

Structural Analysis of Quadrilateral Snake Graphs variants via Reverse Sombor-Based Indices and Their Polynomial Formulations

Abstract

Topological indices are widely used to describe the structural characteristics of graphs through numerical values. In this work, several variants of quadrilateral snake graphs are analyzed. For these graph classes, reverse degree-based indices including the reverse Sombor index, reverse Elliptic Sombor index, reverse Euler Sombor index, and reverse Harmonic index are computed using edge partition techniques. In addition, the reverse Sombor polynomial is formulated to provide a detailed description of the graph structure. **Explicit expressions are obtained for each class of graphs, and corresponding numerical values are presented to illustrate the results. The outcomes of this work contribute to the ongoing development of topological indices and provide further insight into the structural behavior of quadrilateral snake graph models.**

Keywords: Reverse Sombor index; Reverse Elliptic Sombor index; Reverse Euler Sombor index; Reverse Harmonic index; Reverse Sombor polynomial; Quadrilateral snake graph; Degree-based descriptors.

1 Introduction

Topological indices are numerical quantities that capture structural features of a graph by considering how its vertices are connected [11, 17, 26]. Over the years, researches worldwide have proposed numerous indices, each designed with particular goals and computational approaches in mind. The measures play a significant role in Quantity Structure-Property Relationship (QSPR) and Quantity Structure-Activity Relationship (QSAR) studies and have been widely applied in areas such as environmental analysis, pharmaceutical sciences, drug development, and materials engineering. In addition to these established applications, current research continues to identify new domains where such indices can offer valuable interpretations. Representing chemical compounds as graphs allows researchers to estimate molecular properties without relying on complex experimental procedures, thereby saving time, cost and effort [6, 12, 18, 29].

A foundational advancement in this field was achieved by Harold Wiener, who introduced the wiener index, a concept that became central to chemical graph theory [20, 27]. This distance-based index effectively relates molecular structures to physicochemical properties, including boiling and melting points of alkanes. Although many new indices have since been introduced, the wiener index continues to serve as an important benchmark for evaluating newly developed descriptors. Building on this early contribution, a broad range of topological indices has been formulated and applied across disciplines such as materials science, medicinal chemistry, and pharmaceutical research [6, 7, 12, 13, 20, 27].

In more recent developments, Ivan Gutman proposed the Sombor index in 2020 [9], which quickly attracted attention and has been extensively investigated in subsequent studies [4, 5, 22, 24, 25, 30]. This work led to the introduction of several related variants, including the Sombor Polynomial, the Elliptic Sombor index [10], and the Euler Sombor index [15]. Inspired by these advancements, researchers have further explored reverse degree-based adaptations of Sombor-type indices, such as the reverse Euler sombor index, and examined their structural properties across different classes of graphs[16]. In addition, the selection of reverse sombor-based indices is justified due to their enhanced sensitivity in capturing structural variations compared to traditional degree-based indices. These indices provide improved structural interpretation, particularly for complex graph structures such as quadrilateral snake graphs. Snake graphs have attracted attention for their structural importance in graph theory. P. Mahalank et al. (2021) [19] studied Zagreb indices, while Bhairaba Kumar Majhi et al. (2024)[2] introduces Revan indices and their polynomials. Chitra Ramaprakash (2024)[3] examined Total Coloring of Snake Graphs and other Standard Graphs and Gerand Rozario Joseph et al. (2025)[7] focused on Fibonacci prime labelling of snake graphs. More recently, Saranya K. M. and Manimekalai S. (2025)[23] studied Revan Topological Indices of Quadrilateral Snake Graphs: Polynomial Formulations, Python Implementation, and Applications. **Motivated by existing studies on Revan topological indices of quadrilateral snake graphs, the present work extends the analysis by focusing on reverse Sombor-based indices. While previous studies primarily addressed Revan indices and their polynomial formulations, this works investigates the same classes of snake graphs including quadrilateral, alternate quadrilateral, double quadrilateral, alternate double quadrilateral and cycle quadrilateral snake graphs using reverse Sombor, reverse Euler Sombor, reverse Elliptic Sombor, and reverse Harmonic indices. The novelty of this study lies in replacing Revan-based descriptors with reverse Sombor-based indices, which provide enhanced sensitivity and improved structural interpretation. In addition, the reverse Sombor polynomial is formulated to provide a more detailed structural representation. The results show consistent growth behavior with respect to graph parameters, highlighting stability and effectiveness of the proposed indices. These findings contribute to the advancement of reverse degree-based topological indices and provide a new perspective for analyzing complex graph structures.**

2 Basic Notions and Reverse Sombor Indices

Let $G = (V(G), E(G))$ be a finite, simple, and connected graph with vertex set $V(G)$ and edge set $E(G)$. The degree of a vertex $x \in V(G)$ is denoted by $d_G(x)$, and the maximum degree of G is written as $\Delta(G)$.

For each vertex $x \in V(G)$, the reverse degree is defined by

$$c_G(x) = \Delta(G) - d_G(x) + 1.$$

The reverse Sombor index of G is expressed as

$$RSO(G) = \sum_{xy \in E(G)} \sqrt{c_G(x)^2 + c_G(y)^2}.$$

The reverse Euler Sombor index [15] is given by

$$REUSO(G) = \sum_{xy \in E(G)} \sqrt{c_G(x)^2 + c_G(y)^2 + c_G(x)c_G(y)}.$$

The reverse Elliptic Sombor index [10] is defined as

$$RESO(G) = \sum_{xy \in E(G)} (c_G(x) + c_G(y))(\sqrt{c_G(x)^2 + c_G(y)^2}).$$

The reverse Harmonic index is written as

$$RH(G) = \sum_{xy \in E(G)} \frac{2}{c_G(x) + c_G(y)}.$$

The reverse Sombor polynomial [1] of G is expressed as

$$RSO(G) = \sum_{xy \in E(G)} \frac{1}{c_G(x)^2 + c_G(y)^2} X^{[c_G(x)^2 + c_G(y)^2]}.$$

Here, x and y denote the vertices of graph G , and X is a variable used to define the polynomial.

3 Results and Discussion

3.1 Quadrilateral Snake Graph

A Quadrilateral snake graph (QS_n) is derived from a path $v_1, v_2, v_3, v_4, \dots, v_n$ by connecting new vertices u_i and w_i to v_i and v_{i+1} respectively, and then connecting u_i and w_i see [23]. let n denote the number of quadrilateral units in the graph. Hence, the Quadrilateral Snake Graph (QS_n) has $|V| = 3n + 1$ vertices and $|E| = 4n$ edges.

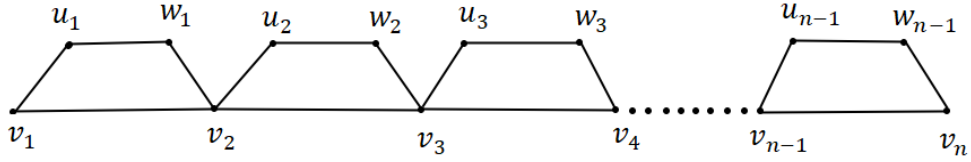


Figure 1: Quadrilateral Snake Graph. Adapted from[23]

Theorem 3.1. *Let G be the Quadrilateral snake graph $Q(S_n)$. Then*

$$\begin{aligned} RSO(G) &= 3\sqrt{2}(n+2) + 2\sqrt{10}n + \sqrt{2}(n-2) \\ REUSO(G) &= 3\sqrt{3}(n+2) + 2\sqrt{13}n + \sqrt{3}(n-2) \\ RESO(G) &= 18\sqrt{2}(n+2) + 8\sqrt{10}n + 2\sqrt{2}(n-2) \\ RH(G) &= \frac{n+2}{3} + 2(n-1) \\ RSO(G, X) &= \frac{1}{8(5n+4)} X^{8(5n+4)} \end{aligned}$$

Proof. Let G be the quadrilateral snake graph $Q(S_n)$ with $|V(G)| = 3n + 1$ and $|E(G)| = 4n$. The edge set can be partitioned as,

$$\begin{aligned} E_1 &= \{xy \in E(G) | d_G(x) = d_G(y) = 2\}, |E_1| = n + 2 \\ E_2 &= \{xy \in E(G) | d_G(x) = 2, d_G(y) = 4\}, |E_2| = 2n \\ E_3 &= \{xy \in E(G) | d_G(x) = d_G(y) = 4\}, |E_3| = n - 2 \end{aligned}$$

Here $\Delta(G) = 4$ and then, $c_G(x) = 5 - d_G(x)$

$$\begin{aligned} CE_1 &= \{xy \in E(G) | c_G(x) = c_G(y) = 3\}, |CE_1| = n + 2 \\ CE_2 &= \{xy \in E(G) | c_G(x) = 3, c_G(y) = 1\}, |CE_2| = 2n \\ CE_3 &= \{xy \in E(G) | c_G(x) = c_G(y) = 1\}, |CE_3| = n - 2 \end{aligned}$$

Using the reverse edge partition of graph $Q(S_n)$ and indices formula, we get

$$\begin{aligned}
 RSO(G) &= (n+2)\sqrt{3^2+3^2} + 2n\sqrt{3^2+1^2} + (n-2)\sqrt{1+1} \\
 &= 3\sqrt{2}(n+2) + 2\sqrt{10}n + \sqrt{2}(n-2) \\
 REUSO(G) &= (n+2)\sqrt{3^2+3^2+3.3} + 2n\sqrt{3^2+1^2+3} + (n-2)\sqrt{1+1+1} \\
 &= 3\sqrt{3}(n+2) + 2\sqrt{13}n + \sqrt{3}(n-2) \\
 RESO(G) &= (n+2)(3+3)\sqrt{3^2+3^2} + 2n(3+1)\sqrt{3^2+1^2} + (n-2)(1+1)\sqrt{1+1} \\
 &= 18\sqrt{2}(n+2) + 8\sqrt{10}n + 2\sqrt{2}(n-2) \\
 \\
 RH(G) &= \frac{2(n+2)}{3+3} + \frac{2(2n)}{3+1} + \frac{2(n-2)}{1+1} \\
 &= \frac{n+2}{3} + 2(n-1) \\
 RSO(G, X) &= \frac{1}{(n+2)(18) + 2n(10) + (n-2)(2)} X^{[(n+2)(18)+2n(10)+(n-2)(2)]} \\
 &= \frac{1}{8(5n+4)} X^{8(5n+4)}
 \end{aligned}$$

□

Example: For $n = 2$ in the Quadrilateral Snake Graph (QS_n), we have $|V| = 7$ vertices and $|E| = 8$ edges. Then, the indices are given by $RSO(G) = 12\sqrt{2} + 4\sqrt{10}$, $REUSO(G) = 12\sqrt{3} + 4\sqrt{13}$, $RESO(G) = 72\sqrt{2} + 16\sqrt{10}$, $RH(G) = \frac{10}{3}$, and $RSO(G, X) = \frac{1}{112}X^{112}$.

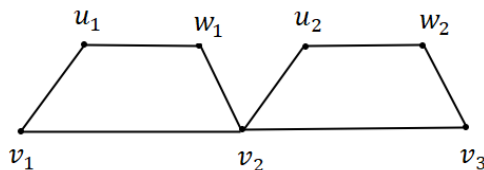


Figure 2: Quadrilateral snake graph (QS_2). Adapted from[23]

Table 1: Comparison of RSO , $REUSO$, $RESO$, and RH of Quadrilateral snake graph (QS_n).

n	RSO(G)	REUSO(G)	RESO(G)	RH(G)
1	17.6383	21.0675	98.8373	1
2	29.6197	35.2068	152.4198	3.3333
3	41.6011	49.3461	206.0023	5.6667
4	53.5825	63.4854	259.5848	8
5	65.5639	77.6247	313.1673	10.3333
6	77.5453	91.7640	366.7498	12.6667
7	89.5267	105.9033	420.3323	15
8	101.5081	120.0426	473.9148	17.3333
9	113.4895	134.1820	527.4973	19.6667
10	125.4709	148.3213	581.0798	22

3.2 Alternating Quadrilateral Snake Graph

An Alternate Quadrilateral snake graph is derived from a path $v_1, v_2, v_3, v_4, \dots, v_n$ by connecting new vertices u_i and w_i to v_i and v_{i+1} (alternatively) respectively, and then connecting u_i

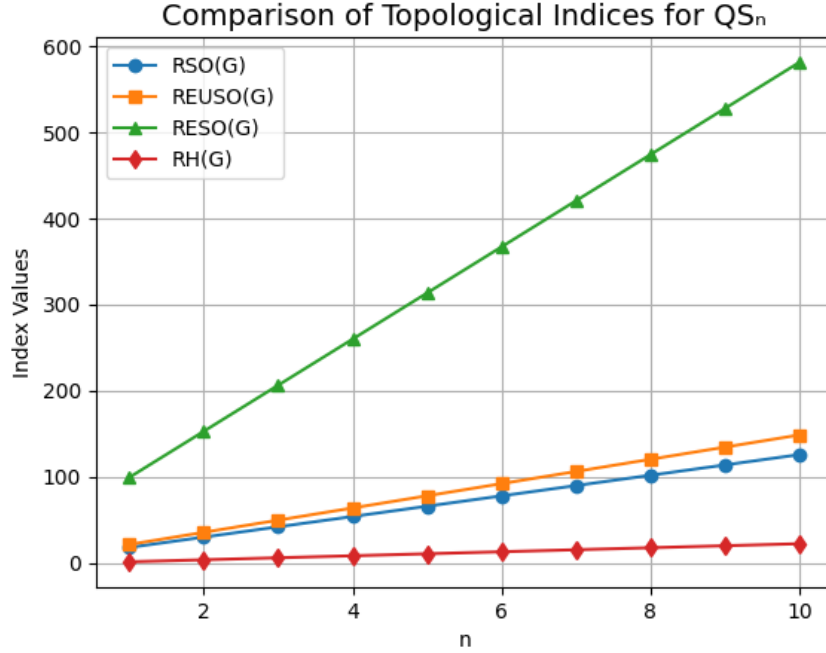


Figure 3: Comparison of RSO , $REUSO$, $RESO$, and RH of QS_n .

and w_i . It is denoted by AQS_n see [23]. let n denote the number of alternating quadrilateral units in the graph. Hence, the Alternating Quadrilateral Snake Graph (AQS_n) has $|V| = 4(n + 1)$ vertices and $|E| = 5n + 4$ edges.

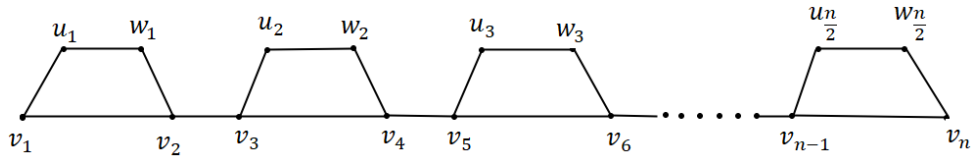


Figure 4: Alternating Quadrilateral Snake Graph. Adapted from[23]

Theorem 3.2. Let G be the alternating quadrilateral snake graph (AQS_n). Then

$$\begin{aligned}
 RSO(G) &= 2\sqrt{2}(n + 3) + 2\sqrt{5}(n + 1) + \sqrt{2}(2n - 1) \\
 REUSO(G) &= 2\sqrt{3}(n + 3) + 2\sqrt{7}(n + 1) + \sqrt{3}(2n - 1) \\
 RESO(G) &= 8\sqrt{2}(n + 3) + 6\sqrt{5}(n + 1) + 2\sqrt{2}(2n - 1) \\
 RH(G) &= \frac{n + 3}{2} + \frac{4(n + 1)}{3} + 2n - 1 \\
 RSO(G, X) &= \frac{1}{2(11n + 16)} X^{2(11n+16)}
 \end{aligned}$$

Proof. Let G be the quadrilateral snake graph $AQ(S_n)$ with $|V| = 4(n + 1)$ vertices and $|E| = 5n + 4$ edges. The edge set can be partitioned as,

$$\begin{aligned}
 E_1 &= \{xy \in E(G) | d_G(x) = d_G(y) = 2\}, |E_1| = n + 3 \\
 E_2 &= \{xy \in E(G) | d_G(x) = 2, d_G(y) = 3\}, |E_2| = 2(n + 1) \\
 E_3 &= \{xy \in E(G) | d_G(x) = d_G(y) = 3\}, |E_3| = 2n - 1
 \end{aligned}$$

Here $\Delta(G) = 3$ and then, $c_G(x) = 4 - d_G(x)$

$$\begin{aligned} CE_1 &= \{xy \in E(G) | c_G(x) = c_G(y) = 2\}, |CE_1| = n + 3 \\ CE_2 &= \{xy \in E(G) | c_G(x) = 2, c_G(y) = 1\}, |CE_2| = 2(n + 1) \\ CE_3 &= \{xy \in E(G) | c_G(x) = c_G(y) = 1\}, |CE_3| = 2n - 1 \end{aligned}$$

Using the reverse edge partition of graph $AQ(S_n)$ and indices formula, we get

$$\begin{aligned} RSO(G) &= (n + 3)\sqrt{2^2 + 2^2} + 2(n + 1)\sqrt{2^2 + 1} + (2n - 1)\sqrt{1 + 1} \\ &= 2\sqrt{2}(n + 3) + 2\sqrt{5}(n + 1) + \sqrt{2}(2n - 1) \\ REUSO(G) &= (n + 3)\sqrt{2^2 + 2^2 + 2.2} + 2(n + 1)\sqrt{2^2 + 1 + 2.1} + (2n - 1)\sqrt{1 + 1 + 1} \\ &= 2\sqrt{3}(n + 3) + 2\sqrt{7}(n + 1) + \sqrt{3}(2n - 1) \\ RESO(G) &= (n + 3)(2 + 2)\sqrt{2^2 + 2^2} + 2(n + 1)(2 + 1)\sqrt{2^2 + 1} + (2n - 1)(1 + 1)\sqrt{1 + 1} \\ &= 8\sqrt{2}(n + 3) + 6\sqrt{5}(n + 1) + 2\sqrt{2}(2n - 1) \\ RH(G) &= \frac{2(n + 3)}{4} + \frac{4(n + 1)}{3} + \frac{2(2n - 1)}{2} \\ &= \frac{n + 3}{2} + \frac{4(n + 1)}{3} + 2n - 1 \\ RSO(G, X) &= \frac{1}{(n + 3)(8) + 2(n + 1)(5) + (2n - 1)(2)} X^{[(n+3)(8)+2(n+1)(5)+(2n-1)(2)]} \\ &= \frac{1}{2(11n + 16)} X^{2(11n+16)} \end{aligned}$$

□

Example: For $n = 1$ in the Alternating Quadrilateral Snake Graph (AQS_n), we have $|V| = 8$ vertices and $|E| = 9$ edges. Then, the indices are given by $RSO(G) = 9\sqrt{2} + 4\sqrt{5}$, $REUSO(G) = 9\sqrt{3} + 4\sqrt{7}$, $RESO(G) = 34\sqrt{2} + 12\sqrt{5}$, $RH(G) = \frac{17}{3}$, and $RSO(G, X) = \frac{1}{54}X^{54}$.

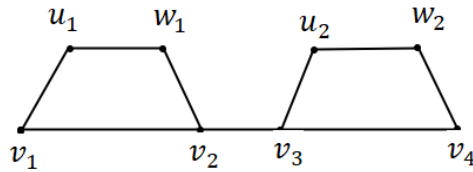


Figure 5: Alternate Quadrilateral snake graph (AQS_1). Adapted from [23]

3.3 Double Quadrilateral Snake Graph

A Double Quadrilateral snake graph consists of two quadrilateral snakes that have a common path and is denoted by DQS_n see [23]. let n denote the number of double quadrilateral units in the graph. Hence, the Double Quadrilateral Snake Graph (DQS_n) has $|V| = 5n + 1$ vertices and $|E| = 7n$ edges.

Table 2: Comparison of RSO , $REUSO$, $RESO$, and RH of Alternate Quadrilateral snake graph (AQS_n).

n	$RSO(G)$	$REUSO(G)$	$RESO(G)$	$RH(G)$
1	21.6722	26.1715	74.9161	5.6667
2	31.8012	38.3912	105.303	9.5
3	41.9302	50.6109	135.69	13.3333
4	52.0592	62.8306	166.077	17.1667
5	62.1882	75.0503	196.464	21
6	72.3171	87.27	226.8509	24.8333
7	82.4461	99.4897	257.2379	28.6667
8	92.5751	111.7094	287.6249	32.5
9	102.7041	123.9291	318.0118	36.3333
10	112.8331	136.1488	348.3988	40.1667

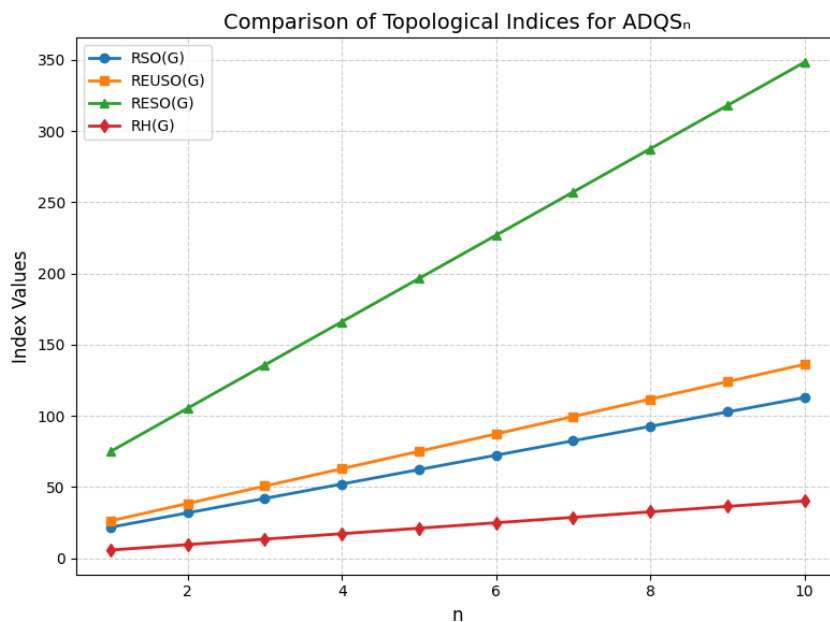


Figure 6: Comparison of RSO , $REUSO$, $RESO$, and RH of AQS_n .

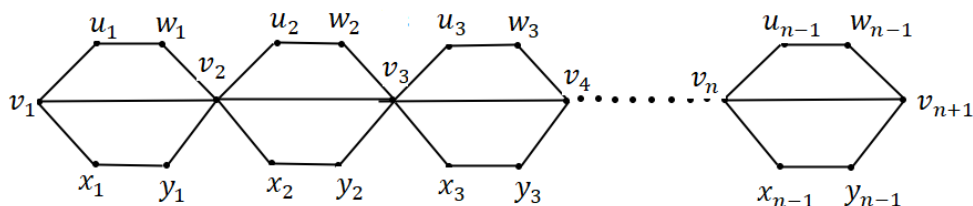


Figure 7: Double Quadrilateral Snake Graph. Adapted from [23]

Theorem 3.3. *Let G be the double quadrilateral snake graph. Then*

$$\begin{aligned}
RSO(G) &= 10\sqrt{2}n + 4\sqrt{41} + 4\sqrt{26}(n-1) + 2\sqrt{17} + (n-2)\sqrt{2} \\
REUSO(G) &= 10\sqrt{3}n + 4\sqrt{61} + 4\sqrt{31}(n-1) + 2\sqrt{21} + (n-2)\sqrt{3} \\
RESO(G) &= 100\sqrt{2}n + 36\sqrt{41} + 24\sqrt{26}(n-1) + 10\sqrt{17} + 2\sqrt{2}(n-2) \\
RH(G) &= \frac{2n}{5} + \frac{8}{9} + \frac{4(n-1)}{3} + \frac{4}{5} + n - 2 \\
RSO(G, X) &= \frac{1}{2(103n+45)} X^{2(103n+45)}
\end{aligned}$$

Proof. Let G be the double quadrilateral snake graph (DQS_n) with $|V| = 5n + 1$ vertices and $|E| = 7n$ edges. The edge set can be partitioned as,

$$\begin{aligned}
E_1 &= \{xy \in E(G) | d_G(x) = d_G(y) = 2\}, |E_1| = 2n \\
E_2 &= \{xy \in E(G) | d_G(x) = 2, d_G(y) = 3\}, |E_2| = 4 \\
E_3 &= \{xy \in E(G) | d_G(x) = 2, d_G(y) = 6\}, |E_3| = 4(n-1) \\
E_4 &= \{xy \in E(G) | d_G(x) = 3, d_G(y) = 6\}, |E_4| = 2 \\
E_5 &= \{xy \in E(G) | d_G(x) = d_G(y) = 6\}, |E_5| = n-2
\end{aligned}$$

Here $\Delta(G) = 6$ and then, $c_G(x) = 7 - d_G(x)$

$$\begin{aligned}
CE_1 &= \{xy \in E(G) | c_G(x) = c_G(y) = 5\}, |CE_1| = 2n \\
CE_2 &= \{xy \in E(G) | c_G(x) = 5, c_G(y) = 4\}, |CE_2| = 4 \\
CE_3 &= \{xy \in E(G) | c_G(x) = 5, c_G(y) = 1\}, |CE_3| = 4(n-1) \\
CE_4 &= \{xy \in E(G) | c_G(x) = 4, c_G(y) = 1\}, |CE_4| = 2 \\
CE_5 &= \{xy \in E(G) | c_G(x) = c_G(y) = 1\}, |CE_5| = n-2
\end{aligned}$$

Using the reverse edge partition of graph $DQ(S_n)$ and indices formula, we get

$$\begin{aligned}
RSO(G) &= 2n\sqrt{5^2 + 5^2} + 4\sqrt{5^2 + 4^2} + 4(n-1)\sqrt{5^2 + 1} + 2\sqrt{4^2 + 1} + (n-2)\sqrt{1+1} \\
&= 10\sqrt{2}n + 4\sqrt{41} + 4\sqrt{26}(n-1) + 2\sqrt{17} + (n-2)\sqrt{2} \\
REUSO(G) &= 2n\sqrt{5^2 + 5^2 + 5.5} + 4\sqrt{5^2 + 4^2 + 5.4} + 4(n-1)\sqrt{5^2 + 1 + 5} + \\
&\quad 2\sqrt{4^2 + 1 + 4} + (n-2)\sqrt{1+1+1} \\
&= 10\sqrt{3}n + 4\sqrt{61} + 4\sqrt{31}(n-1) + 2\sqrt{21} + (n-2)\sqrt{3} \\
RESO(G) &= 2n(5+5)\sqrt{5^2 + 5^2} + 4(5+4)\sqrt{5^2 + 4^2} + 4(n-1)(5+1)\sqrt{5^2 + 1} + \\
&\quad 2(4+1)\sqrt{4^2 + 1} + (n-2)(1+1)\sqrt{1+1} \\
&= 100\sqrt{2}n + 36\sqrt{41} + 24\sqrt{26}(n-1) + 10\sqrt{17} + 2\sqrt{2}(n-2) \\
RH(G) &= \frac{4n}{5+5} + \frac{2(4)}{5+4} + \frac{8(n-1)}{5+1} + \frac{4}{4+1} + \frac{2(n-2)}{2} \\
&= \frac{2n}{5} + \frac{8}{9} + \frac{4(n-1)}{3} + \frac{4}{5} + n - 2 \\
RSO(G, X) &= \frac{1}{2n(50) + 4(41) + 4(n-1)(26) + 2(17) + (n-2)(2)} X^{[2n(50)+4(41)+4(n-1)(26)+2(17)+(n-2)(2)]} \\
&= \frac{1}{2(103n+45)} X^{2(103n+45)}
\end{aligned}$$

□

Example: For $n = 3$ in the Double Quadrilateral Snake Graph (DQS_n), we have $|V| = 16$ vertices and $|E| = 21$ edges. Then, the indices are given by $RSO(G) = 31\sqrt{2} + 4\sqrt{41} + 8\sqrt{26} + 2\sqrt{17}$, $REUSO(G) = 30\sqrt{3} + 4\sqrt{61} + 8\sqrt{31} + 2\sqrt{21}$, $RESO(G) = 302\sqrt{2} + 36\sqrt{41} + 48\sqrt{26} + 10\sqrt{17}$, $RH(G) = \frac{59}{9}$, and $RSO(G, X) = \frac{1}{708} X^{708}$.

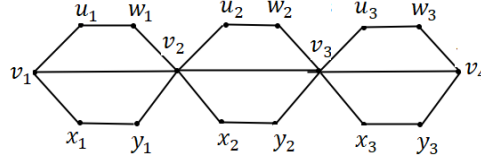


Figure 8: Double Quadrilateral snake graph (DQS_3). Adapted from[23]

Table 3: Comparison of RSO , $REUSO$, $RESO$, and RH of Double Quadrilateral snake graph (DQS_n).

n	RSO(G)	REUSO(G)	RESO(G)	RH(G)
1	46.5866	55.9946	410.3365	1.0889
2	82.5391	97.3182	676.9627	3.8222
3	118.4915	138.6418	943.589	6.5556
4	154.4439	179.9655	1210.2152	9.2889
5	190.3963	221.2891	1476.8414	12.0222
6	226.3488	262.6127	1743.4677	14.7556
7	262.3012	303.9363	2010.094	17.4889
8	298.2536	345.2599	2276.7202	20.2222
9	334.206	386.5835	2543.3465	22.9556
10	370.1585	427.9072	2809.9727	25.6889

3.4 Alternating Double Quadrilateral Snake Graph

An Alternate Double Quadrilateral snake graph consists of two alternate quadrilateral snakes that have a common path and is denoted by $ADQS_n$ see [23]. let n denote the number of alternating double quadrilateral units in the graph. Hence, the Alternating Double Quadrilateral Snake Graph ($ADQS_n$) has $|V| = 6(n + 1)$ vertices and $|E| = 8n + 7$ edges.

Theorem 3.4. Let G be the alternating double quadrilateral snake graph ($ADQS_n$). Then

$$\begin{aligned}
 RSO(G) &= 6\sqrt{2}(n + 1) + 4\sqrt{13} + 4\sqrt{10}n + 2\sqrt{5} + (2n - 1)\sqrt{2} \\
 REUSO(G) &= 6\sqrt{3}(n + 1) + 4\sqrt{19} + 4\sqrt{13}n + 2\sqrt{7} + (2n - 1)\sqrt{3} \\
 RESO(G) &= 36\sqrt{2}(n + 1) + 20\sqrt{13} + 16\sqrt{10}n + 6\sqrt{5} + 2\sqrt{2}(2n - 1) \\
 RH(G) &= \frac{2(n + 1)}{3} + 4n + \frac{29}{15} \\
 RSO(G, X) &= \frac{1}{16(5n + 6)} X^{16(5n+6)}
 \end{aligned}$$

Proof. Let G be the alternating double quadrilateral snake graph $ADQ(S_n)$ with $|V| = 6(n + 1)$

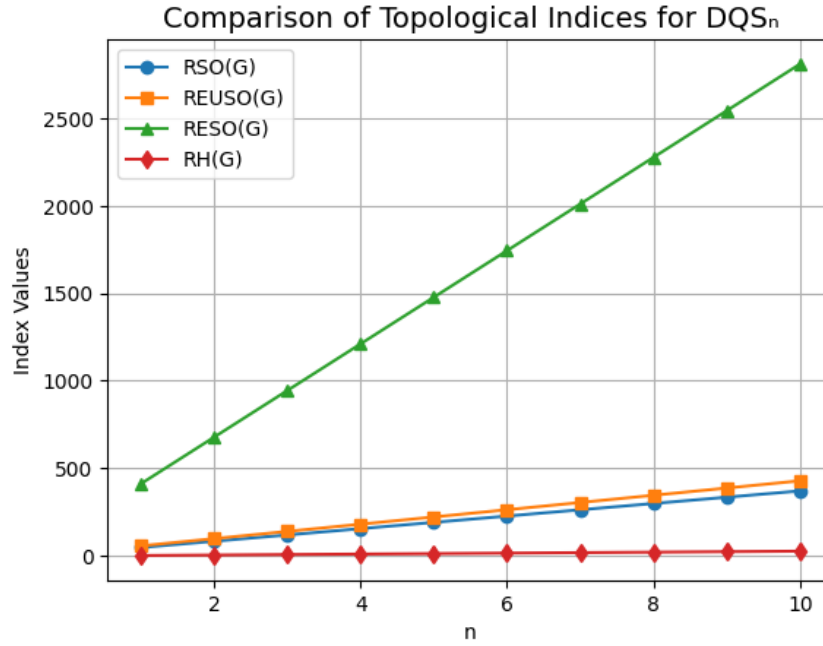


Figure 9: Comparison of RSO , $REUSO$, $RESO$, and RH of DQS_n .

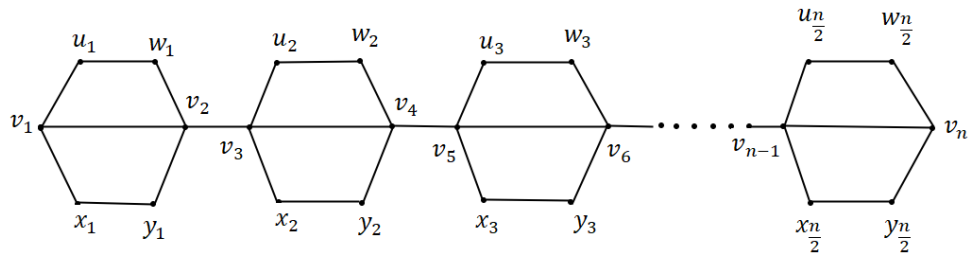


Figure 10: Alternating Double Quadrilateral Snake Graph. Adapted from [23]

vertices and $|E| = 8n + 7$ edges. The edge set can be partitioned as,

$$\begin{aligned}
E_1 &= \{xy \in E(G) | d_G(x) = d_G(y) = 2\}, |E_1| = 2(n+1) \\
E_2 &= \{xy \in E(G) | d_G(x) = 2, d_G(y) = 3\}, |E_2| = 4 \\
E_3 &= \{xy \in E(G) | d_G(x) = 2, d_G(y) = 4\}, |E_3| = 4n \\
E_4 &= \{xy \in E(G) | d_G(x) = 3, d_G(y) = 4\}, |E_4| = 2 \\
E_5 &= \{xy \in E(G) | d_G(x) = d_G(y) = 4\}, |E_5| = 2n - 1
\end{aligned}$$

Here $\Delta(G) = 4$ and then, $c_G(x) = 5 - d_G(x)$

$$\begin{aligned}
CE_1 &= \{xy \in E(G) | c_G(x) = c_G(y) = 3\}, |CE_1| = 2(n+1) \\
CE_2 &= \{xy \in E(G) | c_G(x) = 3, c_G(y) = 2\}, |CE_2| = 4 \\
CE_3 &= \{xy \in E(G) | c_G(x) = 3, c_G(y) = 1\}, |CE_3| = 4n \\
CE_4 &= \{xy \in E(G) | c_G(x) = 2, c_G(y) = 1\}, |CE_4| = 2 \\
CE_5 &= \{xy \in E(G) | c_G(x) = c_G(y) = 1\}, |CE_5| = 2n - 1
\end{aligned}$$

Using the reverse edge partition of graph $ADQ(S_n)$ and indices formula, we get

$$\begin{aligned}
RSO(G) &= 2(n+1)\sqrt{3^2+3^2} + 4\sqrt{3^2+2^2} + 4n\sqrt{3^2+1} + 2\sqrt{2^2+1} + (2n-1)\sqrt{1+1} \\
&= 6\sqrt{2}(n+1) + 4\sqrt{13} + 4\sqrt{10}n + 2\sqrt{5} + (2n-1)\sqrt{2} \\
REUSO(G) &= 2(n+1)\sqrt{3^2+3^2+3.3} + 4\sqrt{3^2+2^2+3.2} + 4n\sqrt{3^2+1+3.3} + \\
&\quad 2\sqrt{2^2+1+2} + (2n-1)\sqrt{1+1+1} \\
&= 6\sqrt{3}(n+1) + 4\sqrt{19} + 4\sqrt{13}n + 2\sqrt{7} + (2n-1)\sqrt{3} \\
RESO(G) &= 2(n+1)(3+3)\sqrt{3^2+3^2} + 4(3+2)\sqrt{3^2+2^2} + 4n(3+1)\sqrt{3^2+1} + \\
&\quad 2(2+1)\sqrt{2^2+1} + (2n-1)(1+1)\sqrt{1+1} \\
&= 36\sqrt{2}(n+1) + 20\sqrt{13} + 16\sqrt{10}n + 6\sqrt{5} + 2\sqrt{2}(2n-1) \\
RH(G) &= \frac{4(n+1)}{3+3} + \frac{2(4)}{3+2} + \frac{8n}{3+1} + \frac{4}{2+1} + \frac{2(2n-1)}{2} \\
&= \frac{2(n+1)}{3} + 4n + \frac{29}{15} \\
RSO(G, X) &= \frac{1}{2(n+1)(18) + 4(13) + 4n(10) + 2(5) + (2n-1)(2)} X^{[2(n+1)(18)+4(13)+4n(10)+2(5)+(2n-1)(2)]} \\
&= \frac{1}{16(5n+6)} X^{16(5n+6)}
\end{aligned}$$

□

Example: For $n = 1$ in the Alternating Double Quadrilateral Snake Graph ($ADQS_n$), we have $|V| = 12$ vertices and $|E| = 15$ edges. Then, the indices are given by $RSO(G) = 13\sqrt{2} + 4\sqrt{13} + 4\sqrt{10} + 2\sqrt{5}$, $REUSO(G) = 13\sqrt{3} + 4\sqrt{19} + 4\sqrt{13} + 2\sqrt{7}$, $RESO(G) = 74\sqrt{2} + 20\sqrt{3} + 16\sqrt{10} + 6\sqrt{5}$, $RH(G) = \frac{109}{15}$, and $RSO(G, X) = \frac{1}{176} X^{176}$.

3.5 Cycle Quadrilateral Snake Graph

A Cycle Quadrilateral snake graph is derived from a cycle $v_1, v_2, v_3, v_4, \dots, v_n$ by connecting new vertices u_i and w_i to v_i and v_{i+1} respectively and then connecting u_i and w_i . It is denoted by CQS_n see [23]. let n denote the number of cycle quadrilateral units in the graph. Hence, the Cycle Quadrilateral Snake Graph (CQS_n) has $|V| = 3n$ vertices and $|E| = 4n$ edges.

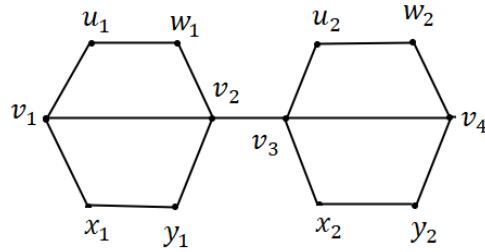


Figure 11: Alternate Double Quadrilateral snake graph ($ADQS_1$). Adapted from[23]

Table 4: Comparison of RSO , $REUSO$, $RESO$, and RH of Alternate Double Quadrilateral snake graph ($ADQS_n$).

n	$RSO(G)$	$REUSO(G)$	$RESO(G)$	$RH(G)$
1	49.9282	59.6660	240.7757	7.2667
2	73.8910	87.9446	347.9405	11.9333
3	97.8539	116.2232	455.1056	16.6
4	121.8167	144.5018	562.2706	21.2667
5	145.7795	172.7804	669.4356	25.9333
6	169.7423	201.059	776.6006	30.6
7	193.7051	229.3376	883.7656	35.2667
8	217.668	257.6162	990.9306	39.9333
9	241.6308	285.8949	1098.0956	44.6
10	265.5936	314.1735	1205.2605	49.2667

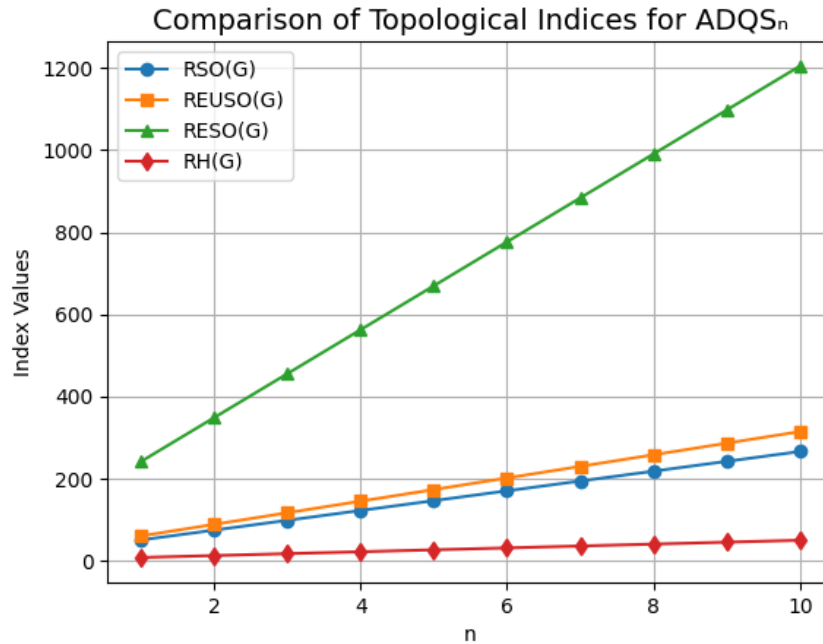


Figure 12: Comparison of RSO , $REUSO$, $RESO$, and RH of $ADQS_n$.

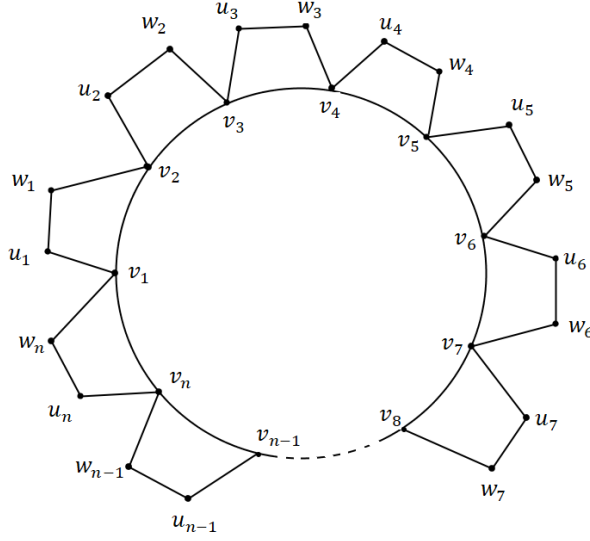


Figure 13: Cycle Quadrilateral Snake Graph. Adapted from[23]

Theorem 3.5. *Let G be the cycle quadrilateral snake graph (CQS_n). Then*

$$\begin{aligned}
 RSO(G) &= 3\sqrt{2n} + 2\sqrt{10n} + \sqrt{2n} \\
 REUSO(G) &= 3\sqrt{3n} + 2\sqrt{13n} + \sqrt{3n} \\
 RESO(G) &= 18\sqrt{2n} + 8\sqrt{10n} + 2\sqrt{2n} \\
 RH(G) &= \frac{7n}{3} \\
 RSO(G, X) &= \frac{1}{40n} X^{40n}
 \end{aligned}$$

Proof. Let G be the cycle quadrilateral snake graph (CQS_n) with $|V| = 3n$ vertices and $|E| = 4n$ edges. The edge set can be partitioned as,

$$\begin{aligned}
 E_1 &= \{xy \in E(G) | d_G(x) = d_G(y) = 2\}, |E_1| = n \\
 E_2 &= \{xy \in E(G) | d_G(x) = 2, d_G(y) = 4\}, |E_2| = 2n \\
 E_3 &= \{xy \in E(G) | d_G(x) = d_G(y) = 4\}, |E_3| = n
 \end{aligned}$$

Here $\Delta(G) = 4$ and then, $c_G(x) = 5 - d_G(x)$

$$\begin{aligned}
 CE_1 &= \{xy \in E(G) | c_G(x) = c_G(y) = 3\}, |CE_1| = n \\
 CE_2 &= \{xy \in E(G) | c_G(x) = 3, c_G(y) = 1\}, |CE_2| = 2n \\
 CE_3 &= \{xy \in E(G) | c_G(x) = c_G(y) = 1\}, |CE_3| = n
 \end{aligned}$$

Using the reverse edge partition of graph $CQ(S_n)$ and indices formula, we get

$$\begin{aligned}
RSO(G) &= n\sqrt{3^2 + 3^2} + 2n\sqrt{3^2 + 1^2} + n\sqrt{1 + 1} \\
&= 3\sqrt{2}n + 2\sqrt{10}n + \sqrt{2}n \\
REUSO(G) &= n\sqrt{3^2 + 3^2 + 3 \cdot 3} + 2n\sqrt{3^2 + 1^2 + 3} + n\sqrt{1 + 1 + 1} \\
&= 3\sqrt{3}n + 2\sqrt{13}n + \sqrt{3}n \\
RESO(G) &= n(3 + 3)\sqrt{3^2 + 3^2} + 2n(3 + 1)\sqrt{3^2 + 1^2} + n(1 + 1)\sqrt{1 + 1} \\
&= 18\sqrt{2}n + 8\sqrt{10}n + 2\sqrt{2}n \\
RH(G) &= \frac{2n}{3 + 3} + \frac{2(2n)}{3 + 1} + \frac{2n}{1 + 1} \\
&= \frac{7n}{3} \\
RSO(G, X) &= \frac{1}{18n + 2n(10) + n(2)} X^{[18n + 2n(10) + n(2)]} \\
&= \frac{1}{40n} X^{40n}
\end{aligned}$$

□

Example: For $n = 3$ in the Cycle Quadrilateral Snake Graph (CQS_n), we have $|V| = 9$ vertices and $|E| = 12$ edges. Then, the indices are given by $RSO(G) = 12\sqrt{2} + 6\sqrt{10}$, $REUSO(G) = 12\sqrt{3} + 6\sqrt{13}$, $RESO(G) = 60\sqrt{2} + 24\sqrt{10}$, $RH(G) = 7$, and $RSO(G, X) = \frac{1}{120} X^{120}$.

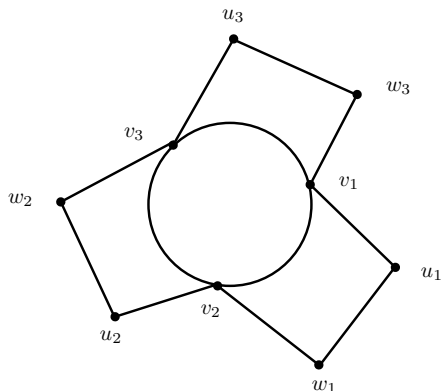


Figure 14: Cycle Quadrilateral Snake Graph CQS_3

4 Conclusion

In this paper, we have studied multiple variants of quadrilateral snake graphs and derived closed form expressions for several reverse degree-based topological indices. The reverse Sombor, reverse Euler Sombor, reverse Elliptic Sombor, and reverse Harmonic indices were computed systematically using edge partition methods. Furthermore, the reverse Sombor polynomial was established for each graph class, providing an extended view of their structural properties. **The obtained results contribute to the theoretical development of reverse topological indices and may support future studies in chemical graph theory, QSPR modeling, and related areas. The scope of the present work is limited to specific snake graph structures, and further research may extend these methods to graph structures and their applications. In particular, future work may focus**

Table 5: Comparison of RSO , $REUSO$, $RESO$, and RH of Cycle Quadrilateral snake graph (CQS_n).

n	$RSO(G)$	$REUSO(G)$	$RESO(G)$	$RH(G)$
1	11.9814	14.1393	53.5825	2.3333
2	23.9628	28.2786	107.165	4.6667
3	35.9442	42.4179	160.7475	7
4	47.9256	56.5572	214.33	9.3333
5	59.9070	70.6965	267.9125	11.6667
6	71.8885	84.8358	321.495	14
7	83.8699	98.9751	375.0774	16.3333
8	95.8513	113.1144	428.6599	18.6667
9	107.8327	127.2538	482.2424	21
10	119.8141	141.3931	535.8249	23.3333

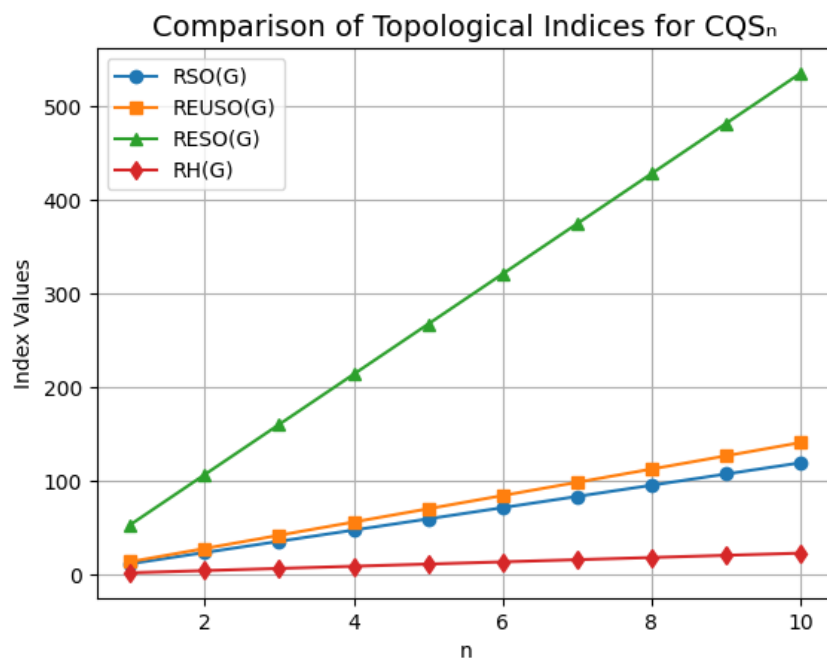


Figure 15: Comparison of RSO , $REUSO$, $RESO$, and RH of CQS_n .

on incorporating fuzzy-based approaches into quadrilateral snake graph models, which could provide a more flexible for handling uncertainty and enhance their applicability in real-world systems.

References

- [1] Alimon, N.I., Sarmin, N.H., Khasraw, S.M.S., Najmuddin, N., and Ismail, G. (2025). The sombor index of a power graph for some finite groups and their sombor polynomial, Malaysian J. of Mathematical Sci., 19(2), 399-410. <https://doi.org/10.47836/mjms.19.2.03>
- [2] Bhairaba Kumar Majhi, Veerabhadrapa Kulli, Ismail Naci Cangul (2024). Revan indices and their polynomials of square snake graphs, Montes Taurus J. Pune Appl. Math., 6(1), 12-17.
- [3] Chitra Ramaprakash (2024). Total Coloring of Snake Graphs and other Standard Graphs, IOSR Journal of Mathematics, 20(2), 14-19.
- [4] Das, K. C., and Shang, Y. (2021). Some extremal graphs with respect to Sombor index. Mathematics, 9(11), 1202.
- [5] Deng, H., Tang, Z., and Wu, R. (2021). Molecular trees with extremal values of Sombor indices. International Journal of Quantum Chemistry, 121(11), e26622.
- [6] Estrada, E., and Uriarte, E. (2001). Recent advances on the role of topological indices in drug discovery research. Current Medicinal Chemistry, 8(13), 1573-1588.
- [7] Gerard Rozario Joseph, Anna S. Varghese, Lawrence Rozario Raj P. (2025). k^{th} Fibonacci Prime Labeling of Snake Graphs. Journal of Mechanics of Continua and Mathematical Sciences, 20(6), 1-12. <https://doi.org/10.26782/jmcms.2025.06.00001>
- [8] Gutman, I. (2013). Degree-based topological indices. Croatica Chemica Acta, 86(4), 351-361.
- [9] Gutman, I., Furtula, B., and Elphick, C. (2014). Three new/old vertex-degree based topological indices. MATCH Communications in Mathematical and in Computer Chemistry, 72, 617-632.
- [10] Gutman, I., Furtula, B., and Oz, M. S. (2024). Geometric approach to vertex degree-based topological indices Elliptic Sombor index, theory and application. International Journal of Quantum Chemistry, 124(2), e27346.
- [11] Harary, F. (1969). Graph Theory, Addison-Wesley, Reading, Mass.
- [12] Havare, O. C. (2019). QSPR analysis with curvilinear regression modeling and topological indices. Iranian Journal of Mathematical Chemistry, 10(4), 331-341.
- [13] Hayat, S., Khan, A., Ali, K., and Liu, J. B. (2024). Structure-property modeling for thermodynamic properties of benzenoid hydrocarbons by temperature-based topological indices. Ain Shams Engineering Journal, 15(3), 102586. <https://doi.org/10.1016/j.asej.2023.102586>
- [14] Hayat, S., Alanazi, S. J., and Liu, J. B. (2024). Two novel temperature-based topological indices with strong potential to predict physicochemical properties of polycyclic aromatic hydrocarbons with applications to silicon carbide nanotubes. Physica Scripta, 99(5), 055027. <https://doi.org/10.1088/1402-4896/ad3ada>

- [15] Ivan, G. (2024). Relating Sombor and Euler indices. *Vojnotehnicki glasnik*, 72(1), 1-12.
- [16] Kirana, B., Shanmukha, M. C., and Usha, A. (2024). Comparative study of Sombor index and its various versions using regression models for top priority polycyclic aromatic hydrocarbons. *Research Square* (preprint). <https://doi.org/10.21203/rs.3.rs-4344427/v1>.
- [17] Kulli, V. R. (2012). *College Graph Theory*, Vishwa Int. Publ., Gulbarga, India.
- [18] Lokesha, V., Shetty, B. S., Ranjini, P. S., Cangul, I. N., and Cevik, A. S. (2013). New bounds for Randic and GA indices. *Journal of inequalities and applications*, 2013, 1-7.
- [19] Mahalank, P., Majhi, B. K., Delen, S., and Cangul, I. N. (2021). Zagreb indices of square snake graphs. *Montes Taurus J. Pune Appl. Math.*, 3(3), 165-171.
- [20] Nikolic, S., and Trinajstic, N. (1995). The Wiener index: Development and applications. *Croatica Chemica Acta*, 68(1), 105-129.
- [21] Prabhu, S., Murugan, G., Arockiaraj, M., Arulperumjothi, M., and Manimozhi, V. (2021). Molecular topological characterization of three classes of polycyclic aromatic hydrocarbons. *Journal of Molecular Structure*, 1229, 129501.
- [22] Redzepovic, I. (2021). Chemical applicability of Sombor indices: Survey. *Journal of the Serbian Chemical Society*, 86(5), 445457.
- [23] Saranya, K. M., and Manimekalai, S (2025). *Revan Topological Indices of Quadrilateral Snake Graphs: Polynomial Formulations, Python Implementation, and Applications*, *The Nepali Mathematical Sciences Report*,42(2),162-177. <https://doi.org/10.3126/nmsr.v42i2.88560>
- [24] Shanmukha, M. C., Usha, A., Kulli, V. R., Shilpa, K. C., (2024). Chemical applicability and curvilinear regression models of vertex degree-based topological index: Elliptic Sombor index. *Int. J. Quantum Chem.* 124(9), e27376.
- [25] Sun, X., and Du, J. (2022). On Sombor index of trees with fixed domination number. *Applied Mathematics and Computation*, 421, 126946.
- [26] Trinajstic, N. (2018). *Chemical Graph Theory*. CRC Press.
- [27] Wiener, H. (1947). Structural determination of paraffin boiling points. *Journal of the American Chemical Society*, 69(1), 17-20.
- [28] Zhao, D., Siddiqui, M. K., Cheema, I. Z., Muhammad, M. H., Rauf, A., and Ishtiaq, M. (2022). On molecular descriptors of polycyclic aromatic hydrocarbon. *Polycyclic Aromatic Compounds*, 42(6), 3422-3433.
- [29] Zaman, S., Yaqoob, H. S. A., Ullah, A., and Sheikh, M. (2023). QSPR analysis of some novel drugs used in blood cancer treatment via degree based topological indices and regression models. *Polycyclic Aromatic Compounds*, 1-17.
- [30] Zhou, T., Lin, Z., and Miao, L. (2021). The Sombor index of trees and unicyclic graphs with given maximum degree. *arXiv preprint arXiv:2103.07947*.