

Edge Induced V_4 – Magic Labeling of Line graphs

Abstract

Let $V_4 = \{0, a, b, c\}$ be the Klein-4-group with identity element 0 and $G = (V(G), E(G))$ be a graph. Let $f : E(G) \rightarrow V_4 \setminus \{0\}$ be an edge labeling and $f^+ : V(G) \rightarrow V_4$ denotes the induced vertex labeling of f defined by $f^+(u) = \sum_{uv \in E(G)} f(uv)$ for all $u \in V(G)$. Then f^+ again induces an edge labeling $f^{++} : E(G) \rightarrow V_4$ defined by $f^{++}(uv) = f^+(u) + f^+(v)$, for all $uv \in E(G)$. A graph $G = (V(G), E(G))$ is said to be an edge induced V_4 -magic graph, if there exists an edge labeling f for which the function f^{++} is a constant function. The function f , so obtained is called an Edge Induced V_4 -Magic Labeling (EIML) of G . The present paper deals with basic results regarding EIML of line graphs and the characterization of EIML of line graph of certain named graphs.

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1 Introduction

Paper [10] discusses the concept of V_4 – magic labeling of graphs. Subsequently, paper [6] studies the concept of induced magic labeling of graphs in the context of a general group. Building on this, paper [7] considers induced V_4 – magic labeling as a special case of the results in [6], obtained by restricting the group to V_4 . Motivated

by the developments in [10], [6], and [7], paper [8] introduces a new concept called edge-induced V_4 -magic labeling of graphs. Further extending this idea, paper [9] investigates edge-induced V_4 -magic labeling for certain classes of subdivision graphs.

The present paper intends to deal exclusively with simple, connected, finite and undirected graphs. Also note that, the Klein 4-group [5] is denoted by $V_4 = \{0, a, b, c\}$ which is a non cyclic abelian group of order 4 with every non identity element has order 2. Let $G = (V(G), E(G))$ be the graph with vertex set $V(G)$ and edge set $E(G)$. The reader may check [1], [2] and [3] for the standard terminology and notations related to Graph theory.

Let $f : E(G) \rightarrow V_4 \setminus \{0\}$ be an edge labeling and $f^+ : V(G) \rightarrow V_4$ denotes the induced vertex labeling of f defined by $f^+(u) = \sum_{uv \in E(G)} f(uv)$ for all $u \in$

$V(G)$. Then f^+ again induces an edge labeling, say, $f^{++} : E(G) \rightarrow V_4$ defined by $f^{++}(uv) = f^+(u) + f^+(v)$, for all $uv \in E(G)$. A graph $G = (V(G), E(G))$ is said to be an edge induced V_4 -magic graph, if there exists an edge labeling f for which $f^{++}(e)$ is a constant for all $e \in E(G)$. If this constant is x , then x is said to be the induced edge sum of the graph G and the function f , so obtained is called an edge induced V_4 -magic labeling (EIML) of G . This paper aims to discuss edge induced V_4 -magic labeling of line graph of some graphs which belong to the following categories:

- (i) $\sigma_a(V_4) :=$ Set of all edge induced V_4 -magic graphs with edge induced magic labeling f satisfying $f^{++}(u) = a$ for all $u \in V$.
- (ii) $\sigma_0(V_4) :=$ Set of all edge induced V_4 -magic graphs with edge induced magic labeling f satisfying $f^{++}(u) = 0$ for all $u \in V$.
- (iii) $\sigma(V_4) := \sigma_a(V_4) \cap \sigma_0(V_4)$.

2 PRELIMINARIES

Definition 2.1. [3] The corona $P_n \circ K_1$ is called the comb graph CB_n .

Definition 2.2. [4] A flag graph is denoted by Fl_n and it is obtained by joining one vertex of C_n to an extra vertex called the root.

Definition 2.3. [4] The sun graph on $m = 2n$ vertices, denoted by Sun_n , is the graph obtained by attaching a pendant vertex to each vertex of a n -cycle.

Definition 2.4. The bistar $B_{m,n}$ is the graph obtained by joining the central or apex vertex of $K_{1,m}$ and $K_{1,n}$ by an edge.

Definition 2.5. [4] A triangular snake graph TS_n is obtained from a path $v_1, v_2, v_3, \dots, v_n$ by joining v_i and v_{i+1} to a new vertex w_i for $i = 1, 2, 3, \dots, n-1$.

Theorem 2.6. [8] Let $G = (V, E)$ be a graph with either each vertex is of odd degree or even degree then $G \in \sigma_0(V_4)$.

Theorem 2.7. [8] (**Induced edge sum theorem**).

For any graph G , f is an edge induced V_4 -Magic labeling of G if and only if the induced edge sum

$$x = f^{++}(uv) = \sum_{u\alpha \in E, \alpha \neq v} f(u\alpha) + \sum_{\beta v \in E, \beta \neq u} f(\beta v), \text{ for all } (u, v) \in E \quad (1)$$

The Equation (1) corresponding to an edge uv in G , is called induced edge sum equation of the edge uv .

Theorem 2.8. [8] For the path graph P_n , we have the following:

- (i) $P_2 \in \sigma_0(V_4)$ and $P_2 \notin \sigma_a(V_4)$.
- (ii) $P_3 \in \sigma_a(V_4)$ and $P_3 \notin \sigma_0(V_4)$.
- (iii) $P_4 \in \sigma_a(V_4)$ and $P_4 \notin \sigma_0(V_4)$.
- (iv) P_n is not an edge induced magic graph for any $n \geq 5$.

Theorem 2.9. [8] For the cycle graph C_n , we have the following:

- (i) $C_n \in \sigma_0(V_4)$ for all n .
- (ii) $C_n \in \sigma_a(V_4)$ if and only if n is a multiple of 4.

Theorem 2.10. [8] Let K_n be the complete graph with n vertices, then $K_n \in \sigma_0(V_4)$ for all n .

Definition 2.11. [1] Let G be a graph, then the line graph of G is denoted by $L(G)$ and it is a graph whose vertex set is in 1-1 correspondence with the edge set of G and two vertices of $L(G)$ are joined by an edge if and only if the corresponding edges of G are adjacent in G .

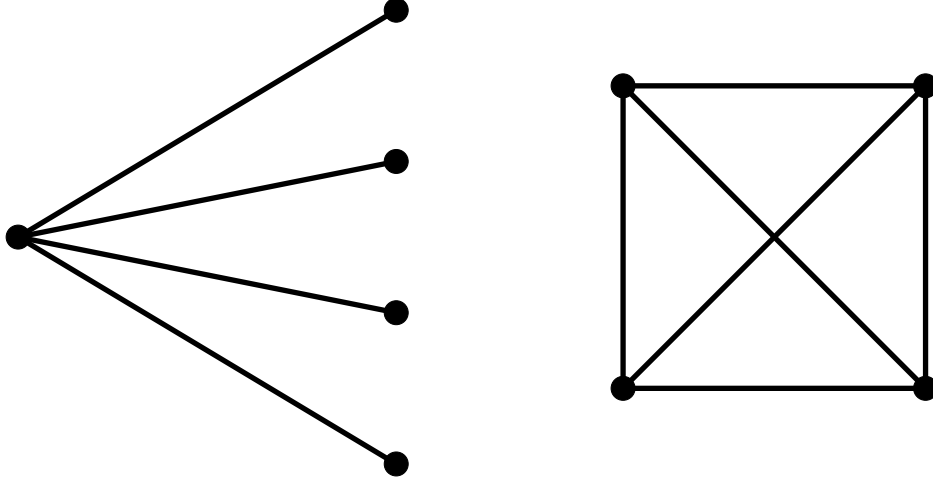


Fig. 1. $K_{1,4}$ Graph and its line graph

3 MAIN RESULTS

Theorem 3.1. *For the line graph of a path graph P_n , we have the following:*

- (i) $L(P_3) \in \sigma_0(V_4)$ and $L(P_3) \notin \sigma_a(V_4)$.
- (ii) $L(P_4) \in \sigma_a(V_4)$ and $L(P_4) \notin \sigma_0(V_4)$.
- (iii) $L(P_5) \in \sigma_a(V_4)$ and $L(P_5) \notin \sigma_0(V_4)$.
- (iv) $L(P_n)$ is not an edge induced magic graph for any $n \geq 6$.

Proof. Consider the path graphs,

- (i) Since $L(P_3) = P_2$, proof of (i) follows directly from Theorem 2.8 (i).
- (ii) Since $L(P_4) = P_3$ proof of (ii) follows directly from Theorem 2.8 (ii).
- (iii) Since $L(P_5) = P_4$, proof of (iii) follows directly from Theorem 2.8 (iii).
- (iv) Note that $L(P_n) = P_{n-1}$. Thus if $n \geq 6$, then $L(P_n) = P_{n-1}$ and $n - 1 \geq 5$, therefore the proof of (iv) follows from Theorem 2.8 (iv).

□

Corollary 3.2. $L(P_n) \notin \sigma(V_4)$ for any $n \geq 3$.

Proof. Proof of the corollary follows from Theorem 3.1. \square

Theorem 3.3. $L(C_n) \in \sigma_0(V_4)$ for all n .

Proof. Since $L(C_n) = C_n$, the proof follows from Theorem 2.9 (i). \square

Theorem 3.4. $L(C_n) \in \sigma_a(V_4)$ if and only if n is a multiple of 4.

Proof. Since $L(C_n) = C_n$, the proof follows from Theorem 2.9 (ii). \square

Corollary 3.5. $L(C_n) \in \sigma(V_4)$ if and only if n is a multiple of 4.

Proof. The proof follows from Theorem 3.3 and Theorem 3.4. \square

Theorem 3.6. Let $K_{1,n}$ be the star graph, then $L(K_{1,n}) \in \sigma_0(V_4)$ for all n .

Proof. Since $L(K_{1,n}) = K_n$, the proof follows from Theorem 2.10. \square

Theorem 3.7. Let CB_n be the comb graph, then we have the following.

(i) $L(CB_n) \notin \sigma_0(V_4)$ for any n .

(ii) $L(CB_n) \notin \sigma_a(V_4)$ for any n .

Proof. Let $\{u_i, v_i : i = 1, 2, 3, \dots, n\}$ be the vertex set of CB_n , where u_i is the pendant vertex adjacent to v_i . Also let $w_i = u_i v_i$, $i = 1, 2, 3, \dots, n$ and $t_k = u_k u_{k+1}$, $k = 1, 2, 3, \dots, n-1$ be the edges in CB_n . Then $\{w_i, t_k : i = 1, 2, 3, \dots, n, k = 1, 2, 3, \dots, n-1\}$ are the vertices of $L(CB_n)$.

(i) : If possible, suppose $L(CB_n) \in \sigma_0(V_4)$ for some n . Then there exists an EIML say $f : E(L(CB_n)) \rightarrow V_4 \setminus \{0\}$ with $f^{++}(e) = 0$ for all $e \in E(L(CB_n))$.

Let $f(t_i w_i) = \alpha_i$, $f(t_i w_{i+1}) = \beta_i$, for $i = 1, 2, 3, \dots, n-1$ and $f(t_j t_{j+1}) = \gamma_j$, for $j = 1, 2, 3, \dots, n-2$. Then the induced edge sum equation of the edges $t_i w_i$ gives the equation,

$$\begin{aligned} \gamma_1 + \beta_1 &= \gamma_1 + \beta_1 + \gamma_2 + \beta_2 \\ &= \gamma_2 + \beta_2 + \gamma_3 + \beta_3 \\ &\dots \\ &= \gamma_{n-3} + \beta_{n-3} + \gamma_{n-2} + \beta_{n-2} \\ &= \gamma_{n-2} + \beta_{n-2} + \beta_{n-1}. \end{aligned}$$

But since $f^{++} \equiv 0$, we get $\gamma_1 + \beta_1 = 0$, using this fact in the above system of equations, we get

$$\gamma_1 + \beta_1 = \gamma_2 + \beta_2 = \gamma_3 + \beta_3 = \cdots = \gamma_{n-2} + \beta_{n-2} = 0.$$

That is $\gamma_i = \beta_i$ for $i = 1, 2, 3, \dots, n-2$. Thus using $\gamma_{n-2} = \beta_{n-2}$ in the equation $\gamma_{n-2} + \beta_{n-2} + \beta_{n-1} = 0$, we get $\beta_{n-1} = 0$. That is $f(t_{n-1}w_n) = 0$, which is a contradiction. Hence our assumption is wrong, that is $L(CB_n) \notin \sigma_0(V_4)$ for all n .

(ii) : If possible, suppose $L(CB_n) \in \sigma_a(V_4)$ for some n . Then there exists an EIML say $g : E(L(CB_n)) \rightarrow V_4 \setminus \{0\}$ with $g^{++}(e) = a$ for all $e \in E(L(CB_n))$.

Let $g(t_i w_i) = \alpha_i$, $g(t_i w_{i+1}) = \beta_i$, for $i = 1, 2, 3, \dots, n-1$ and $g(t_j t_{j+1}) = \gamma_j$, for $j = 1, 2, 3, \dots, n-2$.

Then the induced edge sum of the edge $t_1 w_1$ and $t_2 w_2$ gives $\gamma_1 + \beta_1 = \gamma_1 + \beta_1 + \gamma_2 + \beta_2 = a$. Thus we get $\gamma_1 + \beta_1 = a$ and $\gamma_2 + \beta_2 = 0$.

Similarly the induced edge sum of the edges $t_1 w_2$ and $t_1 t_2$ gives $\alpha_1 + \alpha_2 + \gamma_1 = \alpha_1 + \alpha_2 + \beta_1 + \beta_2 + \gamma_2$. Since $\gamma_2 + \beta_2 = 0$ the above equation reduces to $\alpha_1 + \alpha_2 + \gamma_1 = \alpha_1 + \alpha_2 + \beta_1$ and which implies that $\gamma_1 = \beta_1$. That is $\gamma_1 + \beta_1 = 0$, which is contradiction. Hence our assumption is wrong, that is $L(CB_n) \notin \sigma_a(V_4)$ for any n .

Hence the Proof. □

Theorem 3.8. For the flag graph Fl_n , we have $L(Fl_n) \notin \sigma_a(V_4)$ for any n .

Proof. Let $V(Fl_n) = \{w, v_1, v_2, v_3, \dots, v_n\}$, where $v_1, v_2, v_3, \dots, v_n$ are the vertices of corresponding cycle graph C_n and w is the root vertex adjacent to the vertex v_1 . Also suppose $e_i = v_i v_{i+1}$ and $e = v_1 w$ are the edges in Fl_n . Therefore we can take the vertex set of $L(Fl_n)$ equal to $\{e, e_1, e_2, e_3, \dots, e_n\}$.

If possible, suppose $L(Fl_n) \in \sigma_a(V_4)$ for some n . Then there exists an EIML say $f : E(L(Fl_n)) \rightarrow V_4 \setminus \{0\}$ with $f^{++}(e) = a$ for all $e \in E(L(Fl_n))$.

Let $f(e_i e_{i+1}) = \alpha_i$, for $i = 1, 2, 3, \dots, n$ with $i+1$ is taken modulo n , $f(e_1 e) = \alpha$ and $f(e_n e) = \beta$. Then the induced edge sum equation of the edges $e_n e_1$, $e_1 e$, and $e_n e$ gives the equation:

$$\alpha_{n-1} + \alpha_1 + \alpha + \beta = \alpha_n + \alpha_1 + \beta = \alpha_{n-1} + \alpha_n + \alpha = a.$$

but $\alpha_n + \alpha_1 + \beta = \alpha_{n-1} + \alpha_n + \alpha$ implies $\alpha + \beta = \alpha_1 + \alpha_{n-1}$, thus $\alpha_{n-1} + \alpha_1 + \alpha + \beta = 0$, which is a contradiction. Hence our assumption is wrong, that is $L(Fl_n) \notin \sigma_a(V_4)$ for any n .

Hence the Proof. □

Theorem 3.9. *For the sun graph Sun_n , we have $L(Sun_n) \in \sigma_0(V_4)$ for all n .*

Proof. Note that in $L(Sun_n)$ every vertex is of even degree. Therefore by Theorem 2.6, we have $L(Sun_n) \in \sigma_0(V_4)$ for all n . \square

Theorem 3.10. *Consider bistar graph $B_{m,n}$, then $L(B_{m,n}) \in \sigma_0(V_4)$ for m and n are even.*

Proof. Note that for m and n are even, every vertex in $L(B_{m,n})$ is of even degree. Therefore by Theorem 2.6, we have $L(B_{m,n}) \in \sigma_0(V_4)$ for all n . \square

Theorem 3.11. *For the triangular snake graph TS_n , we have $L(TS_n) \in \sigma_0(V_4)$ for all n .*

Proof. Since every vertex in $L(TS_n)$ is of even degree, by Theorem 2.6 the proof follows. \square

4 Conclusion

This paper has attempted to investigate the key results pertaining to edge-induced V_4 - magic labeling of line graphs of for a certain class of graphs, such as P_n , C_n , $K_{1,n}$, CB_n , Fl_n , Sun_n , $B_{m,n}$, and TS_n .

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