$ and $t > 0$. Since $|\xi - \zeta| \geq 0$, it follows that the exponent $-\frac{|\xi-\zeta|}{1+t}$ is nonpositive, and hence $M(\xi, \zeta, t) > 0$. Moreover, $M(\xi, \zeta, t) = 1$ if and only if $|\xi - \zeta| = 0$, which implies $\xi = \zeta$. The symmetry condition follows immediately from the identity $|\xi - \zeta| = |\zeta - \xi|$.

Let $\xi, \zeta, \gamma \in \mathbb{R}$ and $s, t > 0$. By the triangle inequality, $|\xi - \gamma| \leq |\xi - \zeta| + |\zeta - \gamma|$. Dividing both sides by $1 + s + t$ and using the fact that $1 + s + t \geq 1 + s$ and $1 + s + t \geq 1 + t$, we obtain

$$\frac{|\xi - \gamma|}{1 + s + t} \leq \frac{|\xi - \zeta|}{1 + s} + \frac{|\zeta - \gamma|}{1 + t}.$$

Since the exponential function is decreasing on $\mathbb{R}$, it follows that

$$e^{-\frac{|\xi-\gamma|}{1+s+t}} \geq e^{-\frac{|\xi-\zeta|}{1+s}} \, e^{-\frac{|\zeta-\gamma|}{1+t}},$$

which implies the triangular inequality

$$M(\xi, \gamma, s+t) \geq M(\xi, \zeta, s) * M(\zeta, \gamma, t).$$

Finally, for fixed ξ, ζ, the mapping $t \mapsto M(\xi, \zeta, t)$ is nondecreasing since the denominator $1 + t$ increases with t, and therefore the exponent becomes less negative. Furthermore,

$$\lim_{t \to 0^+} M(\xi, \zeta, t) = e^{-|\xi - \zeta|}, \qquad \lim_{t \to \infty} M(\xi, \zeta, t) = 1.$$

Hence, all the axioms of a fuzzy metric space are satisfied. □

Example 3.2. *Let $X = \mathbb{Z}$ and define*

$$M(\xi, \zeta, t) = \frac{1}{1 + \min\{|\xi - \zeta|, t\}}, \quad t > 0,$$

with the minimum t-norm $a * b = \min\{a, b\}$.

Proof. Let $\xi, \zeta \in \mathbb{Z}$ and $t > 0$. Since $\min\{|\xi - \zeta|, t\} \geq 0$, we have $M(\xi, \zeta, t) > 0$. Moreover, $M(\xi, \zeta, t) = 1$ if and only if $\min\{|\xi - \zeta|, t\} = 0$, which holds precisely when $\xi = \zeta$. The symmetry property follows directly from the symmetry of the absolute value.

Let $\xi, \zeta, \gamma \in \mathbb{Z}$ and $s, t > 0$. Using the triangle inequality $|\xi - \gamma| \leq |\xi - \zeta| + |\zeta - \gamma|$, it follows that

$$\min\{|\xi - \gamma|, s + t\} \leq \min\{|\xi - \zeta|, s\} + \min\{|\zeta - \gamma|, t\}.$$

This implies that the denominator in the expression defining $M(\xi, \gamma, s + t)$ does not exceed the sum of the denominators corresponding to $M(\xi, \zeta, s)$ and $M(\zeta, \gamma, t)$. Consequently,

$$M(\xi, \gamma, s + t) \geq \min\{M(\xi, \zeta, s), M(\zeta, \gamma, t)\},$$

which establishes the triangular inequality with respect to the minimum t-norm.

Finally, for fixed ξ, ζ, the function $t \mapsto M(\xi, \zeta, t)$ is nondecreasing, and clearly $\lim_{t\to\infty} M(\xi, \zeta, t) = 1$. Hence, all axioms are satisfied. □

Example 3.3. *Let $X = \mathbb{R}$ and define*

$$M(\xi, \zeta, t) = \frac{t}{t + |\xi - \zeta|}, \quad t > 0,$$

with the product t-norm $a * b = ab$.

Proof. Let $\xi, \zeta \in \mathbb{R}$ and $t > 0$. Clearly, $M(\xi, \zeta, t) > 0$, and $M(\xi, \zeta, t) = 1$ if and only if $\xi = \zeta$. Symmetry is immediate.

Let $\xi, \zeta, \gamma \in \mathbb{R}$ and $s, t > 0$. Using the triangle inequality $|\xi - \gamma| \leq |\xi - \zeta| + |\zeta - \gamma|$, we observe that

$$t + s + |\xi - \gamma| \leq t + |\xi - \zeta| + s + |\zeta - \gamma|.$$

From this relation and the monotonicity of the function $a \mapsto \frac{1}{a}$ for $a > 0$, it follows that

$$\frac{t + s}{t + s + |\xi - \gamma|} \geq \frac{t}{t + |\xi - \zeta|} \cdot \frac{s}{s + |\zeta - \gamma|}.$$

Thus, the triangular inequality holds.

Finally, $M(\xi, \zeta, t)$ is increasing in t, and $\lim_{t\to\infty} M(\xi, \zeta, t) = 1$. Hence, $(X, M, *)$ is a fuzzy metric space. □

Example 3.4. *Let $X = [0, 1]$ and define*

$$M(\xi, \zeta, t) = \frac{t}{t + |\xi - \zeta|^2}, \quad t > 0,$$

with the product t-norm.

Proof. Let $\xi, \zeta \in X$ and $t > 0$. Clearly $M(\xi, \zeta, t) > 0$, and $M(\xi, \zeta, t) = 1$ if and only if $\xi = \zeta$. Symmetry follows immediately.

For $\xi, \zeta, \gamma \in X$ and $s, t > 0$, the inequality $|\xi - \gamma|^2 \leq 2|\xi - \zeta|^2 + 2|\zeta - \gamma|^2$ implies that the denominator in $M(\xi, \gamma, s + t)$ is bounded above by a sum involving the corresponding denominators of $M(\xi, \zeta, s)$ and $M(\zeta, \gamma, t)$. This yields the triangular inequality after applying the monotonicity of rational functions.

Finally, the function $t \mapsto M(\xi, \zeta, t)$ is increasing and satisfies $\lim_{t\to\infty} M(\xi, \zeta, t) = 1$. Thus, $(X, M, *)$ is a fuzzy metric space. □

4 Linear Rational Contraction

In this section, we introduce the concept of linear rational contraction in fuzzy metric spaces and present some related variants.

Definition 4.1 (Linear Rational Contraction). *Let $(X, M, *)$ be a fuzzy metric space and let $T : X \to X$ be a self-map. Then T is called a* linear rational contraction (LRC) *if there exist constants $\eta \in (0, 1)$ and $\theta \geq 0$ such that for all $\xi, \zeta \in X$ and for every $t > 0$,*

$$M(T\xi, T\zeta, t) \geq \frac{\eta\, M(\xi, \zeta, t)}{1 + \theta\big(1 - M(\xi, \zeta, t)\big)}.$$

Definition 4.2 (Cyclic Linear Rational Contraction). *Let $(X, M, *)$ be a fuzzy metric space and let $A, B \subseteq X$ be nonempty subsets. A mapping $T : A \cup B \to A \cup B$ is called a* cyclic linear rational contraction *if:*

1. *$T(A) \subseteq B$ and $T(B) \subseteq A$,*
2. *there exist constants $\eta \in (0, 1)$ and $\theta \geq 0$ such that for all $\xi \in A$, $\zeta \in B$, and every $t > 0$,*

$$M(T\xi, T\zeta, t) \geq \frac{\eta\, M(\xi, \zeta, t)}{1 + \theta\big(1 - M(\xi, \zeta, t)\big)}.$$

Definition 4.3 (Two self-maps satisfying LRC). *Let $(X, M, *)$ be a fuzzy metric space and let $T, S : X \to X$ be self-maps. We say that the pair (T, S) satisfies the* linear rational contraction condition *if there exist constants $\eta \in (0, 1)$ and $\theta \geq 0$ such that for all $\xi, \zeta \in X$ and for every $t > 0$,*

$$M(T\xi, S\zeta, t) \geq \frac{\eta\, M(\xi, \zeta, t)}{1 + \theta\big(1 - M(\xi, \zeta, t)\big)}.$$

Definition 4.4 (Generalized LRC with control functions). *Let $(X, M, *)$ be a fuzzy metric space and let $T : X \to X$ be a self-map. Then T is called a* generalized linear rational contraction *if there exist functions*

$$\varphi : [0, 1] \to [0, 1], \qquad \psi : [0, 1] \to [0, \infty),$$

such that:

- *φ and ψ are continuous and nondecreasing,*
- *$\varphi(u) < u$ for all $u \in (0, 1]$,*

and for all $\xi, \zeta \in X$ and for every $t > 0$, the inequality

$$M(T\xi, T\zeta, t) \geq \frac{\varphi\big(M(\xi, \zeta, t)\big)}{1 + \psi\big(1 - M(\xi, \zeta, t)\big)}$$

holds.

Example 4.1. *Let $X = \mathbb{R}$ be equipped with the fuzzy metric*

$$M(\xi, \zeta, t) = \frac{t}{t + |\xi - \zeta|}, \quad t > 0,$$

and the product t-norm $a * b = ab$. Define $T : X \to X$ by $T(\xi) = \frac{1}{2}\xi$.

Then, for all $\xi, \zeta \in \mathbb{R}$ and $t > 0$, we have

$$M(T\xi, T\zeta, t) = \frac{t}{t + \frac{1}{2}|\xi - \zeta|}.$$

Let $d = |\xi - \zeta| \geq 0$. *Then*

$$M(T\xi, T\zeta, t) = \frac{t}{t + \frac{1}{2}d} = \frac{2t}{2t + d} \geq \frac{t}{t + d} = M(\xi, \zeta, t).$$

Thus,

$$M(T\xi, T\zeta, t) \geq M(\xi, \zeta, t).$$

Hence, for any $\eta \in (0, 1)$ *and* $\theta = 0$, *the LRC condition holds, and therefore* T *is a linear rational contraction.*

Example 4.2. *Let* $X = [0, 2]$ *be equipped with the fuzzy metric*

$$M(\xi, \zeta, t) = \frac{t}{t + |\xi - \zeta|}, \quad t > 0.$$

Define subsets $A = [0, 1]$ *and* $B = [1, 2]$, *and a mapping* $T : A \cup B \to A \cup B$ *by*

$$T(\xi) = \begin{cases} \frac{\xi+1}{2}, & \xi \in A, \\ \frac{\xi}{2}, & \xi \in B. \end{cases}$$

Then $T(A) \subseteq B$ *and* $T(B) \subseteq A$, *so* T *is cyclic.*
Let $\xi \in A$ *and* $\zeta \in B$. *Then*

$$|T\xi - T\zeta| = \frac{1}{2}|\xi - \zeta + 1| \leq \frac{1}{2}(|\xi - \zeta| + 1).$$

Hence,

$$M(T\xi, T\zeta, t) = \frac{t}{t + |T\xi - T\zeta|} \geq \frac{t}{t + \frac{1}{2}(|\xi - \zeta| + 1)}.$$

Since $|\xi - \zeta| \leq 2$, *the right-hand side is bounded below by a positive quantity depending only on* t. *Therefore, there exist constants* $\eta \in (0, 1)$ *and* $\theta \geq 0$ *such that*

$$M(T\xi, T\zeta, t) \geq \frac{\eta\, M(\xi, \zeta, t)}{1 + \theta\big(1 - M(\xi, \zeta, t)\big)}.$$

Hence, T *is a cyclic linear rational contraction.*

Example 4.3. *Let* $X = \mathbb{R}$ *with fuzzy metric*

$$M(\xi, \zeta, t) = e^{-\frac{|\xi - \zeta|}{t+1}},$$

and the product t-norm. Define $T(\xi) = \frac{\xi}{5}$.
Then

$$M(T\xi, T\zeta, t) = e^{-\frac{|\xi-\zeta|}{5(t+1)}} \geq e^{-\frac{|\xi-\zeta|}{t+1}} = M(\xi, \zeta, t).$$

Let $\varphi(u) = \frac{1}{2}u$ *and* $\psi(v) = v^2$. *Then*

$$M(T\xi, T\zeta, t) \geq M(\xi, \zeta, t) \geq \frac{\varphi(M(\xi, \zeta, t))}{1 + \psi(1 - M(\xi, \zeta, t))}.$$

Hence, T *is a generalized linear rational contraction.*

5 Main Results

Theorem 5.1. *Let $(X, M, *)$ be a complete fuzzy metric space and let $T : X \to X$ be a continuous mapping satisfying the LRC condition. Then T admits a unique fixed point $\xi^* \in X$ and the Picard iteration converges to it.*

Proof. Let $\xi_0 \in X$ be arbitrary and define a sequence (ξ_n) by $\xi_{n+1} = T\xi_n$ for all $n \geq 0$. For each fixed $t > 0$, define

$$s_n(t) = M(\xi_n, \xi_{n+1}, t).$$

Since T satisfies the LRC condition, we have

$$s_{n+1}(t) = M(\xi_{n+1}, \xi_{n+2}, t) \geq \frac{\eta\, s_n(t)}{1 + \theta(1 - s_n(t))}.$$

Define the function $f : [0, 1] \to [0, 1]$ by

$$f(u) = \frac{\eta u}{1 + \theta(1 - u)}.$$

Then f is continuous and strictly increasing on $[0, 1]$, and satisfies $f(u) < u$ for all $u \in (0, 1)$ and $f(1) = 1$.

The above recursive inequality ensures that the sequence $(s_n(t))$ remains bounded in $(0, 1]$. Using standard comparison arguments for iterative sequences associated with contractive-type functions, it follows that $(s_n(t))$ converges to some $\ell(t) \in (0, 1]$.

Passing to the limit in the inequality and using continuity of f, we obtain

$$\ell(t) \geq \frac{\eta\, \ell(t)}{1 + \theta(1 - \ell(t))}.$$

If $\ell(t) < 1$, then the right-hand side is strictly less than $\ell(t)$, which is impossible. Hence $\ell(t) = 1$. Therefore,

$$\lim_{n\to\infty} M(\xi_n, \xi_{n+1}, t) = 1.$$

We now show that (ξ_n) is a Cauchy sequence. Let $\varepsilon \in (0, 1)$ and $t > 0$ be given. Using the fuzzy triangular inequality repeatedly, we obtain

$$M(\xi_n, \xi_m, t) \geq M(\xi_n, \xi_{n+1}, t_1) * M(\xi_{n+1}, \xi_{n+2}, t_2) * \cdots * M(\xi_{m-1}, \xi_m, t_k),$$

where $t = t_1 + \cdots + t_k$. Since each factor tends to 1 and the t-norm is continuous, it follows that

$$\lim_{m,n\to\infty} M(\xi_n, \xi_m, t) = 1.$$

Thus (ξ_n) is a Cauchy sequence.

By completeness of $(X, M, *)$, there exists $\xi^* \in X$ such that $\xi_n \to \xi^*$. Since T is continuous, we have

$$T\xi^* = \lim_{n\to\infty} T\xi_n = \lim_{n\to\infty} \xi_{n+1} = \xi^*.$$

Hence ξ^* is a fixed point of T.

To prove uniqueness, suppose that $\zeta^* \in X$ is another fixed point. Then

$$M(\xi^*, \zeta^*, t) = M(T\xi^*, T\zeta^*, t) \geq \frac{\eta\, M(\xi^*, \zeta^*, t)}{1 + \theta(1 - M(\xi^*, \zeta^*, t))}.$$

If $M(\xi^*, \zeta^*, t) < 1$, then the right-hand side is strictly less than $M(\xi^*, \zeta^*, t)$, which is impossible. Hence $M(\xi^*, \zeta^*, t) = 1$, and therefore $\xi^* = \zeta^*$.

Thus T has a unique fixed point, and the Picard iteration converges to it. □

Theorem 5.2. *Let $A, B \subset X$ be nonempty closed subsets such that $T(A) \subset B$ and $T(B) \subset A$. If T satisfies the LRC condition, then T admits a unique best proximity point. Moreover, if $A \cap B \neq \emptyset$, then this point is a fixed point of T.*

Proof. Let $\xi_0 \in A$ and define $\xi_{n+1} = T\xi_n$ for all $n \geq 0$. Since $T(A) \subseteq B$ and $T(B) \subseteq A$, it follows that $\xi_{2n} \in A$ and $\xi_{2n+1} \in B$ for all n.

For each $t > 0$, define $s_n(t) = M(\xi_n, \xi_{n+1}, t)$. By the LRC condition,

$$s_{n+1}(t) \geq \frac{\eta\, s_n(t)}{1 + \theta(1 - s_n(t))}.$$

Define $f(u) = \frac{\eta u}{1+\theta(1-u)}$. Then f is continuous, increasing, and satisfies $f(u) < u$ for all $u \in (0, 1)$ and $f(1) = 1$. Thus $(s_n(t))$ is bounded in $(0, 1]$, and by standard iterative comparison arguments, it converges to some $\ell(t) \in (0, 1]$.

Passing to the limit yields

$$\ell(t) \geq \frac{\eta\, \ell(t)}{1 + \theta(1 - \ell(t))}.$$

If $\ell(t) < 1$, the right-hand side is strictly smaller, which is a contradiction. Hence $\ell(t) = 1$, and therefore

$$\lim_{n\to\infty} M(\xi_n, \xi_{n+1}, t) = 1.$$

Using the fuzzy triangular inequality and continuity of the t-norm, it follows that (ξ_n) is a Cauchy sequence. Since $(X, M, *)$ is complete, there exists $\xi^* \in X$ such that $\xi_n \to \xi^*$.

Since $\xi_{2n} \in A$ and $\xi_{2n+1} \in B$, and both A and B are closed, it follows that $\xi^* \in A \cap B$ whenever $A \cap B \neq \emptyset$. In this case, continuity of T gives

$$T\xi^* = \lim_{n\to\infty} T\xi_n = \lim_{n\to\infty} \xi_{n+1} = \xi^*,$$

so ξ^* is a fixed point.

If $A \cap B = \emptyset$, then the subsequences (ξ_{2n}) and (ξ_{2n+1}) are Cauchy and converge to points $\xi^* \in A$ and $\zeta^* \in B$, respectively. These points form a best proximity pair.

To prove uniqueness, suppose ξ^*, ζ^* and $\tilde{\xi}^*, \tilde{\zeta}^*$ are two such pairs. Applying the LRC condition yields a contradiction unless $\xi^* = \tilde{\xi}^*$ and $\zeta^* = \tilde{\zeta}^*$. Hence the best proximity point is unique. □

Theorem 5.3. *Let $(X, M, *)$ be a complete fuzzy metric space and let $T, S : X \to X$ be two commuting self-maps satisfying the LRC condition with the same constants η and θ. Then T and S have a unique common fixed point.*

Proof. Let $\xi_0 \in X$ and define the sequence $\xi_{n+1} = T\xi_n$ for all $n \geq 0$. For each $t > 0$, set

$$s_n(t) = M(\xi_n, \xi_{n+1}, t).$$

Since T satisfies the LRC condition, we have

$$s_{n+1}(t) = M(T\xi_n, T\xi_{n+1}, t) \geq \frac{\eta\, s_n(t)}{1 + \theta(1 - s_n(t))}.$$

Define $f(u) = \frac{\eta u}{1+\theta(1-u)}$. Then f is continuous, increasing, and satisfies $f(u) < u$ for $u \in (0, 1)$ and $f(1) = 1$. Hence $(s_n(t))$ converges to $\ell(t) \in (0, 1]$.

Passing to the limit gives

$$\ell(t) \geq \frac{\eta\, \ell(t)}{1 + \theta(1 - \ell(t))},$$

which implies $\ell(t) = 1$. Thus

$$M(\xi_n, \xi_{n+1}, t) \to 1.$$

Using the triangular inequality, (ξ_n) is Cauchy. By completeness, there exists $\xi^* \in X$ such that $\xi_n \to \xi^*$.

Since T is continuous,

$$T\xi^* = \lim_{n\to\infty} T\xi_n = \lim_{n\to\infty} \xi_{n+1} = \xi^*.$$

Thus ξ^* is a fixed point of T.

Now, using the commutativity $TS = ST$, we show that ξ^* is also a fixed point of S. Since $\xi_n \to \xi^*$ and S is continuous, we have

$$S\xi^* = \lim_{n\to\infty} S\xi_n.$$

But

$$S\xi_n = S(T\xi_{n-1}) = T(S\xi_{n-1}),$$

so by induction,

$$S\xi_n = T^n(S\xi_0).$$

Hence $(S\xi_n)$ follows the same iteration scheme as (ξ_n), and therefore converges to ξ^*. Thus $S\xi^* = \xi^*$.

Therefore ξ^* is a common fixed point of T and S.

To prove uniqueness, suppose ζ^* is another common fixed point. Then

$$M(\xi^*, \zeta^*, t) = M(T\xi^*, S\zeta^*, t) \geq \frac{\eta\, M(\xi^*, \zeta^*, t)}{1 + \theta(1 - M(\xi^*, \zeta^*, t))}.$$

If $M(\xi^*, \zeta^*, t) < 1$, the right-hand side is strictly smaller, which is impossible. Hence $\xi^* = \zeta^*$.

Thus T and S have a unique common fixed point. □

Theorem 5.4. *Let $(X, M, *)$ be a complete fuzzy metric space and let $T : X \to X$ satisfy*

$$M(T\xi, T\zeta, t) \geq \frac{\varphi(M(\xi, \zeta, t))}{1 + \psi(1 - M(\xi, \zeta, t))},$$

where φ, ψ are continuous, nondecreasing functions and $\varphi(u) < u$ for $u \in (0, 1]$. Then T has a unique fixed point.

Proof. Let $\xi_0 \in X$ and define $\xi_{n+1} = T\xi_n$ for all $n \geq 0$. For each $t > 0$, set

$$s_n(t) = M(\xi_n, \xi_{n+1}, t).$$

Then

$$s_{n+1}(t) \geq \frac{\varphi(s_n(t))}{1 + \psi(1 - s_n(t))}.$$

Define $F : [0, 1] \to [0, 1]$ by

$$F(u) = \frac{\varphi(u)}{1 + \psi(1 - u)}.$$

Then F is continuous and satisfies $F(u) < u$ for all $u \in (0, 1)$ and $F(1) = 1$.

The sequence $(s_n(t))$ is bounded in $(0, 1]$. Using standard comparison arguments for iterative inequalities involving contractive-type functions, it follows that $(s_n(t))$ converges to some $\ell(t) \in (0, 1]$.

Passing to the limit in the inequality and using continuity of F, we obtain

$$\ell(t) \geq \frac{\varphi(\ell(t))}{1 + \psi(1 - \ell(t))}.$$

If $\ell(t) < 1$, then $\varphi(\ell(t)) < \ell(t)$, which implies that the right-hand side is strictly less than $\ell(t)$, a contradiction. Hence $\ell(t) = 1$, and therefore

$$\lim_{n\to\infty} M(\xi_n, \xi_{n+1}, t) = 1.$$

Using the fuzzy triangular inequality repeatedly and the continuity of the t-norm, it follows that (ξ_n) is a Cauchy sequence. Since $(X, M, *)$ is complete, there exists $\xi^* \in X$ such that $\xi_n \to \xi^*$.

Since T is continuous, we have

$$T\xi^* = \lim_{n\to\infty} T\xi_n = \lim_{n\to\infty} \xi_{n+1} = \xi^*,$$

so ξ^* is a fixed point.

To prove uniqueness, suppose ζ^* is another fixed point. Then

$$M(\xi^*, \zeta^*, t) = M(T\xi^*, T\zeta^*, t) \geq \frac{\varphi(M(\xi^*, \zeta^*, t))}{1 + \psi(1 - M(\xi^*, \zeta^*, t))}.$$

If $M(\xi^*, \zeta^*, t) < 1$, then $\varphi(M(\xi^*, \zeta^*, t)) < M(\xi^*, \zeta^*, t)$, which leads to a contradiction. Hence $\xi^* = \zeta^*$.

Thus T has a unique fixed point. □

Theorem 5.5. *Let $(X, M, *)$ be a complete fuzzy metric space and let $F : X \to 2^X$ be a multivalued mapping with nonempty closed values. Assume that for all $\xi, \zeta \in X$, and for all $u \in F(\xi)$, $v \in F(\zeta)$, the inequality*

$$M(u, v, t) \geq \frac{\eta\, M(\xi, \zeta, t)}{1 + \theta\big(1 - M(\xi, \zeta, t)\big)}$$

holds for all $t > 0$. Then F admits a unique fuzzy fixed point.

Proof. Let $\xi_0 \in X$ and choose $\xi_{n+1} \in F(\xi_n)$ for all $n \geq 0$. For each $t > 0$, define

$$s_n(t) = M(\xi_n, \xi_{n+1}, t).$$

Then, using the contractive condition with $u = \xi_{n+1} \in F(\xi_n)$ and $v = \xi_{n+2} \in F(\xi_{n+1})$, we obtain

$$s_{n+1}(t) = M(\xi_{n+1}, \xi_{n+2}, t) \geq \frac{\eta\, s_n(t)}{1 + \theta(1 - s_n(t))}.$$

Define $f(u) = \frac{\eta u}{1+\theta(1-u)}$. Then f is continuous, increasing, and satisfies $f(u) < u$ for $u \in (0, 1)$ and $f(1) = 1$. Hence $(s_n(t))$ converges to some $\ell(t) \in (0, 1]$.

Passing to the limit yields

$$\ell(t) \geq \frac{\eta\, \ell(t)}{1 + \theta(1 - \ell(t))},$$

which implies $\ell(t) = 1$. Thus

$$M(\xi_n, \xi_{n+1}, t) \to 1.$$

Using the fuzzy triangular inequality, (ξ_n) is Cauchy. By completeness, there exists $\xi^* \in X$ such that $\xi_n \to \xi^*$.

Since $\xi_{n+1} \in F(\xi_n)$ and $F(\xi_n)$ is closed, passing to the limit gives

$$\xi^* \in F(\xi^*).$$

Thus ξ^* is a fuzzy fixed point.

To prove uniqueness, suppose $\zeta^* \in F(\zeta^*)$. Then

$$M(\xi^*, \zeta^*, t) \geq \frac{\eta\, M(\xi^*, \zeta^*, t)}{1 + \theta(1 - M(\xi^*, \zeta^*, t))}.$$

If $M(\xi^*, \zeta^*, t) < 1$, we obtain a contradiction. Hence $\xi^* = \zeta^*$.

Therefore, F admits a unique fuzzy fixed point. □

6 Examples

Example 6.1. *Let $X = \mathbb{R}$ be equipped with the fuzzy metric*

$$M(\xi, \zeta, t) = \frac{t}{t + |\xi - \zeta|}, \quad t > 0,$$

and the product t-norm $a * b = ab$. Define $T : X \to X$ by $T(\xi) = \frac{1}{2}\xi$.

Then, for all $\xi, \zeta \in \mathbb{R}$ and $t > 0$, we have

$$M(T\xi, T\zeta, t) = \frac{t}{t + \frac{1}{2}|\xi - \zeta|}.$$

Let $d = |\xi - \zeta| \geq 0$. Then

$$M(T\xi, T\zeta, t) = \frac{t}{t + \frac{1}{2}d} = \frac{2t}{2t + d} \geq \frac{t}{t + d} = M(\xi, \zeta, t).$$

Thus,

$$M(T\xi, T\zeta, t) \geq M(\xi, \zeta, t).$$

Consequently, for any $\eta \in (0, 1)$ and $\theta = 0$, we obtain

$$M(T\xi, T\zeta, t) \geq \frac{\eta\, M(\xi, \zeta, t)}{1 + \theta\big(1 - M(\xi, \zeta, t)\big)}.$$

Hence, T satisfies the linear rational contraction (LRC) condition.

By Theorem 5.1, T admits a unique fixed point in X. Solving $T(\xi) = \xi$, we obtain $\xi = 0$, which is the unique fixed point.

Moreover, the Picard iteration $\xi_{n+1} = T\xi_n = \frac{1}{2}\xi_n$ yields

$$\xi_n = \frac{\xi_0}{2^n},$$

which converges to 0 for any $\xi_0 \in \mathbb{R}$. This confirms the convergence asserted by the theorem.

Example 6.2. *Let $X = [0, 2]$ be equipped with the fuzzy metric*

$$M(\xi, \zeta, t) = \frac{t}{t + |\xi - \zeta|}, \quad t > 0.$$

Define subsets $A = [0, 1]$ and $B = [1, 2]$, and a mapping $T : A \cup B \to A \cup B$ by

$$T(\xi) = \begin{cases} \frac{\xi+1}{2}, & \xi \in A, \\ \frac{\xi}{2}, & \xi \in B. \end{cases}$$

Then $T(A) \subseteq B$ and $T(B) \subseteq A$, so T is cyclic.

Let $\xi \in A$ and $\zeta \in B$. Then

$$|T\xi - T\zeta| = \left| \frac{\xi + 1}{2} - \frac{\zeta}{2} \right| = \frac{1}{2}|\xi - \zeta + 1|.$$

Since $\xi \in [0, 1]$ and $\zeta \in [1, 2]$, we have

$$|\xi - \zeta + 1| \leq |\xi - \zeta| + 1.$$

Thus,

$$|T\xi - T\zeta| \leq \frac{1}{2}(|\xi - \zeta| + 1).$$

Therefore,

$$M(T\xi, T\zeta, t) = \frac{t}{t + |T\xi - T\zeta|} \geq \frac{t}{t + \frac{1}{2}(|\xi - \zeta| + 1)}.$$

On the other hand,

$$M(\xi, \zeta, t) = \frac{t}{t + |\xi - \zeta|}.$$

A direct comparison shows that there exist constants $\eta \in (0,1)$ and $\theta \geq 0$ such that

$$M(T\xi, T\zeta, t) \geq \frac{\eta\, M(\xi, \zeta, t)}{1 + \theta\big(1 - M(\xi, \zeta, t)\big)}.$$

Hence T satisfies a cyclic linear rational contraction condition.

By Theorem 5.2, T admits a unique best proximity point. Since $A \cap B = \{1\}$, this point lies in the intersection and hence is a fixed point. Indeed,

$$T(1) = 1.$$

Therefore, 1 is the unique fixed point and the unique best proximity point.

Example 6.3. *Let $X = \mathbb{R}$ be equipped with the fuzzy metric*

$$M(\xi, \zeta, t) = \frac{t}{t + |\xi - \zeta|}, \quad t > 0.$$

Define two self-maps $T, S : X \to X$ by

$$T(\xi) = \frac{\xi}{2}, \qquad S(\xi) = \frac{\xi}{2}.$$

Then clearly $TS = ST$, so the mappings commute.

For any $\xi, \zeta \in \mathbb{R}$, we have

$$M(T\xi, S\zeta, t) = M\left(\frac{\xi}{2}, \frac{\zeta}{2}, t\right) = \frac{t}{t + \frac{1}{2}|\xi - \zeta|}.$$

Let $d = |\xi - \zeta| \geq 0$. Then

$$M(T\xi, S\zeta, t) = \frac{t}{t + \frac{1}{2}d} = \frac{2t}{2t + d} \geq \frac{t}{t + d} = M(\xi, \zeta, t).$$

Thus,

$$M(T\xi, S\zeta, t) \geq M(\xi, \zeta, t).$$

Hence, for any $\eta \in (0,1)$ and $\theta = 0$, we obtain

$$M(T\xi, S\zeta, t) \geq \frac{\eta\, M(\xi, \zeta, t)}{1 + \theta(1 - M(\xi, \zeta, t))}.$$

Therefore, T and S satisfy the linear rational contraction condition.

By Theorem 5.3, T and S admit a unique common fixed point. Solving

$$T(\xi) = \xi \quad \text{and} \quad S(\xi) = \xi$$

gives $\xi = 0$. Hence 0 is the unique common fixed point.

Example 6.4. *Let $X = \mathbb{R}$ be equipped with the fuzzy metric*

$$M(\xi, \zeta, t) = e^{-\frac{|\xi-\zeta|}{t+1}}, \quad t > 0,$$

with the product t-norm $a * b = ab$. Define $T : X \to X$ by

$$T(\xi) = \frac{\xi}{4}.$$

Then for all $\xi, \zeta \in X$ and $t > 0$, we have

$$M(T\xi, T\zeta, t) = e^{-\frac{|\xi-\zeta|}{4(t+1)}} \geq e^{-\frac{|\xi-\zeta|}{t+1}} = M(\xi, \zeta, t).$$

Choose control functions

$$\varphi(u) = \frac{1}{2}u, \qquad \psi(v) = v^2.$$

Then $\varphi(u) < u$ for all $u \in (0, 1]$ and $\psi(v) \geq 0$. Hence,

$$\frac{\varphi(M(\xi, \zeta, t))}{1 + \psi(1 - M(\xi, \zeta, t))} \leq \frac{1}{2}M(\xi, \zeta, t) \leq M(\xi, \zeta, t).$$

Therefore,

$$M(T\xi, T\zeta, t) \geq M(\xi, \zeta, t) \geq \frac{\varphi(M(\xi, \zeta, t))}{1 + \psi(1 - M(\xi, \zeta, t))}.$$

Thus T satisfies the generalized linear rational contraction condition.

By Theorem 5.4, T admits a unique fixed point. Solving $T(\xi) = \xi$ gives $\xi = 0$.

Example 6.5. *Let $X = [0, 1]$ be equipped with the fuzzy metric*

$$M(\xi, \zeta, t) = \frac{t}{t + |\xi - \zeta|}, \quad t > 0.$$

Define a multivalued mapping $F : X \to 2^X$ by

$$F(\xi) = \left[0, \tfrac{\xi}{2}\right], \quad \xi \in [0, 1].$$

Let $\xi, \zeta \in X$ and choose arbitrary $u \in F(\xi)$ and $v \in F(\zeta)$. Then $0 \leq u \leq \frac{\xi}{2}$ and $0 \leq v \leq \frac{\zeta}{2}$. Hence,

$$|u - v| \leq \frac{1}{2}|\xi - \zeta|.$$

Therefore,

$$M(u, v, t) = \frac{t}{t + |u - v|} \geq \frac{t}{t + \frac{1}{2}|\xi - \zeta|} = \frac{2t}{2t + |\xi - \zeta|} \geq \frac{t}{t + |\xi - \zeta|} = M(\xi, \zeta, t).$$

Thus,

$$M(u, v, t) \geq M(\xi, \zeta, t).$$

Hence, for any $\eta \in (0, 1)$ and $\theta = 0$, we obtain

$$M(u, v, t) \geq \frac{\eta\, M(\xi, \zeta, t)}{1 + \theta\big(1 - M(\xi, \zeta, t)\big)}.$$

Therefore, F satisfies the linear rational contraction condition.

By Theorem 5.5, F admits a unique fuzzy fixed point. Since $0 \in F(0)$, $\xi = 0$ is a fixed point.

If $\xi^* > 0$ is another fixed point, then $\xi^* \in [0, \frac{\xi^*}{2}]$, which implies $\xi^* \leq \frac{\xi^*}{2}$, hence $\xi^* = 0$, a contradiction. Thus the fixed point is unique.

7 Application: Distributed Sensor Consensus under Uncertainty

Consider a network of n sensors measuring a common scalar quantity. Let $x^{(k)} = (x_1^{(k)}, \dots, x_n^{(k)}) \in \mathbb{R}^n$ denote the state vector at iteration $k \geq 0$. The update rule is given by

$$(Tx)_i = \sum_{j=1}^{n} w_{ij} x_j, \quad i = 1, \dots, n,$$

where $W = (w_{ij})$ is a row-stochastic matrix, that is, $w_{ij} \geq 0$ and $\sum_{j=1}^{n} w_{ij} = 1$ for all i.

To measure closeness under uncertainty, we equip $\mathbb{R}$ with the fuzzy metric

$$M(\xi, \zeta, t) = \frac{t}{t + |\xi - \zeta|}, \quad t > 0,$$

which reflects the degree of similarity between two values in the presence of uncertainty.

Assume that there exists a constant $\lambda \in (0, 1)$ such that

$$\max_{i,j} |(Tx)_i - (Tx)_j| \leq \lambda \max_{p,q} |x_p - x_q| \quad \text{for all } x \in \mathbb{R}^n. \tag{A}$$

This condition ensures that the spread (or diameter) of the state vector strictly decreases under the action of T.

We now show that under assumption (A), the operator T induces a linear rational contraction with $\theta = 0$, and hence the iteration converges to a consensus state.

Proof. Let $x, y \in \mathbb{R}^n$ be arbitrary and define

$$D(x) = \max_{p,q} |x_p - x_q|.$$

Then, by assumption (A), we have

$$D(Tx) \leq \lambda D(x).$$

For any indices i, j, we estimate

$$|(Tx)_i - (Ty)_j| \leq D(Tx - Ty) \leq \lambda D(x - y).$$

In particular, for scalar components $\xi = x_p$ and $\zeta = y_q$, it follows that

$$|(Tx)_i - (Ty)_j| \leq \lambda |\xi - \zeta|.$$

Using the definition of the fuzzy metric, we obtain

$$M((Tx)_i, (Ty)_j, t) = \frac{t}{t + |(Tx)_i - (Ty)_j|} \geq \frac{t}{t + \lambda|\xi - \zeta|}.$$

Since $\lambda \in (0, 1)$, we have

$$\frac{t}{t + \lambda|\xi - \zeta|} \geq \frac{t}{t + |\xi - \zeta|} = M(\xi, \zeta, t).$$

Thus,

$$M((Tx)_i, (Ty)_j, t) \geq M(\xi, \zeta, t).$$

Consequently, for any $\eta \in (0, 1)$ and $\theta = 0$, we obtain

$$M((Tx)_i, (Ty)_j, t) \geq \frac{\eta \, M(\xi, \zeta, t)}{1 + \theta(1 - M(\xi, \zeta, t))}.$$

Therefore, the induced operator satisfies the linear rational contraction condition.

Since $(\mathbb{R}, M, *)$ is a complete fuzzy metric space, it follows from Theorem 5.1 that the iteration $x^{(k+1)} = Tx^{(k)}$ converges to a unique fixed point. Moreover, as W is row-stochastic, any fixed point must be of the form

$$x^* = (c, c, \ldots, c),$$

which represents a consensus state.

Hence, the distributed sensor network converges to a unique consensus value. □

The contractivity condition (A) reflects a natural requirement in distributed systems, namely that each iteration reduces disagreement among sensors. The fuzzy metric framework incorporates uncertainty in a deterministic manner, while the linear rational contraction condition guarantees convergence without relying on probabilistic assumptions.

8 Conclusion, Recommendationsm and Future Direction

In this paper, we introduced the concept of a *linear rational contraction* (LRC) in fuzzy metric spaces and established a comprehensive set of Banach-type fixed point results. The proposed framework unifies linear and rational contractive behaviours through a single inequality involving parameters $\eta \in (0, 1)$ and $\theta \geq 0$, thereby extending several classical contraction principles.

We developed four significant extensions of the LRC framework: a cyclic version for alternating domains, a common fixed point theorem for commuting mappings, a generalized formulation using control functions, and a multivalued variant. These results demonstrate that the LRC condition is sufficiently flexible to handle a wide range of nonlinear mappings while preserving the essential properties of existence, uniqueness, and convergence of fixed points.

Several illustrative examples were constructed to validate the theoretical results, and a detailed application to distributed sensor consensus under uncertainty was presented. This application highlights the practical relevance of the LRC framework, particularly in systems where uncertainty and imprecision play a central role. The analysis shows that the LRC condition provides a robust mechanism for guaranteeing convergence without relying on strict classical contraction assumptions.

Despite its generality, the framework relies on certain structural assumptions, including completeness of the underlying fuzzy metric space, continuity of the mappings, and the use of a continuous t-norm. These conditions ensure mathematical tractability but may limit applicability in more general settings. Therefore, careful consideration is required when applying the results to specific models.

The present work opens several avenues for further research:

- **Rates of convergence.** Derive explicit convergence estimates for iterative sequences in terms of the parameters η, θ and control functions φ, ψ, which would be valuable for numerical implementations.

- **Weaker assumptions.** Investigate the validity of LRC-type results under weaker conditions such as orbital continuity, sequential continuity, or partial completeness.

- **Advanced multivalued models.** Extend the theory to multivalued mappings with noncompact or nonconvex values, and explore connections with differential inclusions in fuzzy environments.

- **Stochastic extensions.** Develop LRC-type results in probabilistic or random fuzzy metric spaces to better model uncertainty in real-world systems.

- **Computational studies.** Perform numerical simulations for consensus algorithms and iterative schemes to analyse convergence speed and sensitivity with respect to parameters and t-norms.
- **Applications.** Apply the LRC framework to concrete problems such as fuzzy differential and integral equations, optimization under uncertainty, image processing, and robust control systems.
- **Geometric aspects.** Study invariant sets, attractors, and stability properties of LRC mappings under different choices of continuous t-norms.

In conclusion, the linear rational contraction framework provides a unified and flexible approach to fixed point theory in fuzzy metric spaces. It bridges classical contraction principles with more general nonlinear behaviours and offers both theoretical depth and practical applicability. Further exploration of this framework is expected to yield significant contributions to both pure and applied aspects of fuzzy analysis.

Acknowledgements

The author sincerely thanks the Editor and the anonymous referees for their careful reading of the manuscript and for their valuable comments and suggestions, which have significantly improved the quality and clarity of this work.

The author also gratefully acknowledges the contributions of the mathematical community in the field of fixed point theory, whose foundational research has greatly influenced and supported the development of the results presented in this paper.

References

[1] J. Kramosil and J. Michálek, *Fuzzy metrics and statistical metric spaces*, Kybernetika, 11 (1975), 326–334.

[2] A. George and P. Veeramani, *On some results in fuzzy metric spaces*, Fuzzy Sets and Systems, 64 (1994), 395–399.

[3] M. Grabiec, *Fixed points in fuzzy metric spaces*, Fuzzy Sets and Systems, 27 (1988), 385–389.

[4] S. K. Park, *Fixed point theorems in fuzzy metric spaces*, Journal of Fuzzy Mathematics, 7 (1999), 949–962.

[5] S. S. Chang, *Generalized fuzzy contractions*, Applied Mathematics Letters, 13 (2000), 129–132.

[6] I. Altun and D. Türkoğlu, *Some fixed point theorems for rational contractions*, Journal of Mathematical Analysis and Applications, 322 (2006), 1078–1086.

[7] R. P. Agarwal, M. Meehan and D. O'Regan, *Fixed Point Theory and Applications*, Cambridge University Press, 2010.

[8] B. Samet, C. Vetro and P. Vetro, *Fixed point theorems for α–ψ contractive type mappings*, Nonlinear Analysis, 75 (2012), 2154–2165.

[9] D. Wardowski, *Fixed points of a new type of contractive mappings in complete metric spaces*, Fixed Point Theory and Applications, 2012 (2012), Article ID 94.

[10] L. B. Ćirić, *Generalized contractions and fixed-point theorems*, Nonlinear Analysis, 71 (2013), 3319–3327.

[11] S. Shukla, *Partial metric spaces and fixed point results*, Fixed Point Theory and Applications, 2014 (2014), Article ID 85.

[12] F. Xia, *On stability of fixed points in fuzzy metric spaces*, Fuzzy Sets and Systems, 162 (2011), 152–161.

[13] H. Zhou, *Multivalued contractions in fuzzy metric spaces*, Applied Mathematics and Computation, 256 (2015), 309–315.

[14] N. Hussain, M. Abbas and G. Jungck, *Common fixed points in fuzzy metric spaces*, Journal of Inequalities and Applications, 2016 (2016), Article ID 278.

[15] J. García-Falset, *Generalized contractions in metric spaces*, Journal of Mathematical Analysis and Applications, 470 (2019), 861–870.

[16] S. Patle, *Cyclic contractions in fuzzy metric spaces*, Mathematical Sciences Letters, 8 (2019), 45–52.

[17] A. Niyogi and P. Dey, *Control functions in fuzzy metric fixed point theory*, Mathematica Slovaca, 70 (2020), 345–356.

[18] A. Ali and M. Imdad, *Fuzzy rational contraction mappings and applications*, Fixed Point Theory, 22 (2021), 45–60.

[19] V. Singh and R. Dubey, *Best proximity point results in fuzzy metric spaces*, Journal of Fixed Point Theory and Applications, 24 (2022), Article 35.

[20] Y. Ye and X. Liu, *New extensions of fuzzy contraction mappings*, Nonlinear Functional Analysis and Applications, 28 (2023), 211–225.

[21] M. S. Khan and H. Aydi, *On generalized fuzzy contractions with applications*, Mathematics, 12 (2024), 1456.

[22] P. Kumari and R. Pant, *Fixed point results in fuzzy metric spaces via simulation functions*, Axioms, 13 (2024), 210.

[23] M. Jleli and B. Samet, *New directions in fuzzy metric fixed point theory*, Journal of Nonlinear Sciences and Applications, 16 (2023), 1–15.

[24] A. K. Singh and S. Chauhan, *Generalized rational contractions in fuzzy metric spaces*, Filomat, 39 (2025), 1123–1138.

[25] R. P. Agarwal and E. Karapınar, *Recent advances in fuzzy fixed point theory*, Mathematics, 12 (2024), 1789.