Fractional Crank-Nicolson Galerkin FEM for Coupled Time-Fractional Nonlocal Parabolic Problems

Abstract

In this paper, we introduce a second-order numerical scheme designed for addressing a coupled time-fractional nonlocal diffusion problem. The scheme is composed of the fractional Crank-Nicolson method integrated with the Galerkin finite element method (FEM) and Newton's method. We establish *a priori* error estimates for a fully-discrete solution in L^2 and H_0^1 norms. Additionally, we provide results based on the standard FEM to verify the theoretical estimates.

Keywords: Nonlocal problem, Uniform mesh, Fractional Crank-Nicolson method, Error estimate

AMS(MOS): 65M12, 65M60, 35R11

1 Introduction

There are various real-world issues that involve multiple unknown functions. For instance, the authors in [18] explored a mathematical model concerning two species, rabbits and foxes, on an island. In this model, the change in population of one species is influenced by the change in population of the other species.

In recent years, numerous researchers have focused on the investigation of nonlocal partial differential equations (PDEs) [2–6, 13, 14, 17]. In [3, 4], a coupled nonlocal parabolic problem was addressed using the FEM. Additionally, for time discretization, the Backward Euler method [3] and the Crank-Nicolson method [4] were implemented.

In this article, we consider the following coupled time-fractional nonlocal diffusion equation with unknowns u and v:
Let $\alpha \in (0,1)$. Find u and v such that

$$
\begin{aligned}
{}^{C}D_t^{\alpha}u(x,t) - M_1(l(u),l(v))\Delta u(x,t) &= f_1(u,v) && \text{in } \Omega\times(0,T], && \text{(1a)}\\
{}^{C}D_t^{\alpha}v(x,t) - M_2(l(u),l(v))\Delta v(x,t) &= f_2(u,v) && \text{in } \Omega\times(0,T], && \text{(1b)}\\
u(x,t) \;=\; v(x,t) &= 0 && \text{on } \partial\Omega\times(0,T], && \text{(1c)}\\
u(x,0) \;=\; v(x,0) &= 0 && \text{in } \Omega, && \text{(1d)}
\end{aligned}
$$

where Ω is a bounded domain in $\mathbb{R}^d$ $(d = 1, 2, 3)$ with smooth boundary $\partial\Omega$, $f_1(u, v)$ and $f_2(u, v)$ represent the forcing terms, $l(u) = \int_\Omega u\,dx$, $l(v) = \int_\Omega v\,dx$, and ${}^C D_t^\alpha$ represents the α^{th} order Caputo fractional derivative [7, Definition 3.1].

The above problem (1a)-(1d) can be seen as a generalisation of the integer order parabolic problem considered in [3, 4] to fractional order. Problems of this kind have application in Biology, where u, v can represent the densities of two populations, which are interacting through the nonlocal functions M_1 and M_2 [1, 9].

In [12], a coupled time-fractional nonlinear diffusion system is solved using a Galerkin finite element scheme along with the fractional Crank-Nicolson method. In [15, 16], the FEM with the $L1$ scheme on a uniform mesh is utilised to find the numerical solution of the time-fractional nonlocal PDE

$$\begin{aligned} {}^C D_t^\alpha u(x,t) - \nabla\Big(a\Big(\int_\Omega u\,dx\Big)\nabla u(x,t)\Big) &= f(x,t) && \text{in } \Omega \times (0,T], \\ u(x,t) &= 0 && \text{on } \partial\Omega \times (0,T], \\ u(x,0) &= u_0 && \text{in } \Omega. \end{aligned}$$

This scheme gives $O(\tau^{2-\alpha})$ (where τ denotes the time step size) convergence in time.

In this work, we use the fractional Crank-Nicolson method on a uniform mesh to discretise the Caputo fractional derivative, which gives $O(\tau^2)$ convergence in time direction. Moreover, the availability of the weights b_k in a simple form (equation (5)) enables the easy implementation of the overall scheme. This method was first proposed by Dimitrov [8] under the compatibility conditions $u \in C^4[0, T]$ and $u(0) = u_t(0) = u_{tt}(0) = 0$. Here, we also assume the same conditions for unknown functions u and v. To discretise the space variable, we use the FEM with a linear basis. To handle the nonlocal term and nonlinearity, Newton's method has been used.

Throughout the paper, $C > 0$ denotes the generic constant independent of mesh parameters h and τ. Let $(\cdot, \cdot)$ denotes the inner product and $\|\cdot\|$ denotes the norm on space $L^2(\Omega)$. For $m \in \mathbb{N}$, $H^m(\Omega)$ represents the standard Sobolev space with the norm $\|\cdot\|_m$ and $H_0^1(\Omega) := \Big\{w \in H^1(\Omega) : w = 0 \text{ on } \partial\Omega\Big\}$.

Moreover, for the existence and uniqueness results as well as numerical analysis, we make the following hypotheses on the given data.

- H1: $M_i : \mathbb{R}^2 \to \mathbb{R}$ is bounded with $0 < m_1 \le M_i(x, y) \le m_2$, x, $y \in \mathbb{R}$, $i = 1, 2$.
- H2: $M_i : \mathbb{R}^2 \to \mathbb{R}$ is Lipschitz continuous with Lipschitz constants $L_i', K_i' > 0$.

  $|M_i(x_1, y_1) - M_i(x_2, y_2)| \le L_i'|x_1 - x_2| + K_i'|y_1 - y_2|, \quad x_1,\ x_2,\ y_1,\ y_2 \in \mathbb{R}, \quad i = 1, 2.$
- H3: $f_i : \mathbb{R}^2 \to \mathbb{R}$ is Lipschitz continuous with Lipschitz constants $L_i, K_i > 0$.

  $|f_i(x_1, y_1) - f_i(x_2, y_2)| \le L_i\|x_1 - x_2\| + K_i\|y_1 - y_2\|, \quad x_1,\ x_2,\ y_1,\ y_2 \in \mathbb{R}, \quad i = 1, 2.$

2 Fully-discrete Formulation

In this section, first we write the weak formulation of (1) and then discretise it in both space and time variables. The weak formulation of problem (1) is given below. Find $u(\cdot,t), v(\cdot,t) \in H_0^1(\Omega)$ for each $t \in (0,T]$ such that $\forall w \in H_0^1(\Omega)$,

$$({}^C D_t^\alpha u, w) + M_1(l(u), l(v))\,(\nabla u, \nabla w) \;=\; \big(f_1(u,v), w\big), \tag{3a}$$

$$({}^C D_t^\alpha v, \omega) + M_2(l(u), l(v))\,(\nabla v, \nabla \omega) \;=\; \big(f_2(u,v), \omega\big), \tag{3b}$$

$$u(x,0) = v(x,0) \;=\; 0. \tag{3c}$$

By using the Faedo-Galerkin method, one can prove that under the hypotheses H1-H3, problem (3) has a unique solution $\{u,\, v\}$ [15, Theorems 2.4 and 2.5].

Now, to discretise equation (1a)-(1d) in space, we use the finite element method. For this, let Ω_h be a quasi uniform partition of Ω into disjoint intervals in $\mathbb{R}^1$ or triangles in $\mathbb{R}^2$ with step size h. Consider the M-dimensional subspace X_h of $H_0^1(\Omega)$ such that

$$X_h \;:=\; \Big\{ w \in C^0(\bar{\Omega}) : w_{|T_k} \in P_1(T_k),\ \forall\, T_k \in \Omega_h \text{ and } w = 0 \text{ on } \partial\Omega \Big\}.$$

Let $\tau_N := \big\{ t_n : t_n = n\tau, \text{ for } n = 0, \ldots, N \big\}$ be a uniform partition of $[0,T]$ into N number of sub-intervals with step size $\tau = \frac{T}{N}$. For each $n = 1, \ldots, N$, we denote $u(t_n)$ and $v(t_n)$ by u^n and v^n, respectively. Let $U^n \approx u^n$ and $V^n \approx v^n$. Also, we set $U^{n,\alpha} = (1-\frac{\alpha}{2})U^n + \frac{\alpha}{2}U^{n-1}$, $V^{n,\alpha} = (1-\frac{\alpha}{2})V^n + \frac{\alpha}{2}V^{n-1}$.

Now, we write an approximation to the Caputo fractional derivative using the fractional Crank-Nicolson method. We know that for any function w, if $w(0) = 0$, then ${}^C D_{t_n}^\alpha w = {}^R D_{t_n}^\alpha w$ [7, P. 53], where ${}^R D_{t_n}^\alpha w$ is the α^{th} order Riemann-Liouville fractional derivative of w [7, Definition 2.2]. Author in [8] derived the following approximation to ${}^R D_{t_{n-\frac{\alpha}{2}}}^\alpha w$.

$${}^C D_{t_{n-\frac{\alpha}{2}}}^\alpha w \;=\; {}^R D_{t_{n-\frac{\alpha}{2}}}^\alpha w \;\;\approx\; D_\tau^\alpha w^n \;:=\; \tau^{-\alpha} \textstyle\sum_{j=0}^{n} b_{n-j}\, \phi^j, \quad n = 1, \ldots, N, \tag{4}$$

where

$$b_k = (-1)^k \frac{\Gamma(\alpha+1)}{\Gamma(k+1)\Gamma(\alpha-k+1)}. \tag{5}$$

Lemma 2.1. *[8, Theorem 1] If $w \in C^4[0,T]$ and $w(0) = 0$, $w_t(0) = 0$, $w_{tt}(0) = 0$, then for $n = 1, \ldots, N$, the error $\|\, {}^C D_{t_{n-\frac{\alpha}{2}}}^\alpha w - D_\tau^\alpha w^n \|$ satisfies*

$$\|\, {}^C D_{t_{n-\frac{\alpha}{2}}}^\alpha w - D_\tau^\alpha w^n \| \le C\tau^2.$$

The fully-discrete scheme for (1) is: for each $n = 1, \ldots, N$, find U^n, $V^n \in X_h$ such that $\forall w \in X_h$,

$$\big({}^C D_\tau^\alpha U^n, w\big) + M_1\big(l(U^{n,\alpha}), l(V^{n,\alpha})\big)\big(\nabla U^{n,\alpha}, \nabla w\big) = \big(f_1(U^{n,\alpha}, V^{n,\alpha}), w\big), \tag{6a}$$

$$\big({}^C D_\tau^\alpha V^n, \omega\big) + M_2\big(l(U^{n,\alpha}), l(V^{n,\alpha})\big)\big(\nabla V^{n,\alpha}, \nabla \omega\big) = \big(f_2(U^{n,\alpha}, V^{n,\alpha}), \omega\big), \tag{6b}$$

$$U^0 = 0, \quad V^0 = 0. \tag{6c}$$

Using the definition of ${}^C D_\tau^\alpha$, (6) can be rewrite as

$$\tau^{-\alpha} b_0(U^n, w) + M_1\big(l(U^{n,\alpha}), l(V^{n,\alpha})\big)(\nabla U^{n,\alpha}, \nabla w) = \big(f_1(U^{n,\alpha}, V^{n,\alpha}), w\big) - \tau^{-\alpha}\sum_{j=1}^{n-1} b_{n-j}(U^{n,\alpha}, w_h), \tag{7a}$$

$$\tau^{-\alpha} b_0(V^n, \omega) + M_2\big(l(U^{n,\alpha}), l(V^{n,\alpha})\big)(\nabla V^{n,\alpha}, \nabla \omega) = \big(f_2(U^{n,\alpha}, V^{n,\alpha}), \omega\big) - \tau^{-\alpha}\sum_{j=1}^{n-1} b_{n-j}\big(V^j, \omega\big). \tag{7b}$$

Let $\{\psi_i(x)\}_{i=1}^M$ be a basis of X_h associated with nodes of Ω_h. Therefore, for U^n, $V^n \in X_h$, we can find some $\beta_i^n,\ \gamma_i^n \in \mathbb{R}$ such that

$$U^n = \sum_{i=1}^M \beta_i^n \psi_i, \quad V^n = \sum_{i=1}^M \gamma_i^n \psi_i\ . \tag{8}$$

Define $\beta^n := [\beta_1^n, \ldots, \beta_M^n]'$ and $\gamma^n := [\gamma_1^n, \ldots, \gamma_M^n]'$.

Now, substituting the values of $U^n,\ V^n$ from (8) into (7), for each $1 \le i \le M$, we obtain the nonlinear algebraic equations

$$G_i(\beta^n, \gamma^n) \ = \ G_i(U^n, V^n) \ = \ 0, \tag{9a}$$

$$H_i(\beta^n, \gamma^n) \ = \ H_i(U^n, V^n) \ = \ 0, \tag{9b}$$

where

$$\begin{aligned}
G_i(U^n, V^n) \ &= \ \tau^{-\alpha} b_0\,(U^n, \psi_i) \ - \ \big(f_1(U^{n,\alpha}, V^{n,\alpha}), \psi_i\big) \\
&\quad + \tau^{-\alpha}\sum_{j=1}^{n-1} b_{n-j}(U^j, \psi_i) \ + \ M_1\big(l(U^{n,\alpha}), l(V^{n,\alpha})\big)(\nabla U^{n,\alpha}, \nabla \psi_i), \\
H_i(U^n, V^n) \ &= \ \tau^{-\alpha} b_0\,(V^n, \psi_i) \ - \ \big(f_2(U^{n,\alpha}, V^{n,\alpha}), \psi_i\big) \\
&\quad + \tau^{-\alpha}\sum_{j=1}^{n-1} b_{n-j}(V^j, \psi_i) \ + \ M_2\big(l(U^{n,\alpha}), l(V^{n,\alpha})\big)(\nabla V^{n,\alpha}, \nabla \psi_i).
\end{aligned}$$

If we apply Newton's method to solve (9), we get the Jacobian matrix J_1 as follows:

$$J_1 = \begin{bmatrix} A & B \\ C & D \end{bmatrix},$$

where the elements of the matrices A, B, C and D take the form

$$\begin{aligned}
(A)_{ki} \;=\; \tfrac{\partial G_i}{\partial \beta_k^n} \;=\; & \tau^{-\alpha}\, b_0(\psi_k,\psi_i) \;+\; \left(1-\frac{\alpha}{2}\right) M_1\big(l(U^{n,\alpha}), l(V^{n,\alpha})\big)(\nabla\psi_k,\nabla\psi_i) \\
& + \left(1-\frac{\alpha}{2}\right)\left(\frac{\partial M_1\big(l(U^{n,\alpha}), l(V^{n,\alpha})\big)}{\partial l(U^n)}\right)\left(\int_\Omega \psi_k\,dx\right)(\nabla U^{n,\alpha},\nabla\psi_i) \\
& - \left(1-\frac{\alpha}{2}\right)\left(\frac{\partial f_1(U^n,V^n)}{\partial U^n}\psi_k,\psi_i\right),
\end{aligned} \tag{10}$$

$$\begin{aligned}
(B)_{ki} \;=\; \tfrac{\partial G_i}{\partial \gamma_k^n} \;=\; & \left(1-\frac{\alpha}{2}\right)\left(\frac{\partial M_1\big(l(U^{n,\alpha}), l(V^{n,\alpha})\big)}{\partial l(V^n)}\right)\left(\int_\Omega \psi_k\,dx\right)(\nabla U^{n,\alpha},\nabla\psi_i) \\
& - \left(1-\frac{\alpha}{2}\right)\left(\frac{\partial f_1(U^n,V^n)}{\partial V^n}\psi_k,\psi_i\right),
\end{aligned} \tag{11}$$

$$\begin{aligned}
(C)_{pi} \;=\; \tfrac{\partial H_i}{\partial \beta_p^n} \;=\; & \left(1-\frac{\alpha}{2}\right)\left(\frac{\partial M_2\big(l(U^{n,\alpha}), l(V^{n,\alpha})\big)}{\partial l(U^n)}\right)\left(\int_\Omega \psi_p\,dx\right)(\nabla V^{n,\alpha},\nabla\psi_i) \\
& - \left(1-\frac{\alpha}{2}\right)\left(\frac{\partial f_2(U^n,V^n)}{\partial U^n}\psi_p,\psi_i\right),
\end{aligned} \tag{12}$$

$$\begin{aligned}
(D)_{pi} \;=\; \tfrac{\partial H_i}{\partial \gamma_p^n} \;=\; & \tau^{-\alpha}\, b_0(\psi_p,\psi_i) \;+\; M_2\big(l(U^{n,\alpha}), l(V^{n,\alpha})\big)(\nabla\psi_p,\nabla\psi_i) \\
& + \left(1-\frac{\alpha}{2}\right)\left(\frac{\partial M_2\big(l(U^{n,\alpha}), l(V^{n,\alpha})\big)}{\partial l(V^n)}\right)\left(\int_\Omega \psi_p\,dx\right)(\nabla V^{n,\alpha},\nabla\psi_i) \\
& - \left(1-\frac{\alpha}{2}\right)\left(\frac{\partial f_2(U^n,V^n)}{\partial V^n}\psi_p,\psi_i\right),
\end{aligned} \tag{13}$$

where $1 \leq i,\ k,\ p \leq M$. From equations (10)-(13), we can observe that the matrices A, B, C, D are not sparse and therefore the Jacobian matrix J_1 is not sparse [2–5]. We follow the idea given in [2–5] to overcome the above issue of sparsity. This idea was first proposed in [10] to solve a nonlocal elliptic boundary value problem. The modified problem is defined as follows:
Find $d_1, d_2 \in \mathbb{R}$ and $U^n, V^n \in X_h$ such that

$$\mathcal{G}_i(U^n, V^n, d_1, d_2) \;=\; 0, \quad 1 \leq i \leq M+1, \tag{14a}$$

$$\mathcal{H}_i(U^n, V^n, d_1, d_2) \;=\; 0, \quad 1 \leq i \leq M+1, \tag{14b}$$

where for $1 \le i \le M$,

$$\begin{aligned}
\mathcal{G}_i(U^n, V^n, d_1, d_2) &= \tau^{-\alpha}\, b_0(U^n, \psi_i) \;-\; \big(f_1(U^{n,\alpha}, V^{n,\alpha}), \psi_i\big) \\
&\quad + \tau^{-\alpha} \sum_{j=1}^{n-1} b_{n-j}(U^j, \psi_i) \;+\; M_1(d_1, d_2)(\nabla U^{n,\alpha}, \nabla \psi_i), \\
\mathcal{H}_i(U^n, V^n, d_1, d_2) &= \tau^{-\alpha}\, b_0(V^n, \psi_i) \;-\; \big(f_2(U^{n,\alpha}, V^{n,\alpha}), \psi_i\big) \\
&\quad + \tau^{-\alpha} \sum_{j=1}^{n-1} b_{n-j}(V^j, \psi_i) \;+\; M_2(d_1, d_2)(\nabla V^{n,\alpha}, \nabla \psi_i),
\end{aligned}$$

and
$$\begin{aligned}
\mathcal{G}_{(M+1)}(U^n, V^n, d_1, d_2) &= l(U^{n,\alpha}) - d_1, \\
\mathcal{H}_{(M+1)}(U^n, V^n, d_1, d_2) &= l(V^{n,\alpha}) - d_2.
\end{aligned}$$

Now, applying Newton's method to the system of equations (14), we get the following matrix equation:

$$J \begin{bmatrix} \beta^n \\ \gamma^n \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} A_1 & B_1 & C_1 & D_1 \\ A_2 & B_2 & C_2 & D_2 \\ A_3 & B_3 & C_3 & D_3 \\ A_4 & B_4 & C_4 & D_4 \end{bmatrix} \begin{bmatrix} \beta^n \\ \gamma^n \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} \bar{\mathcal{G}} \\ \bar{\mathcal{H}} \\ \mathcal{G}_{(M+1)} \\ \mathcal{H}_{(M+1)} \end{bmatrix}, \tag{15}$$

where J denotes the Jacobian matrix, $\beta^n = [\beta_1^n, \dots, \beta_M^n]'$, $\gamma^n = [\gamma_1^n, \dots, \gamma_M^n]'$, $\bar{\mathcal{G}} = [\mathcal{G}_1, \dots, \mathcal{G}_M]'$, $\bar{\mathcal{H}} = [\mathcal{H}_1, \dots, \mathcal{H}_M]'$, and entries of matrices A_r, B_r, C_r, D_r, $(r = 1, 2, 3, 4)$ are given below. For $1 \le i, k \le M$,

$$\begin{aligned}
(A_1)_{ik} &= \tau^{-\alpha}\, b_0(\psi_k, \psi_i) + \big(1 - \frac{\alpha}{2}\big)\Big\{M_1(d_1, d_2)(\nabla \psi_k, \nabla \psi_i) - \Big(\frac{\partial f_1(U^n, V^n)}{\partial U^n}\psi_k, \psi_i\Big)\Big\}, \\
(B_1)_{ik} &= -\,\big(1 - \frac{\alpha}{2}\big)\Big(\frac{\partial f_1(U^n, V^n)}{\partial V^n}\psi_k, \psi_i\Big), \\
(C_1)_{i1} &= \frac{\partial M_1}{\partial d_1}(d_1, d_2)\,(\nabla U^{n,\alpha}, \nabla \psi_i), \qquad (D_1)_{i1} = \frac{\partial M_1}{\partial d_2}(d_1, d_2)\,(\nabla U^{n,\alpha}, \nabla \psi_i).
\end{aligned}$$

$$\begin{aligned}
(A_2)_{ik} &= -\,\big(1 - \frac{\alpha}{2}\big)\Big(\frac{\partial f_2(U^n, V^n)}{\partial U^n}\psi_k, \psi_i\Big), \\
(B_2)_{ik} &= \tau^{-\alpha}\, b_0(\psi_k, \psi_i) + \big(1 - \frac{\alpha}{2}\big)\Big\{M_2(d_1, d_2)(\nabla \psi_k, \nabla \psi_i) - \Big(\frac{\partial f_2(U^n, V^n)}{\partial V^n}\psi_k, \psi_i\Big)\Big\}, \\
(C_2)_{i1} &= \frac{\partial M_2}{\partial d_1}(d_1, d_2)\,(\nabla V^{n,\alpha}, \nabla \psi_i), \qquad (D_2)_{i1} = \frac{\partial M_2}{\partial d_2}(d_1, d_2)\,(\nabla V^{n,\alpha}, \nabla \psi_i).
\end{aligned}$$

$$(A_3)_{1k} = \Big(1 - \frac{\alpha}{2}\Big)\int_\Omega \psi_k\, dx, \qquad (B_3)_{1k} = 0, \qquad (C_3)_{11} = -1, \qquad (D_3)_{11} = 0.$$

$$(A_4)_{1k} = 0, \qquad (B_4)_{1k} = \Big(1 - \frac{\alpha}{2}\Big)\int_\Omega \psi_k\, dx, \qquad (C_4)_{11} = 0, \qquad (D_4)_{11} = -1.$$

Here, it can be seen that A_1, B_1, A_2, B_2 are sparse matrices [4,5,12] and hence J is a sparse matrix.

Note that if (d_1, d_2, U^n, V^n) is the solution of the problem (14), then $\{U^n, V^n\}$ is the solution of the problem (7) and the converse is also true [5, Theorem 3.1].

3 *A priori* Bound

In this section, we provide *a priori* bound for the fully-discrete scheme (6). For this, we need the following lemma.

Lemma 3.1. *[11, Lemma 4.4] For any function $\phi(\cdot,t)$ defined on τ_N, one has*

$$\frac{1}{2}\,{}^{C}D_{\tau}^{\alpha}\|\phi^n\|^2 \le \big({}^{C}D_{\tau}^{\alpha}\phi^n,\, \phi^{n,\alpha}\big),$$

where $\phi^{n,\alpha} := (1-\frac{\alpha}{2})\phi^n + \frac{\alpha}{2}\phi^{n-1}$, *for* $n = 1, \ldots, N$.

For the derivation of *a priori* bound and error estimate, we also use the following discrete fractional Grönwall type inequality.

Lemma 3.2. *[11, Lemma 4.3] Suppose the nonnegative sequences $\{\omega^n, \phi^n : n \ge 0\}$ satisfy*

$${}^{C}D_{\tau}^{\alpha}\omega^n \le \lambda_1\omega^n + \lambda_2\omega^{n-1} + \phi^n, \quad n \ge 1,$$

where λ_1 and λ_2 are nonnegative constants. Then there exists a positive constant $\tau^\star$ such that when $\tau \le \tau^\star$,

$$\omega^n \le 2\,E_\alpha(2\lambda t_n^\alpha)\left(\omega^0 + \frac{t_n^\alpha}{\Gamma(1+\alpha)}\max_{0\le j\le n}\phi^j\right), \quad 1 \le n \le N,$$

where $E_\alpha(z)$ is the Mittag-Leffler function and $\lambda = \lambda_1 + \frac{\lambda_2}{2-2^{(1-\alpha)}}$.

In the following theorem, we derive *a priori* bound for the fully-discrete solution.

Theorem 3.3. *Let (U^n, V^n) (for $1 \le n \le N$) be the solution of (6). Then there exists a positive constant τ^* (independent of h) such that when $\tau \le \tau^*$, U^n, V^n satisfy*

$$\|U^n\| + \|V^n\| \le C, \tag{16}$$

$$\|\nabla U^n\| + \|\nabla V^n\| \le C. \tag{17}$$

Proof. Choosing $w = U^{n,\alpha}$ in (6a) and then using Hypothesis H1, the Cauchy-Schwartz inequality, and the inequality $ab \leq \frac{1}{2}a^2 + \frac{1}{2}b^2$, we can obtain

$$\left({}^C D_\tau^\alpha U^n, U^{n,\alpha}\right) + m_1 \|\nabla U^{n,\alpha}\|^2 \leq \frac{1}{2}\|f_1(U^{n,\alpha}, V^{n,\alpha})\|^2 + \frac{1}{2}\|U^{n,\alpha}\|^2. \quad (18)$$

Since f_1 is Lipschitz continuous, we have

$$\begin{aligned}\Big|\|f_1(U^{n,\alpha}, V^{n,\alpha})\| - \|f_1(0,0)\|\Big| &\leq \|f_1(U^{n,\alpha}, V^{n,\alpha}) - f_1(0,0)\| \\ &\leq L_1\|U^{n,\alpha}\| + K_1\|V^{n,\alpha}\|.\end{aligned}$$

Therefore,

$$\|f_1(U^{n,\alpha}, V^{n,\alpha})\| \leq C\left(1 + \|U^{n,\alpha}\| + \|V^{n,\alpha}\|\right). \quad (19)$$

An application of Lemma 3.1 and (19) into (18), gives

$${}^C D_\tau^\alpha \|U^n\|^2 \leq C\big(1 + \|U^{n,\alpha}\|^2 + \|V^{n,\alpha}\|^2\big). \quad (20)$$

Similarly, we can get the estimate for V^n using (6b) as follows:

$${}^C D_\tau^\alpha \|V^n\|^2 \leq C\big(1 + \|V^{n,\alpha}\|^2 + \|U^{n,\alpha}\|^2\big). \quad (21)$$

Adding (20) and (21), we get

$${}^C D_\tau^\alpha (\|U^n\|^2 + \|V^n\|^2) \leq C\big(1 + \|U^{n,\alpha}\|^2 + \|V^{n,\alpha}\|^2\big). \quad (22)$$

Using Lemma 3.2 in (22), we can arrive at

$$\|U^n\|^2 + \|V^n\|^2 \leq C. \quad (23)$$

For $a, b \geq 0$, using $\frac{1}{2}(a+b)^2 \leq a^2 + b^2$ in (23), we obtain (16).

One can prove (17) by choosing $w = {}^C D_\tau^\alpha U^n$ in (6a) and $w = {}^C D_\tau^\alpha V^n$ in (6b), and then using the similar arguments as above.
This completes the proof. □

Theorem 3.4. *Let $U^0, \ldots, U^{n-1}$ and $V^0, \ldots, V^{n-1}$ are given. Then there exists a positive constant τ^* (independent of h) such that when $\tau \leq \tau^*$, the problem (6) has a unique solution U^n, $V^n \in X_h$, for all $1 \leq n \leq N$.*

Proof. The proof is similar to the proof of [12, Theorem 1]. □

4 *A priori* Error Estimate

In this section, we derive the convergence estimate for the fully-discrete solution. Before this, we recall the definition of the Ritz projection operator [19, P.8].

$$(\nabla\phi, \nabla w) = (\nabla R_h\phi, \nabla w), \quad \forall \phi \in H_0^1(\Omega),\ w \in X_h. \tag{24}$$

In the following lemma, we state an approximation property for the operator R_h [19, Lemma 1.1], which will be useful in the derivation of *a priori* error estimate.

Lemma 4.1. *There exists $C > 0$, independent of h such that $\forall \phi \in H^2(\Omega) \cap H_0^1(\Omega)$,*

$$\|\phi - R_h\phi\|_{L^2(\Omega)} + h\,\|\nabla(\phi - R_h\phi)\|_{L^2(\Omega)} \ \leq \ Ch^2\,\|\Delta\phi\|_{L^2(\Omega)}. \tag{25}$$

Using the intermediate projection R_h, we split the error into two parts as

$$\begin{aligned} u^n - U^n &= (u^n - R_hu^n) + (R_hu^n - U^n) = \zeta_1^n + \chi_1^n, \\ v^n - V^n &= (v^n - R_hv^n) + (R_hv^n - V^n) = \zeta_2^n + \chi_2^n. \end{aligned}$$

Next theorem is one of the main results of this paper.

Theorem 4.2. *Let (u^n, v^n) and (U^n, V^n) be the solution of (1) and (6), respectively. Assume that $u,\ v \in C^4\big([0,T]; L^2(\Omega)\big) \cap C^2\big([0,T]; H^2(\Omega) \cap H_0^1(\Omega)\big)$ and $\left(\frac{\partial^i u}{\partial t^i}\right)_{t=0} = \left(\frac{\partial^i v}{\partial t^i}\right)_{t=0} = 0$, for $i = 0,1,2$. Then there exists a positive constant τ^* (independent of h) such that when $\tau \leq \tau^*$, the following estimates hold.*

$$\begin{aligned} \|u^n - U^n\| + \|v^n - V^n\| &\leq C\left(h^2 + \tau^2\right), &(26)\\ \|\nabla u^n - \nabla U^n\| + \|\nabla v^n - \nabla V^n\| &\leq C\left(h + \tau^2\right), &(27) \end{aligned}$$

where $n = 1, \dots, N$.

Proof. For any $w \in X_h$, we have

$$\begin{aligned} &\big({}^C D_\tau^\alpha \chi_1^n, w\big) + M_1\big(l(U^{n,\alpha}), l(V^{n,\alpha})\big)\,(\nabla\chi_1^{n,\alpha}, \nabla w) \\ &= \big({}^C D_\tau^\alpha R_hu^n - {}^C D_{t_{n-\frac{\alpha}{2}}}^\alpha u, w\big) + M_1\big(l(u^{n-\frac{\alpha}{2}}), l(v^{n-\frac{\alpha}{2}})\big)(\nabla u^{n,\alpha} - \nabla u^{n-\frac{\alpha}{2}}, \nabla w) \\ &\quad + \Big\{M_1\big(l(U^{n,\alpha}), l(V^{n,\alpha})\big) - M_1\big(l(u^{n-\frac{\alpha}{2}}), l(v^{n-\frac{\alpha}{2}})\big)\Big\}(\nabla u^{n,\alpha}, \nabla w) \\ &\quad + \big(f_1(u^{n-\frac{\alpha}{2}}, v^{n-\frac{\alpha}{2}}) - f_1(U^{n,\alpha}, V^{n,\alpha}), w\big). \end{aligned} \tag{28}$$

We set $w = \chi_1^{n,\alpha}$ in (28), and then use the bound of M_1, the Cauchy-Schwartz inequality in the resulting equation to arrive at

$$\begin{aligned} &\big({}^C D_\tau^\alpha \chi_1^n, \chi_1^{n,\alpha}\big) + m_1\,\|\nabla\chi_1^{n,\alpha}\|^2 \\ &\leq \|{}^C D_\tau^\alpha R_hu^n - {}^C D_{t_{n-\frac{\alpha}{2}}}^\alpha u\|\,\|\chi_1^{n,\alpha}\| + m_2\,\|\nabla u^{n,\alpha} - \nabla u^{n-\frac{\alpha}{2}}\|\,\|\nabla\chi_1^{n,\alpha}\| \\ &\quad + R_u\left|M_1\big(l(U^{n,\alpha}), l(V^{n,\alpha})\big) - M_1\big(l(u^{n-\frac{\alpha}{2}}), l(v^{n-\frac{\alpha}{2}})\big)\right|\,\|\nabla\chi_1^{n,\alpha}\| \\ &\quad + \left\|f_1(u^{n-\frac{\alpha}{2}}, v^{n-\frac{\alpha}{2}}) - f_1(U^{n,\alpha}, V^{n,\alpha})\right\|\,\|\chi_1^{n,\alpha}\|. \end{aligned} \tag{29}$$

Applying the Poincar$\acute{e}$ inequality and the inequality $ab \leq \frac{\epsilon}{2}a^2 + \frac{1}{2\epsilon}b^2$ into (29), we can obtain

$$\begin{aligned}
&\left({}^C D_\tau^\alpha \chi_1^n, \chi_1^{n,\alpha}\right) + m_1 \|\nabla \chi_1^{n,\alpha}\|^2 \\
&\leq \frac{2}{m_1} \|{}^C D_\tau^\alpha R_h u^n - {}^C D_{t_{n-\frac{\alpha}{2}}}^\alpha u\|^2 + \frac{2}{m_1} \left\| f_1(u^{n-\frac{\alpha}{2}}, v^{n-\frac{\alpha}{2}}) - f_1(U^{n,\alpha}, V^{n,\alpha}) \right\|^2 \\
&\quad + \frac{2R_u^2}{m_1} \left| M_1\big(l(U^{n,\alpha}), l(V^{n,\alpha})\big) - M_1\big(l(u^{n-\frac{\alpha}{2}}), l(v^{n-\frac{\alpha}{2}})\big) \right|^2 \\
&\quad + \frac{2\, m_2^2}{m_1} \|\nabla u^{n,\alpha} - \nabla u^{n-\frac{\alpha}{2}}\|^2 + \frac{m_1}{2} \|\nabla \chi_1^{n,\alpha}\|^2. \qquad (30)
\end{aligned}$$

Lipschitz continuity of functions M_1 and f_1 gives

$$\begin{aligned}
&\left| M_1\big(l(U^{n,\alpha}), l(V^{n,\alpha})\big) - M_1\big(l(u^{n-\frac{\alpha}{2}}), l(v^{n-\frac{\alpha}{2}})\big) \right| \\
&\quad \leq L_1' C \, \|U^{n,\alpha} - u^{n-\frac{\alpha}{2}}\| + K_1' C \, \|V^{n,\alpha} - v^{n-\frac{\alpha}{2}}\| \\
&\quad \leq C \left\{ \|\zeta_1^{n,\alpha}\| + \|\chi_1^{n,\alpha}\| + \|\zeta_2^{n,\alpha}\| + \|\chi_2^{n,\alpha}\| + \|u^{n,\alpha} - u^{n-\frac{\alpha}{2}}\| + \|v^{n,\alpha} - v^{n-\frac{\alpha}{2}}\| \right\}, \quad (31)
\end{aligned}$$

and

$$\begin{aligned}
&\left\| f_1(u^{n-\frac{\alpha}{2}}, v^{n-\frac{\alpha}{2}}) - f_1(U^{n,\alpha}, V^{n,\alpha}) \right\| \\
&\quad \leq L_1 C \, \|U^{n,\alpha} - u^{n-\frac{\alpha}{2}}\| + K_1 C \, \|V^{n,\alpha} - v^{n-\frac{\alpha}{2}}\| \\
&\quad \leq C \left\{ \|\zeta_1^{n,\alpha}\| + \|\chi_1^{n,\alpha}\| + \|\zeta_2^{n,\alpha}\| + \|\chi_2^{n,\alpha}\| + \|u^{n,\alpha} - u^{n-\frac{\alpha}{2}}\| + \|v^{n,\alpha} - v^{n-\frac{\alpha}{2}}\| \right\}. \quad (32)
\end{aligned}$$

Thus, from equations (30)-(32) and Lemma 3.1, we have

$$\begin{aligned}
{}^C D_\tau^\alpha \|\chi_1^n\|^2 \leq C \big\{ &\|{}^C D_\tau^\alpha R_h u^n - {}^C D_{t_{n-\frac{\alpha}{2}}}^\alpha u\|^2 + \|\zeta_2^{n,\alpha}\|^2 + \|\chi_2^{n,\alpha}\|^2 + \|\zeta_1^{n,\alpha}\|^2 + \|\chi_1^{n,\alpha}\|^2 \\
&+ \|\nabla u^{n,\alpha} - \nabla u^{n-\frac{\alpha}{2}}\|^2 + \|u^{n,\alpha} - u^{n-\frac{\alpha}{2}}\|^2 + \|v^{n,\alpha} - v^{n-\frac{\alpha}{2}}\|^2 \big\}, \qquad (33)
\end{aligned}$$

where C is dependent on m_1, m_2, R_u, L_1', K_1', L_1, K_1.

Now, it follows from Taylor's theorem that

$$\|u^{n,\alpha} - u^{n-\frac{\alpha}{2}}\| + \|v^{n,\alpha} - v^{n-\frac{\alpha}{2}}\| + \|\nabla u^{n,\alpha} - \nabla u^{n-\frac{\alpha}{2}}\| \leq C\,\tau^2. \qquad (34)$$

From Lemma 2.1 and (25), we have

$$\begin{aligned}
\|{}^C D_\tau^\alpha R_h u^n - {}^C D_{t_{n-\frac{\alpha}{2}}}^\alpha u\| &\leq \|{}^C D_\tau^\alpha R_h u^n - {}^C D_{t_{n-\frac{\alpha}{2}}}^\alpha R_h u\| + \|{}^C D_{t_{n-\frac{\alpha}{2}}}^\alpha R_h u - {}^C D_{t_{n-\frac{\alpha}{2}}}^\alpha u\| \\
&\leq C\left(\tau^2 + h^2\right). \qquad (35)
\end{aligned}$$

Using (25) and (34)-(35) in (33), we get

$${}^C D_\tau^\alpha \|\chi_1^n\|^2 \leq C\left(\|\chi_1^{n,\alpha}\|^2 + \|\chi_2^{n,\alpha}\|^2 + \left(h^2 + \tau^2\right)^2\right). \qquad (36)$$

Similarly, we can get an estimate for $\|\chi_2^{n,\alpha}\|$ as follows:

$$ {}^C D_\tau^\alpha \|\chi_2^n\|^2 \leq C\left(\|\chi_1^{n,\alpha}\|^2 + \|\chi_2^{n,\alpha}\|^2 + \left(h^2+\tau^2\right)^2\right). \tag{37} $$

Therefore, from (36) and (37)

$$ \begin{aligned} {}^C D_\tau^\alpha\left(\|\chi_1^n\|^2 + \|\chi_2^n\|^2\right) &\leq C\left(h^2+\tau^2\right)^2 + C\left(1-\frac{\alpha}{2}\right)^2\left(\|\chi_1^n\|^2 + \|\chi_2^n\|^2\right) \\ &\quad + \frac{C\,\alpha^2}{4}\left(\|\chi_1^{n-1}\|^2 + \|\chi_2^{n-1}\|^2\right). \end{aligned} \tag{38} $$

An application of Lemma 3.2 into (38) leads to

$$ \|\chi_1^n\|^2 + \|\chi_2^n\|^2 \leq C\left(h^2+\tau^2\right)^2. \tag{39} $$

Therefore,

$$ \|\chi_1^n\| + \|\chi_2^n\| \leq C\left(h^2+\tau^2\right). \tag{40} $$

Finally, using the Triangle inequality together with the estimates (40) and (25), we can get (26).

Now, in order to derive an error estimate in H_0^1-norm, we take $w = {}^C D_\tau^\alpha \chi_1^n$ in (28), and then perform the similar steps as above. This completes the proof. □

Remark 4.3. *In this study, we demonstrate the convergence of the proposed method without taking into account the weak singularity of the solution. We will address the convergence analysis for the case of weak singularity in future research work.*

5 Numerical Experiments

In this section, we perform some numerical experiments by considering two different problems with known exact solution. In both problems, we take the final time $T = 1$ and tolerance $\epsilon = 10^{-7}$ for stopping the Newton's iteration. We denote the number of sub-intervals in time by N. Moreover, let $(M_s + 1)$ be the number of node points in each spatial direction. In order to obtain the order of convergence in spatial direction in L^2 and H_0^1 norms, we take $N = M_s$ for different values of M_s. Similarly, to calculate the convergence rate in temporal direction in L^2-norm, we take $M_s = N$ for different values of N.

Example 5.1. *For first example, we consider* (1) *with the spatial domain* $\Omega = (0,1)$, $M_1(z,w) = 3 + \sin z + \cos w$, $M_2(z,w) = 5 + \cos z + \sin w$. *The functions* f_1 *and* f_2 *are chosen such that the analytical solution of the equation* (1) *be* $u(x,t) = t^{2+\alpha}\sin 2\pi x$ *and* $v(x,t) = t^{3-\alpha}\sin \pi x$.

Error and convergence rate in the spatial direction in L^2 and H_0^1 norms are given in Tables 1 and 2, respectively. Furthermore, Table 3 shows error and convergence rate in the temporal direction in L^2-norm.

	M_s	$\|u^n - U^n\|$	Rate	$\|v^n - V^n\|$	Rate
$\alpha = 0.4$	2^6	7.17E-04	1.999438471	1.60E-04	2.000114366
	2^7	1.79E-04	1.999759291	3.99E-05	2.000074895
	2^8	4.48E-05	1.99989373	9.98E-06	2.000038509
	2^9	1.12E-05	-	2.49E-06	-
$\alpha = 0.7$	2^6	7.21E-04	1.999524396	1.58E-04	2.000010343
	2^7	1.80E-04	1.999830784	3.95E-05	2.00001483
	2^8	4.51E-05	1.999928113	9.89E-06	2.000000769
	2^9	1.13E-05	-	2.47E-06	-

Table 1: (Example-5.1) *Error and convergence rate in spatial direction in L^2-norm.*

	M_s	$\|u^n - U^n\|$	Rate	$\|v^n - V^n\|$	Rate
$\alpha = 0.4$	2^6	1.26E-01	0.999784124	3.15E-02	0.99994853
	2^7	6.30E-02	0.999946033	1.57E-02	0.999987136
	2^8	3.15E-02	0.999986509	7.87E-03	0.999996784
	2^9	1.57E-02	-	3.93E-03	-
$\alpha = 0.7$	2^6	1.26E-01	0.999784473	3.15E-02	0.999948918
	2^7	6.30E-02	0.99994612	1.57E-02	0.999987232
	2^8	3.15E-02	0.99998653	7.87E-03	0.999996808
	2^9	1.57E-02	-	3.93E-03	-

Table 2: (Example-5.1) *Error and convergence rate in spatial direction in H_0^1-norm.*

	N	$\|u^n - U^n\|$	Rate	$\|v^n - V^n\|$	Rate
$\alpha = 0.4$	2^6	7.17E-04	1.999438471	1.60E-04	2.000114366
	2^7	1.79E-04	1.999759291	3.99E-05	2.000074895
	2^8	4.48E-05	1.99989373	9.98E-06	2.000038509
	2^9	1.12E-05	-	2.49E-06	-
$\alpha = 0.7$	2^6	7.21E-04	1.999524396	1.58E-04	2.000010343
	2^7	1.80E-04	1.999830784	3.95E-05	2.00001483
	2^8	4.51E-05	1.999928113	9.89E-06	2.000000769
	2^9	1.13E-05	-	2.47E-06	-

Table 3: (Example-5.1) *Error and convergence rate in temporal direction in L^2-norm.*

Example 5.2. *In this example, we take* $\Omega = (0,1)\times(0,1)$, $M_1(z,w) = 3+\sin z+\cos w$, $M_2(z,w) = 5 + \cos z + \sin w$. *We choose* f_1 *and* f_2 *such that the analytical solution of equation* (1) *be* $u(x,y,t) = t^3 \sin 2\pi x\, \sin 2\pi y$ *and* $v(x,y,t) = t^4 \sin \pi x\, \sin \pi y$.

Error and convergence rate in the spatial direction in L^2 and H_0^1 norms are given in Tables 4 and 5, respectively. Table 6 shows the error and convergence rate in the temporal direction in L^2-norm. For $\alpha = 0.5$, the graphs of exact and numerical solutions are shown in Figures 1 and 2.

	M_s	$\|u^n - U^n\|$	Rate	$\|v^n - V^n\|$	Rate
	2^3	8.76E-02	1.910229823	2.05E-02	1.983174711
$\alpha = 0.5$	2^4	2.33E-02	1.976537254	5.18E-03	1.996676573
	2^5	5.92E-03	1.993876169	1.30E-03	1.999788928
	2^6	1.49E-03	-	3.24E-04	-
	2^3	8.77E-02	1.910535849	1.99E-02	1.981811644
$\alpha = 0.9$	2^4	2.33E-02	1.976942804	5.04E-03	1.995308014
	2^5	5.92E-03	1.994131588	1.26E-03	1.998953039
	2^6	1.49E-03	-	3.16E-04	-

Table 4: (Example-5.2) *Error and convergence rate in spatial direction in* L^2*-norm.*

	M_s	$\|u^n - U^n\|$	Rate	$\|v^n - V^n\|$	Rate
	2^3	1.28E+00	0.986482547	3.24E-01	0.99590991
$\alpha = 0.5$	2^4	6.48E-01	0.996624338	1.62E-01	0.99896682
	2^5	3.25E-01	0.999158027	8.13E-02	0.999740748
	2^6	1.63E-01	-	4.06E-02	-
	2^3	1.28E+00	0.986646867	3.24E-01	0.995636078
$\alpha = 0.9$	2^4	6.48E-01	0.996659575	1.62E-01	0.998904181
	2^5	3.25E-01	0.999164853	8.13E-02	0.999725974
	2^6	1.63E-01	-	4.06E-02	-

Table 5: (Example-5.2) *Error and convergence rate in spatial direction in* H_0^1*-norm.*

	N	$\|u^n - U^n\|$	Rate	$\|v^n - V^n\|$	Rate
$\alpha = 0.5$	2^3	8.76E-02	1.910229823	2.05E-02	1.983174711
	2^4	2.33E-02	1.976537254	5.18E-03	1.996676573
	2^5	5.92E-03	1.993876169	1.30E-03	1.999788928
	2^6	1.49E-03	-	3.24E-04	-
$\alpha = 0.9$	2^3	8.77E-02	1.910535849	1.99E-02	1.981811644
	2^4	2.33E-02	1.976942804	5.04E-03	1.995308014
	2^5	5.92E-03	1.994131588	1.26E-03	1.998953039
	2^6	1.49E-03	-	3.16E-04	-

Table 6: (Example-5.2) *Error and convergence rate in temporal direction in* L^2*-norm.*

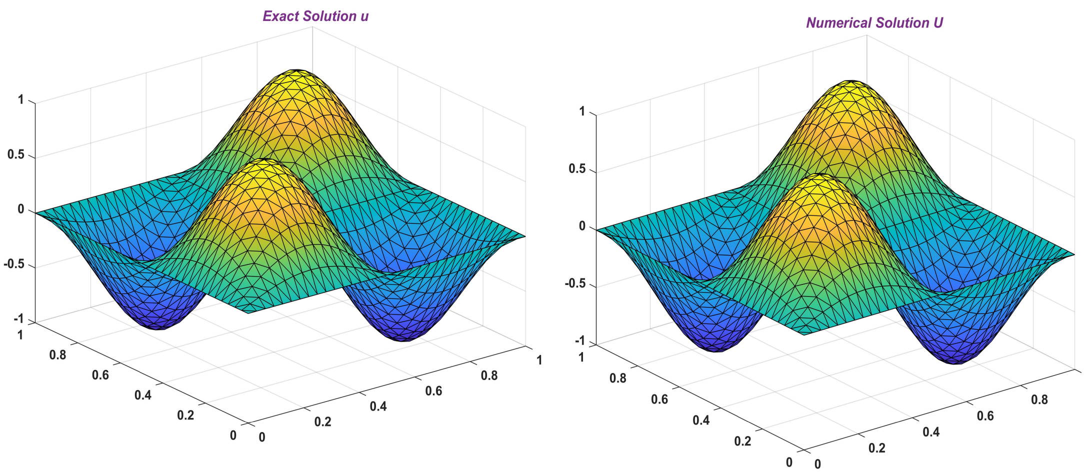

Figure 1: (Example-5.2) *The exact solution* u *and numerical solution* U *for* $\alpha = 0.5$.

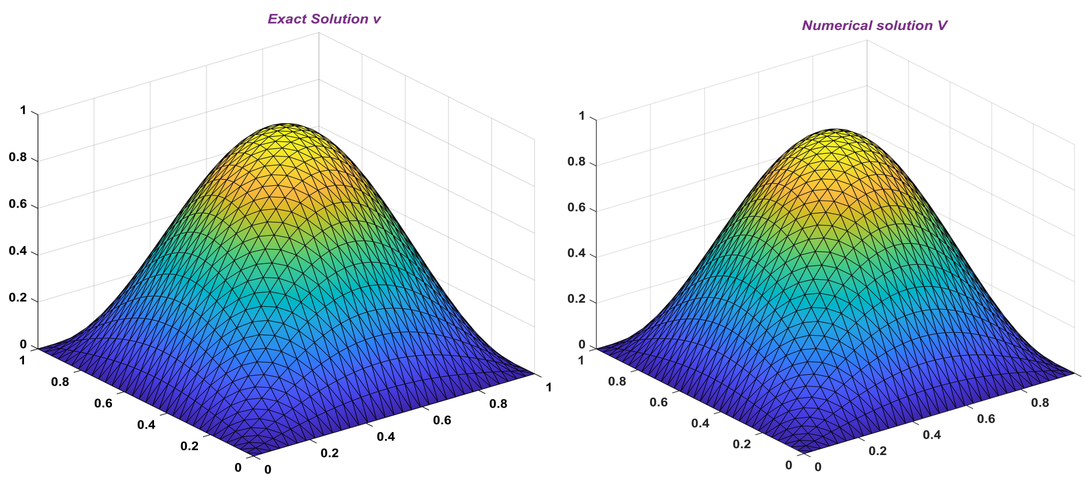

Figure 2: (Example-5.2) *The exact solution* v *and numerical solution* V *for* $\alpha = 0.5$.

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