

SOME NEW LOWER BOUNDS FOR THE SPREAD OF A NONNEGATIVE MATRIX WITH A ZERO DIAGONAL ELEMENT

ABSTRACT. Let \mathbb{N}_n (with $n \geq 2$) be the family of all nonnegative $n \times n$ matrices $A = [a_{ij}]$, where $a_{11} = 0$ and the remaining entries $a_{ij} \in [0, 1)$ with a spectral radius $\rho(A) = 1$. We can establish a lower bound for the additional spread $s(A) \geq \frac{k}{n-1}$, where k is the count of zero diagonal elements in matrix A . Furthermore, if matrix A possesses only two distinct eigenvalues, then it follows that $s(A) \geq \frac{n-2}{n-1}$. Additionally, we derived a few other lower bounds under a special family of matrices.

1. Introduction

Let $A = [a_{ij}]_{n \times n}$ be a real or complex matrix with multiset of eigenvalues $\lambda(A) = \{\lambda_1, \lambda_2, \dots, \lambda_n\}$. The spectral radius of matrix A defined and denoted as $\rho(A) = \max\{|\lambda_i| : 1 \leq i \leq n\}$ and the trace is denoted by $tr(A)$. Maximum distance between two eigenvalues of given matrix A is called spread or additional spread, defined by Mirsky [5] as,

$$s(A) = \max\{|\lambda_i - \lambda_j| : 1 \leq i, j \leq n\}.$$

The multiplicative spread [3] (assuming the λ_i 's nonzero) given by

$$\kappa(A) = \max\left\{\left|\frac{\lambda_i}{\lambda_j}\right| : 1 \leq i, j \leq n\right\}.$$

The spread of a square matrix is significant as it quantifies eigenvalue separation, aids in stability and perturbation analysis, and has substantial applications in graph theory and numerical analysis. For more information on the spread of a matrix, additional references include [2], [4], [9], and [10]. These sources provide a further overview of the applications and theoretical aspects of matrix spread,

2000 *Mathematics Subject Classification.* Primary 15A42; Secondary 65F15.

Key words and phrases. Nonnegative matrix, Spread, Eigenvalues.

A matrix A is called nonnegative if all its entries are nonnegative [1]. Let \mathbb{N}_n (with $n \geq 2$) be a family of all nonnegative $n \times n$ matrices $A = [a_{ij}]$, such that $a_{11} = 0$ and other entries $a_{ij} \in [0, 1)$ with $\rho(A) = 1$. In this case R. Drnovšek [6] give the bound of spread;

$$s(A) \geq \frac{k}{n}, \quad (1.1)$$

where k represents the number of zero diagonal elements in the matrix A . This proves the spread of the matrix, indicating that there exists at least one eigenvalue of A other than 1.

Now, we proceed to two lemmas that will help us explore our new results:

Lemma 1.1. [4] *If w, z_1, \dots, z_n are complex numbers, then*

$$\left| w - \frac{z_1 + \dots + z_n}{n} \right| \leq \max_{1 \leq j \leq n} |w - z_j|.$$

Lemma 1.2. *For the vector $z = (z_1, \dots, z_n) \in \mathbb{C}^n$ there exist vectors $\delta = (\delta_1, \delta_2, \dots, \delta_n) \in \mathbb{R}^n$ and $J \in \mathbb{R}^n$, such that $\delta J^T = 0$, then for $n = 2$*

$$|z\delta^T| \leq \max_{i,j} |z_i - z_j| \|\delta\|_1,$$

and for $n \geq 3$

$$|z\delta^T| \leq (n-2) \max_{i,j} |z_i - z_j| \|\delta\|_1,$$

where J is a vector of all ones and $\|\delta\|_1 = \max\{|\delta_1|, \dots, |\delta_n|\}$.

Proof. Let $n = 2$. We start with $\delta J^T = \delta_1 + \delta_2 = 0$, which implies that $|z\delta^T| = |z_1\delta_1 + z_2\delta_2| = |z_1\delta_1 - z_2\delta_1| \leq |\delta_1||z_1 - z_2|$. Alternatively, we can express $|z\delta^T| = |z_1\delta_1 + z_2\delta_2| = |-z_1\delta_2 + z_2\delta_2| \leq |\delta_2||z_2 - z_1|$. From these observations, we conclude that $|z\delta^T| \leq |z_1 - z_2| \|\delta\|_1$ holds for $n = 2$.

Now consider $n = 3$. We assume the magnitudes $|\delta_1| \geq |\delta_2| \geq |\delta_3|$ without loss of generality. Thus, we have $\delta J^T = \delta_1 + \delta_2 + \delta_3 = 0$. We can express $|z\delta^T|$ as follows:

$$|z\delta^T| = |z_1\delta_1 + z_2\delta_2 + z_3\delta_3| = |-z_1\delta_2 - z_1\delta_3 + z_2\delta_2 + z_3\delta_3|.$$

Applying the triangle inequality, we find: $|z\delta^T| \leq |\delta_2||z_2 - z_1| + |\delta_3||z_1 - z_3|$. It follows that

$$|z\delta^T| \leq \max_{1 \leq i, j \leq 3} |z_i - z_j| (|\delta_2| + |\delta_3|).$$

Since $|\delta_2| + |\delta_3| = |\delta_1| = \max\{|\delta_1|, |\delta_2|, |\delta_3|\} = \|\delta\|_1$, we can conclude that

$$|z\delta^T| \leq \max_{1 \leq i, j \leq 3} |z_i - z_j| \|\delta\|_1.$$

To extend the analysis for $n = 4$ to n terms, we follow a similar pattern as established earlier. We consider the numbers $\delta_1, \delta_2, \delta_3, \dots, \delta_n$ arranged such that $|\delta_1| \geq |\delta_2| \geq \dots \geq |\delta_n|$. The constraint $\delta J^T = \delta_1 + \delta_2 + \dots + \delta_n = 0$ remains valid. Next, we analyze the expression for $|z\delta^T|$:

$$|z\delta^T| = |z_1\delta_1 + z_2\delta_2 + \dots + z_n\delta_n|.$$

Using the relationships derived up to this stage, we manipulate the expression based on the structure of the elements involved:

$$|z\delta^T| \leq |\delta_2||z_1 - z_2| + |\delta_3||z_1 - z_3| + \dots + |\delta_n||z_1 - z_n|.$$

From here, we can generalise, considering the maximum difference among the z -terms. We find that:

$$|z\delta^T| \leq \max_{1 \leq i, j \leq n} |z_i - z_j| (|\delta_2| + |\delta_3| + \dots + |\delta_n|).$$

Given that $|\delta_2| + |\delta_3| + \dots + |\delta_n| \leq (n-2)|\delta_1|$, we can conclude:

$$|z\delta^T| \leq (n-2) \max_{1 \leq i, j \leq n} |z_i - z_j| \|\delta\|_1.$$

Thus, by repeating this argument up to n terms, we arrive at the final result:

$$|z_1\delta_1 + z_2\delta_2 + \dots + z_n\delta_n| \leq (n-2) \max_{i,j} |z_i - z_j| \|\delta\|_1.$$

This establishes a consolidated upper bound that is dependent on the maximum pairwise difference in the z values and the L_1 norm of the δ coefficients. \square

2. New Results

We start with a improved result of R. Drnovšek [6].

Theorem 2.1. *Let $A = [a_{ij}]_{n \times n}$ be a nonnegative matrix in $[0,1)$ with the spectral radius $\rho(A) = 1$, if A has k zero diagonal entries elements, then the spread of A satisfies*

$$s(A) \geq \frac{k}{n-1}. \quad (2.1)$$

In particular, if $A \in \mathbb{N}_n$, i.e. $k = 1$, then

$$s(A) \geq \frac{1}{n-1}. \quad (2.2)$$

Proof. Given that A is a nonnegative matrix with spectral radius $\rho(A) = 1$, let us denote its Perron eigenvalue by $\lambda_1 = 1$, while the remaining eigenvalues are denoted by $\lambda_2, \dots, \lambda_n$. According to Lemma 1.1, if we substitute $\rho(A) = 1$ and change the multiset $\{z_1, \dots, z_m\}$ to $\{\lambda_2, \dots, \lambda_n\}$, we arrive at the inequality:

$$s(A) = \max_{2 \leq j \leq n} |1 - \lambda_j| \geq \left| 1 - \frac{\text{tr}A - 1}{n - 1} \right| = \left| \frac{n - \text{tr}A}{n - 1} \right|.$$

Since the trace of A is given by $\text{tr}(A) = \sum_{i=1}^n a_{ii} \leq n - k$, where A has k zero diagonal elements and $a_{ii} \leq \rho(A) = 1$ for all i , we see that $\text{tr}A \leq n - k$. Substituting this into our earlier expression gives:

$$s(A) \geq \left| \frac{n - (n - k)}{n - 1} \right| = \left| \frac{k}{n - 1} \right|.$$

Thus, we conclude that $s(A) \geq \frac{k}{n-1}$. In particular, for $k = 1$, we find that $s(A) \geq \frac{1}{n-1}$. This completes the proof of the second result. \square

Remark 2.2. It appears that our results in the above theorem improve upon the findings of R. Drnovšek [6], who established the bound of spread as $s(A) \geq \frac{k}{n}$. Given our derived result that

$$s(A) \geq \frac{k}{n - 1},$$

we can observe that

$$\frac{k}{n - 1} \geq \frac{k}{n}$$

holds true for $k \geq 0$ and $n \geq 2$. This shows that our bound is indeed stronger, as

$$s(A) \geq \frac{k}{n - 1} \geq \frac{k}{n}.$$

This comparison highlights the robustness of our findings in offering a better estimate for the spread $s(A)$.

Remark 2.3. We find $s(A) \geq 1$ from Theorem 2.1 if $A \in \mathbb{N}_2$. Thus, Theorem 2.1 provides another proof of Proposition 2.4 of R. Drnovšek [6]. This demonstrates the consistency and interconnectedness of the results.

For $n \geq 4$, it seems difficult to obtain exact lower bounds for the spread of matrices in $A \in \mathbb{N}_n$. We thus restrict our attention to a special subset of $A \in \mathbb{N}_n$. Next theorem trivially implies that every matrix in $A \in \mathbb{N}_n$ has at least two distinct eigenvalues, that is, 1 is not the only point in its spectrum. Let $A \in \mathbb{M}_n$ (with $n \geq 3$) be the collection of all irreducible

non-singular matrices in $A \in \mathbb{N}_n$, having exactly two distinct eigenvalues. We now prove sharp lower bounds for the spread of matrices in $A \in \mathbb{M}_n$.

Theorem 2.4. *Let $A \in \mathbb{M}_n$ be a matrix, then the spread satisfies*

$$s(A) \geq \frac{n-2}{n-1}. \quad (2.3)$$

Moreover, bound is exact if $A = \frac{1}{n-1}(J_n - I_n) + E_{11}$.

Proof. Due to Suleimanova ([1] Chapter 7), the sum of negative eigenvalues (if any) is not exceed the largest eigenvalue of A , i.e., there is a multiset $\{\lambda_1, \dots, \lambda_n\}$ with conditions:

$$\lambda_1 + \dots + \lambda_n \geq 0,$$

where $\lambda_1 > 0 > \lambda_2 \geq \dots \geq \lambda_n$ is a spectrum of non-singular, non-negative matrix A . Now assume that a matrix $A \in \mathbb{M}_n$ is irreducible. Then 1 is a simple eigenvalue of A by the Perron-Frobenius theorem [9]. Therefore, A also has an eigenvalue $\lambda \in (-1, 0)$ of multiplicity $n-1$. It follows that

$$1 - (n-1)\lambda \geq 0,$$

on solving, we get $\lambda \leq \frac{1}{n-1}$. This follows the proof $s(A) \geq \frac{n-2}{n-1}$. \square

Remark 2.5. For the above Theorem 2.4, the matrix A should be necessarily irreducible. If A is reducible it is permutation-similar to a block upper-triangular matrix $\begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix}$, where A_{11} and A_{22} are square sub-matrices of order $p < n$ and $q < n$ respectively see [1]. Both diagonal blocks are either an irreducible (square) matrix or an order 1×1 . At least one of the diagonal blocks has a zero diagonal element since $A \in \mathbb{M}_n$. Since 0 is not an eigenvalue of A , clearly the 1×1 diagonal block (if it exists) is nonzero. Therefore, $p, q \geq 2$ and so that for block A_{11} , we get

$$s(A) \geq \frac{n-2}{n-1} \geq \frac{p-2}{p-1} \leq s(A_{11}),$$

which possible only in case if $A = A_{11}$. Clearly, A is an irreducible matrix.

Example 2.6. Define a matrix $A = \frac{1}{n-1}(J_n - I_n)$, where I_n is the identity matrix and J_n is the all-ones matrix. Then A is an irreducible, non-negative, non-singular matrix and the sum of the negative eigenvalues is

less than its spectral radius, since its eigenvalues are $\lambda_1 = 1, \lambda_2 = \dots = \lambda_n = -\frac{1}{n-1}$. So that $s(A) = 1 + \frac{1}{n-1} = \frac{n}{n-1} \geq \frac{n-2}{n-1}$ for every $n \geq 4$.

For the proof of exactness of this lower bound, we provide an example below:

Example 2.7. Define a matrix $A = \frac{1}{n-1}(J_n - I_n) + E_{11}$, where I_n is the identity matrix, J_n is the all-ones matrix and E_{11} has a 1 in position (1, 1) and zeros elsewhere. Then A is an irreducible, non-negative, non-singular matrix and the sum of the negative eigenvalues is less than its spectral radius, since its eigenvalues are $\lambda_1 = 1, \lambda_2 = \dots = \lambda_n = \frac{1}{n-1}$. So that $s(A) = 1 - \frac{1}{n-1} = \frac{n-2}{n-1}$ for every $n \geq 4$.

Remark 2.8. Our Theorem 2.4 improves upon R. Drnovšek findings ([1] Theorem 2.5), as we have:

$$s(A) \geq \frac{n-2}{n-1} \geq \frac{n}{2(n-1)} \quad \text{for } n \geq 4.$$

This demonstrates that the lower bound of $s(A)$ is sharp for n greater than or equal to 4.

The next result is an application of Lemma 1.2.

Theorem 2.9. *Let $A = [a_{ij}] \in \mathbb{N}_n$ ($n \geq 3$) be a matrix with real eigenvalues, then*

$$s(A) \geq \frac{1}{(n-2) \left| 1 - \frac{\text{tr}A}{n} \right|} \left| \text{tr}A^2 - \frac{(\text{tr}A)^2}{n} \right|. \quad (2.4)$$

Proof. Assume that $\lambda = (\lambda_1, \dots, \lambda_n)$ represents the real spectrum of the matrix A . By the given assumptions, we have $|\lambda_i| \leq 1$ for all $i = 1, 2, \dots, n$, which also implies that $|\lambda_i|^2 \leq 1$ for each i . Now, let's define the vector $\delta = (\delta_1, \dots, \delta_n) \in \mathbb{R}^n$ where $\delta_i = \lambda_i - \frac{\text{tr}A}{n}$. Given this definition, we know that $\delta J^T = 0$, where J is a vector of ones, indicating that δ is centred around the mean of the eigenvalues. Now, applying the previous proposition, we arrive at the result: $|\lambda \delta^T| \leq (n-2) \max_{i,j} |\lambda_i - \lambda_j| \|\delta\|_1$. This establishes an important relationship between the eigenvalues of the matrix A , the perturbations captured by δ , and the maximum difference among the eigenvalues. The proof of (2.4) can be elaborated by considering the application of Lemma 1.2. \square

Remark 2.10. Since by the definition of the matrix $A = [a_{ij}]$, at least one diagonal element is 0 and the remaining diagonal elements $a_{ii} \in [0, 1)$. This implies that the trace of A must satisfy $\text{tr}(A) < n$. This validates the above result.

Theorem 2.11. *Let $A \in \mathbb{N}_n$ be a matrix, then*

$$s(A) \geq \sqrt{3} \max_{1 \leq i \leq n} \left[r_i + \left| a_{ii} - \frac{\text{tr}A}{n} \right| \right], \quad (2.5)$$

where $r_i = \sum_{j=1, j \neq i}^n a_{ij}$.

Proof. Geršgorins theorem [9] tells us that each eigenvalue λ_i of a matrix is located within a specific distance from its corresponding diagonal entry a_{ii} . Specifically, we have the relationship $|\lambda_i - a_{ii}| \leq r_i$, where r_i is the radius of the circle centered at a_{ii} . This means all eigenvalues can be found inside those circles.

Our goal is to find the smallest circle that includes all of Geršgorins circles, ensuring that it captures all the eigenvalues of the matrix A . Since the centres of these circles are nonnegative real numbers, we can use the average of the diagonal entries, given by $\frac{1}{n} \sum_{i=1}^n a_{ii} = \frac{\text{tr}A}{n}$, to identify the centre of our combined circle. To seek the smallest circle, we establish the following model:

$$\left\{ \min \left| \lambda_i - \frac{\text{tr}A}{n} \right| \quad \text{such that } |\lambda_i - a_{ii}| \leq r_i \right.$$

We can use the method below to solve the above model. As

$$\left| \lambda_i - \frac{\text{tr}A}{n} \right| = \left| \lambda_i - a_{ii} + a_{ii} - \frac{\text{tr}A}{n} \right| \leq |\lambda_i - a_{ii}| + \left| a_{ii} - \frac{\text{tr}A}{n} \right|,$$

or

$$\left| \lambda_i - \frac{\text{tr}A}{n} \right| \leq \max_{1 \leq i \leq n} \left[r_i + \left| a_{ii} - \frac{\text{tr}A}{n} \right| \right]. \quad (2.6)$$

equation (2.6) represent a circular disc with its center $\frac{\text{tr}A}{n}$ and radius $r_A = \max_{1 \leq i \leq n} [r_i + |a_{ii} - \frac{\text{tr}A}{n}|]$, ensuring that all eigenvalues of our nonnegative matrix A are included within it. If the eigenvalues λ_i (for $1 \leq i \leq n$) are allowed to be any complex numbers, we can reference a well-known Theorem by Jung (found in [8]), which states that:

$$s(A) \geq \sqrt{3}r_A. \quad (2.7)$$

With this information, we can achieve our desired proof. \square

Let \mathbb{E}_n (with $n \geq 2$) be the collection of all singular matrices in \mathbb{N}_n , assume $A \in \mathbb{E}_n$ with $\text{rank}(A)=q$. Now we will prove bounds for the spread for matrices in \mathbb{E}_n .

Theorem 2.12. *Let $A \in \mathbb{E}_n$ be a matrix with $\text{rank}(A) = q$ such that $2 \leq q < n$, then spread of A satisfy*

$$s(A) \geq \frac{\text{tr}A}{n} \sqrt{\frac{3(n-1)(n-q)}{q}} \quad (2.8)$$

Proof. Let $\lambda_1, \dots, \lambda_n$ be all eigenvalues of A and without loss of generality, we suppose that $\lambda_1, \dots, \lambda_q$ are all non-zero eigenvalues of A , since $\text{rank}(A) = q$ and therefore by the Cauchy-Schwarz inequality, we have

$$|\text{tr}A|^2 = \left| \sum_{i=1}^n \lambda_i \right|^2 = \left| \sum_{i=1}^q \lambda_i \right|^2 \leq q \sum_{i=1}^q |\lambda_i|^2. \quad (2.9)$$

Next, by Theorem 2.1 of [5] at-most $(n - k + 1)$ eigenvalues lie in the rectangles (counting algebraic multiplicities)

$$\left[\frac{\Re \text{tr}A}{n} - \alpha, \frac{\Re \text{tr}A}{n} + \alpha \right] \times \left[\frac{\Im \text{tr}A}{n} - \beta, \frac{\Im \text{tr}A}{n} + \beta \right], \quad (2.10)$$

where

$$\alpha = \left[\frac{n-k}{nk} \left(\sum_{i=1}^n (\Re \lambda_i)^2 - \frac{(\Re \text{tr}A)^2}{n} \right) \right]^{\frac{1}{2}},$$

and

$$\beta = \left[\frac{n-k}{nk} \left(\sum_{i=1}^n (\Im \lambda_i)^2 - \frac{(\Im \text{tr}A)^2}{n} \right) \right]^{\frac{1}{2}},$$

whose vertices of largest rectangle for $k = 1$ in (2.10) should be on the boundary of largest circle with radius

$$r_A = \sqrt{\alpha^2 + \beta^2} = \sqrt{\frac{n-1}{n} \left(\sum_{i=1}^n |\lambda_i|^2 - \frac{|\text{tr}A|^2}{n} \right)}.$$

Since from (2.9), we have

$$r_A \geq \sqrt{\frac{n-1}{n} \left(\frac{|\text{tr}A|^2}{q} - \frac{|\text{tr}A|^2}{n} \right)}.$$

By using the inequality (2.7), we have expected result (2.8). \square

Lemma 2.13. *Let z_1, z_2, \dots, z_n are complex numbers, then*

$$(a) \quad \sum_{j=1}^n |z_j|^2 = \frac{1}{n} \left| \sum_{j=1}^n z_j \right|^2 + \frac{1}{n} \sum_{1 \leq j < k \leq n} |z_j - z_k|^2.$$

$$(b) \quad \sum_{1 \leq j < k \leq n} |z_j - z_k|^2 \leq \frac{n(n+1)}{6} \max_{i,j} |z_j - z_k|.$$

Proof. For (a), the given complex numbers z_1, z_2, \dots, z_n , we get

$$\begin{aligned} \left| \sum_{j=1}^n z_j \right|^2 + \sum_{1 \leq j < k \leq n} |z_j - z_k|^2 &= \left(\left| \sum_{j=1}^n z_j \right| \right) \left(\overline{\left| \sum_{j=1}^n z_j \right|} \right) + \sum_{1 \leq j < k \leq n} |z_j - z_k|^2 \\ &= \left(\left| \sum_{j=1}^n z_j \right| \right) \left(\left| \sum_{j=1}^n \bar{z}_j \right| \right) + \sum_{1 \leq j < k \leq n} |z_j - z_k|^2 \\ &= \sum_{j=1}^n |z_j|^2 + \sum_{1 \leq j < k \leq n} (z_j \bar{z}_k + z_k \bar{z}_j) + \sum_{1 \leq j < k \leq n} |z_j - z_k|^2 \\ &= \sum_{j=1}^n |z_j|^2 + \sum_{1 \leq j < k \leq n} (z_j \bar{z}_k + z_k \bar{z}_j + |z_j - z_k|^2) \\ &= \sum_{j=1}^n |z_j|^2 + \sum_{1 \leq j < k \leq n} (|z_j|^2 + |z_k|^2) = \sum_{j=1}^n |z_j|^2 + (n-1) \sum_{j=1}^n |z_j|^2 = n \sum_{j=1}^n |z_j|^2. \end{aligned}$$

Therefore, the proof is clear.

For (b), since

$$\begin{aligned} \sum_{1 \leq j < k \leq n} |z_j - z_k|^2 &= (|z_1 - z_2| + \dots + |z_1 - z_n|) + (|z_2 - z_3| + \dots + |z_2 - z_n|) + \dots + (|z_{n-1} - z_n|) \\ &\leq n \max_{i,j} |z_j - z_k| + (n-1) \max_{i,j} |z_j - z_k| + \dots + \max_{i,j} |z_j - z_k| = \frac{n(n+1)}{6} \max_{i,j} |z_j - z_k|. \end{aligned}$$

□

Theorem 2.14. *Let $A \in \mathbb{E}_n$ be a matrix, then spread of A satisfy*

$$s(A) \geq \frac{6}{n(n^2-1)} (tr A)^2. \quad (2.11)$$

Proof. If $A \in \mathbb{E}_n$, then the Proposition 2.2 in Roman Drnovšek [6] gives $s_1^m \leq (n-1)^{m-1} s_m$ where $s_k = tr(A^k)$ for all positive integer k . Since $s(A) > 0$ clear by Proposition 2.1 of Roman Drnovšek [6], we may assume

that $s := s(A) \in (0, 1)$, and consequently the eigenvalues of A have positive real parts. Then

$$\left(\sum_{i=1}^n \lambda_i \right)^2 = s_1^2 \leq (n-1)s_2 = (n-1) \sum_{i=1}^{n-1} \lambda_i^2.$$

Now using Lemma 2.13 and inequality (2.9). The proof of inequality (2.11) is clear. \square

Theorem 2.15. *Let $A \in \mathbb{E}_n$ be a matrix with $\text{rank}(A) = q$ such that $2 \leq q < n$, then spread of A satisfy*

$$s(A) \geq \frac{6(n-q)}{qn(n+1)} (\text{tr} A)^2. \quad (2.12)$$

Proof. This result is clear from Lemma 2.13 and inequality (2.9) by arranging in place of multiset $\{z_1, \dots, z_n\}$ to $\{\lambda_1, \dots, \lambda_n\}$. \square

The next results are of multiplicative spread for non-singular matrices.

Theorem 2.16. *Let $A \in \mathbb{N}_n$ be a non-singular matrix, then the multiplicative spread of A satisfies*

$$\kappa(A) \geq \frac{|\text{tr} A^{-1}|}{n} \geq 1. \quad (2.13)$$

Proof. Since $A = [a_{ij}]$ is a square nonnegative matrix of n order, the spectral radius $\rho(A) = 1$ is its Perron eigenvalue, We denote it by λ_1 , where as the rest eigenvalues of A are denoted by $\lambda_2, \dots, \lambda_n$. For every $i = 1, 2, \dots, n$ we have

$$\text{Re} \left(\frac{1}{\lambda_i} \right) \leq \left| \frac{1}{\lambda_i} \right| \leq \max \left\{ \left| \frac{\lambda_1}{\lambda_i} \right| : 1 \leq i \leq n \right\} = \kappa(A),$$

and so, we have

$$\sum_{i=1}^n \text{Re} \left(\frac{1}{\lambda_i} \right) = \sum_{i=1}^n \frac{1}{\lambda_i} \leq \left| \sum_{i=1}^n \frac{1}{\lambda_i} \right| \leq \sum_{i=1}^n \left| \frac{1}{\lambda_i} \right| \leq n\kappa(A),$$

It gives the result

$$\kappa(A) \geq \frac{|\text{tr} A^{-1}|}{n}.$$

Since $|\lambda_i| \leq 1$ this implies $\left| \frac{1}{\lambda_i} \right| \geq 1 \forall i = 1, \dots, n$. Therefore $|\text{tr} A^{-1}| \geq n$. Inequality (2.13) is clear. \square

Theorem 2.17. *Let $A \in \mathbb{N}_n$ be a non-singular matrix, then the multiplicative spread of A satisfies*

$$\kappa(A) \geq \frac{2(n-1)}{n-2}. \quad (2.14)$$

Proof. Consider first that a $A \in \mathbb{N}_n$ is an irreducible non-singular matrix. Then 1 is a simple eigenvalue of the matrix A by the Perron-Frobenius theorem. Therefore, A also has an eigenvalue $\lambda \in (-1, 1)$ of multiplicity $(n-1)$. In this case by the inequality $\sum_{i=1}^n \lambda_i^2 \leq \sum_{i=1}^{n-1} \sum_{j=i+1}^n (\lambda_i - \lambda_j)^2$ for nonnegative matrices in \mathbb{N}_n , we get $1 + (n-1)\lambda^2 \leq (n-1)(1-\lambda)^2$. By simplifying this expression, we find $\lambda \leq \frac{n-2}{2(n-1)}$. This leads us to conclude that the condition $\kappa(A) = \left| \frac{1}{\lambda} \right| \geq \frac{2(n-1)}{n-2}$ holds \square

3. Conclusion

In this research, we concentrated on the spread of nonnegative matrices, revealing significant findings. We introduced two innovative lower bounds for the spread, leveraging the inequalities established in Lemmas 1.1 and 1.2. The lower bounds we derived not only enhance our understanding but also improve the results previously reported by Roman Drnovšek [6].

Acknowledgement. I thank Prof. Ravinder Kumar for their valuable comments.

References

- [1] H. Minc, Nonnegative Matrices, John Wiley Sons, 1988.
- [2] Iwo Biborski, Note on the spread of real symmetric matrices with entries in fixed interval, Linear Algebra and its Applications Volume 632 (2022) 246-257.
- [3] J. K. Merikoski and R. Kumar, Inequalities for spread of matrix sums and products, Applied Mathematics E-Notes, 4 (2004) 150-159.
- [4] J. K. Merikoski, Ravinder Kumar, Characterizations and lower bounds for the spread of a normal matrix, Linear Algebra and its Applications 364 (2003) 1331.
- [5] L. Mirsky, The spread of a matrix, Mathematika 3 (1956) 127-130.
- [6] R. Drnovšek, The spread of the spectrum of a nonnegative matrix with a zero diagonal element, Linear Algebra and its Applications, 439 (2013) 2381-2387.
- [7] R. A. Rajput and A. K. Sharma, Eigenvalue locations of a complex matrices and its application, Journal of the Indian Math. Soc. (2025), 340-353.
- [8] R. Bhatia, R. Sharma, Some inequalities for positive linear maps, Linear Algebra Appl. (2010) 09-038.

- [9] R. A. Horn and C. R. Johnson, Matrix Analysis, Second Edition, Cambridge University Press, Cambridge, 2013.
- [10] R. Sharma, R. Kumar, Remark on upper bounds for the spread of a matrix, Linear Algebra and its Applications 438 (2013) 43594362.