Original Research Article
Matrix Methods for Connectivity and Reachability in Transportation Networks

Abstract

This paper presents a matrix-based approach for modelling and analysing transportation networks using concepts from graph theory and linear algebra. The incidence matrix and its transformation into the adjacency matrix through the product MM^T are employed to represent structural relationships within the network. Matrix operations and their powers are used to study both direct and indirect connectivity, while the reachability matrix provides an effective algebraic criterion for determining accessibility among nodes. The theoretical results establish a connection between matrix formulations and graph connectivity, offering a systematic framework for network analysis. The applicability of the proposed method is demonstrated through several transportation models, including regional and large-scale networks, where key hubs, connectivity patterns, and efficiency are identified. The study shows that matrix-based techniques provide a scalable and practical tool for transportation planning, route optimization, and analysis of complex network systems.

Keywords: Graph Theory, Adjacency Matrix, Incidence Matrix, Transportation Network.

MSC (2020): 05C50, 15A18, 90B10, 05C82.

1 Introduction

The mathematical modelling of complex systems has emerged as a central theme in contemporary applied mathematics. Among the various available tools, matrix theory and graph theory have proven to be particularly effective in representing and analysing structured interactions in real-world networks such as transportation, communication, and logistics systems. Matrix representations enable compact encoding of relationships among system components, facilitating both computational efficiency and analytical clarity [3].

The evolution of graph-theoretic models has significantly contributed to understanding complex network behaviour. Early work on random graphs by Erdős and Rényi [4] laid the foundation for probabilistic network analysis. Later developments, including small-world models [5, 11] and scale-free networks [6], revealed that many real systems exhibit non-trivial topological features such as clustering and heavy-tailed degree distributions. These insights motivated the development of algebraic and spectral methods for investigating structural and dynamical properties of networks [8, 10, 13].

Within this framework, matrices play a fundamental role in encoding network structure. The adjacency matrix captures direct connections between nodes, while higher powers of the matrix provide information about indirect interactions and path structures. Measures such as communicability, efficiency, and graph energy have been effectively characterized using matrix functions and eigenvalue-based techniques [7, 12, 2]. Spectral graph theory, in particular, offers a rigorous approach to analysing connectivity, robustness, and flow behaviour in networks [9].

Recent advancements in matrix-based network analysis, including spectral decomposition and graph neural network approaches, have significantly expanded the applicability of algebraic methods to large-scale and intelligent transportation systems [24, 25, 27]. These developments highlight the growing importance of integrating algebraic techniques with modern computational frameworks.

Apart from adjacency representations, incidence matrices provide an alternative perspective by explicitly describing relationships between vertices and edges. This representation is especially useful in applications where edge properties are critical, such as transportation and distribution systems. Algebraic transformations, including products of the form MM^T, allow the derivation of adjacency-like structures and enable further analysis using linear algebraic tools [1].

In recent years, matrix-based methodologies have gained increasing importance in the study of large and complex networks. Applications in transportation systems demonstrate that matrix models can effectively evaluate connectivity, optimize routing, and assess system resilience [16, 18]. Advances in spectral analysis have further improved the understanding of network structure and performance [17, 19]. Additionally, techniques based on matrix powers and reachability have been widely applied to analyse multi-step interactions and connectivity patterns [20, 21].

The relevance of these approaches is particularly evident in transportation networks, where efficient connectivity and optimization are essential. Recent contributions show that matrix-based optimization techniques can enhance system performance and sustainability [22, 23]. Furthermore, modern developments in smart city modelling and network flow optimization emphasize the growing role of advanced matrix methods in real-world applications [26, 28]. Studies focusing on robustness and structural stability also provide valuable insights into the resilience of interconnected systems [15, 14].

Despite these advancements, many existing studies focus primarily on computational or simulation-based approaches, with comparatively less emphasis on a unified algebraic framework that combines incidence transformations and matrix power analysis. This gap motivates the present study.

Motivated by these developments, the present work investigates the application of matrix operations to the analysis of regional transportation networks. In particular, we examine the transformation of incidence matrices into adjacency matrices and utilize matrix powers to study connectivity, reachability, and path structures. The proposed approach provides a systematic and scalable framework for identifying key nodes, evaluating network efficiency, and detecting structural gaps.

Main Contributions of the Paper: The key contributions of this study are summarized as follows:

- A unified matrix-based framework is developed for analysing connectivity and reachability in transportation networks.
- A novel interpretation of the transformation MM^T is provided, linking incidence-based representations to adjacency structures in a systematic manner.
- An algebraic criterion for connectivity using reachability matrices is established and theoretically justified.
- The proposed methodology is applied to real-world transportation networks, demonstrating its effectiveness in identifying key hubs and connectivity patterns.
- The framework provides a scalable approach that can be extended to large and complex network systems.

Practical Significance: The proposed matrix-based framework has direct applications in transportation planning, including route optimization, identification of critical hubs, and detection of connectivity gaps. It can assist policymakers in improving infrastructure design, minimizing travel time, and enhancing network resilience. Furthermore, the methodology can be extended to other domains such as communication networks, supply chains, and smart city systems.

The novelty of this work lies in combining incidence-based transformations and matrix power techniques into a unified analytical framework with both theoretical and practical relevance.

The remainder of the paper is organized as follows. Section 2 introduces the necessary preliminaries and fundamental matrix operations. Section 3 develops matrix operations associated with incidence matrices and examines their structural properties. Section 4 presents the main theoretical results along with applications to transportation networks. Finally, Section 5 concludes the study and outlines potential directions for future research.

2 Preliminaries and Matrix Operations

This section presents the fundamental concepts from graph theory and matrix algebra that form the basis for modelling and analysing transportation networks. These tools provide a rigorous mathematical framework for representing connectivity, analysing structural properties, and understanding the flow of information or resources within complex systems [8, 3, 21]. The integration of graph-theoretic and matrix-based approaches enables both theoretical analysis and practical applications in network modelling.

Definition 2.1. *A graph is an ordered pair $G = (V, E)$, where V is a finite non-empty set of vertices and $E \subseteq V \times V$ is a set of edges connecting pairs of vertices.*

Remark 2.2. *Graphs serve as mathematical models for a wide variety of real-world systems, including transportation, communication, and social networks. In such representations, vertices correspond to entities such as cities or stations, while edges represent direct connections or interactions between them [5, 6]. The abstraction provided by graphs allows complex systems to be analysed using well-established mathematical techniques.*

Definition 2.3. *Let $G = (V, E)$ be a graph with n vertices. The adjacency matrix $A = [a_{ij}]$ is defined as an $n \times n$ matrix such that*

$$a_{ij} = \begin{cases} 1, & \text{if there exists an edge between vertices } v_i \text{ and } v_j, \\ 0, & \text{otherwise.} \end{cases}$$

Remark 2.4. *For undirected graphs, the adjacency matrix is symmetric, i.e., $a_{ij} = a_{ji}$. The powers of the adjacency matrix provide deeper insight into network structure: the (i, j) entry of A^k represents the number of distinct walks of length k between vertices v_i and v_j. This property is widely used in analysing connectivity, information flow, and communication efficiency in networks [7, 13].*

Definition 2.5. *For a graph with n vertices and m edges, the incidence matrix $M = [m_{ij}]$ is an $n \times m$ matrix defined by*

$$m_{ij} = \begin{cases} 1, & \text{if vertex } v_i \text{ is incident with edge } e_j, \\ 0, & \text{otherwise.} \end{cases}$$

Remark 2.6. *The incidence matrix provides a representation of vertex–edge relationships and is particularly useful in applications where edge-based interactions are significant. In transportation systems, for example, edges may represent routes, and the incidence matrix helps analyse how routes connect different locations [1, 16]. This representation is often more flexible than adjacency matrices when dealing with network transformations.*

Definition 2.7. *Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be matrices of the same order. Their sum and difference are defined as*

$$(A + B)_{ij} = a_{ij} + b_{ij}, \quad (A - B)_{ij} = a_{ij} - b_{ij}.$$

Remark 2.8. *Matrix addition and subtraction allow the comparison and integration of multiple network structures. These operations are useful in analysing variations between different configurations, such as seasonal changes in transportation routes or alternative network designs [16, 18].*

Definition 2.9. *Let A be an $m \times n$ matrix and B be an $n \times p$ matrix. The product AB is defined as an $m \times p$ matrix given by*

$$(AB)_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}.$$

Remark 2.10. *Matrix multiplication is a fundamental operation in graph theory. It enables the computation of indirect connections and path structures within networks. In particular, it plays a crucial role in deriving adjacency relations from incidence matrices and in analysing multi-step interactions [7, 17].*

Definition 2.11. *Let $A = [a_{ij}]$ be a matrix. The transpose of A, denoted by A^T, is defined by*

$$(A^T)_{ij} = a_{ji}.$$

Remark 2.12. *The transpose operation is essential in studying symmetry and structural properties of networks. For undirected graphs, symmetry of the adjacency matrix reflects mutual connections between vertices. The transpose is also used in constructing derived matrices such as MM^T, which play a central role in network analysis [16].*

Definition 2.13. *Let A be a square matrix. The k-th power of A is defined as*

$$A^k = \underbrace{A \cdot A \cdots A}_{k \text{ times}}, \quad k \in \mathbb{N}.$$

Remark 2.14. *Matrix powers are used to analyse higher-order connectivity in networks. The (i, j) entry of A^k gives the number of walks of length k between vertices, providing insight into indirect relationships and network depth [7, 20].*

Definition 2.15. *Given an incidence matrix M, the matrix*

$$A = MM^T$$

is called the derived adjacency matrix.

Remark 2.16. *The matrix MM^T transforms vertex–edge relationships into vertex–vertex connectivity. Its diagonal entries correspond to vertex degrees, while off-diagonal entries indicate adjacency relations. This dual representation makes it a powerful tool for analysing structural properties of networks [16, 17].*

Definition 2.17. *Let A and B be matrices with non-negative entries. The Boolean product of A and B is defined as*

$$(A \odot B)_{ij} = \begin{cases} 1, & \text{if } \sum_k a_{ik} b_{kj} > 0, \\ 0, & \text{otherwise.} \end{cases}$$

Remark 2.18. *The Boolean product focuses on the existence of paths rather than their number. It is particularly useful in reachability analysis, where the primary concern is whether a connection exists between nodes in a network [19].*

Definition 2.19. *Let A be the adjacency matrix of a graph. The reachability matrix is defined as*

$$S_k = A + A^2 + A^3 + \cdots + A^k.$$

Remark 2.20. *The reachability matrix provides cumulative connectivity information. If the (i, j) entry of S_k is non-zero, then there exists a path between vertices v_i and v_j of length at most k. This concept is essential for analysing accessibility, efficiency, and robustness of networks [15, 14].*

Definition 2.21. *A graph is said to be connected if there exists a path between every pair of vertices.*

Remark 2.22. *Connectivity is a fundamental property of networks. A connected network ensures that all nodes can communicate directly or indirectly. Matrix-based techniques, particularly those involving powers of adjacency matrices and reachability matrices, provide effective tools for verifying connectivity in large-scale systems [15, 14].*

3 Matrix Operations on Incidence Matrix

Matrix operations provide a systematic and efficient framework for analysing complex network structures. When applied to incidence matrices, these operations enable a precise algebraic interpretation of relationships between vertices and edges, which is particularly useful in modelling transportation and logistics systems [16, 21].

Definition 3.1. *Let M_1 and M_2 be two incidence matrices of the same order representing different network configurations. The addition and subtraction of these matrices are defined as*

$$(M_1 + M_2)_{ij} = (M_1)_{ij} + (M_2)_{ij}, \quad (M_1 - M_2)_{ij} = (M_1)_{ij} - (M_2)_{ij}.$$

Remark 3.2. *These operations facilitate the comparison and integration of multiple network layers, such as different transportation routes or distribution systems [18].*

To illustrate these operations, consider a simple network consisting of three nodes:

$$V_1 \text{ (Manufacturer)}, \quad V_2 \text{ (Warehouse)}, \quad V_3 \text{ (Retailer)}.$$

Let the corresponding incidence matrices be

$$M_1 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad M_2 = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 1 & 0 \end{bmatrix}.$$

The matrix sum is given by

$$M_1 + M_2 = \begin{bmatrix} 1 & 1 \\ 2 & 2 \\ 1 & 1 \end{bmatrix},$$

which reflects the combined connectivity of the two systems.

Remark 3.3. *Entries greater than one indicate shared connections across multiple network configurations, whereas unit entries correspond to connections present in only one configuration [16].*

Similarly, the matrix difference is

$$M_1 - M_2 = \begin{bmatrix} 1 & -1 \\ 0 & 0 \\ -1 & 1 \end{bmatrix}.$$

Remark 3.4. *Matrix subtraction highlights structural differences between networks. Positive and negative entries indicate the presence or absence of connections in one configuration relative to another [18].*

Definition 3.5. *Let M be an incidence matrix of order $n \times m$. The matrix*

$$A = MM^T$$

is called the derived adjacency matrix.

Remark 3.6. *Since the incidence matrix is generally rectangular, its powers are not directly defined. The transformation $A = MM^T$ produces a square matrix that captures vertex connectivity and serves as a foundation for further analysis [17, 16].*

Definition 3.7. *Let A be the derived adjacency matrix. The k-th power of A is defined as*

$$A^k = \underbrace{A \cdot A \cdots A}_{k \text{ times}}, \quad k \in \mathbb{N}.$$

Remark 3.8. *The (i, j) entry of A^k represents the number of walks of length k between vertices v_i and v_j. This property is fundamental in analysing multi-step connectivity and flow within networks [7, 20].*

For illustration, consider a three-node system. The $(1, 3)$ entry of A^2 is given by

$$(A^2)_{13} = \sum_{k=1}^{3} a_{1k}a_{k3},$$

which counts all possible two-step paths between vertices v_1 and v_3.

Remark 3.9. *If $(A^2)_{13} = n$, then there exist n distinct paths of length two connecting the corresponding vertices [7].*

Definition 3.10. *Let A be a matrix with non-negative entries. The Boolean product of A is defined as*

$$(A \odot A)_{ij} = \begin{cases} 1, & \text{if } \sum_k a_{ik}a_{kj} > 0, \\ 0, & \text{otherwise.} \end{cases}$$

Remark 3.11. *The Boolean product determines the existence of paths rather than their count and is particularly useful in reachability analysis of large-scale networks [19].*

Definition 3.12. *The reachability matrix is defined as*

$$S_k = A + A^2 + A^3 + \cdots + A^k.$$

Remark 3.13. *The (i, j) entry of S_k indicates whether there exists a path between vertices v_i and v_j within k steps. This provides a practical tool for evaluating connectivity and accessibility in networks [15, 14].*

Remark 3.14. ***Shortest Path Interpretation:*** *The smallest value of k for which $(S_k)_{ij} > 0$ determines the shortest path length between vertices v_i and v_j.*

Remark 3.15. *If S_k contains zero entries even for $k = n - 1$, then the graph is disconnected. This indicates the presence of isolated components or structural gaps in the network [15].*

4 Main Results

This section presents the principal theoretical results that establish a rigorous connection between matrix operations and structural properties of networks. The results developed here provide an algebraic framework for analysing connectivity, degree distribution, and path structure in transportation networks.

Theorem 4.1. *Let $G = (V, E)$ be a finite graph with adjacency matrix A of order n. Define*

$$S_k = A + A^2 + A^3 + \cdots + A^k, \quad k \in \mathbb{N}.$$

Then the graph G is connected if and only if there exists an integer $k \leq n-1$ such that every entry of S_k is non-zero.

Proof. Assume that G is connected. Then for any pair of vertices v_i and v_j, there exists a path joining them. Let $d(i, j)$ denote the length of the shortest path between v_i and v_j. Since a simple graph with n vertices cannot contain a path of length exceeding $n-1$, it follows that $d(i, j) \leq n-1$.

By the algebraic interpretation of matrix powers, the (i, j) entry of A^k represents the number of walks of length k from v_i to v_j. In particular, $(A^{d(i,j)})_{ij} > 0$. Therefore, for $k = n-1$, the matrix

$$S_{n-1} = A + A^2 + \cdots + A^{n-1}$$

has strictly positive entries in every position. This shows that each pair of vertices is connected by a path of length at most $n-1$.

Conversely, suppose that there exists an integer $k \leq n-1$ such that every entry of S_k is non-zero. Then for each pair (v_i, v_j), there exists an integer $r \leq k$ such that $(A^r)_{ij} > 0$. This implies the existence of a walk of length r connecting v_i to v_j, and hence a path between them. Consequently, every pair of vertices in G is connected.

Thus, the graph G is connected if and only if there exists $k \leq n-1$ such that S_k has no zero entries. □

Remark 4.2. *This result provides a practical algebraic criterion for connectivity. In applications such as transportation networks, the absence of zero entries in the reachability matrix ensures that every location is accessible within a finite number of steps [15, 14].*

Theorem 4.3. *Let $G = (V, E)$ be an undirected graph with incidence matrix M of order $n \times m$, and define*

$$A = MM^T = [a_{ij}].$$

Then the matrix A satisfies the following:

(i) A is symmetric.

(ii) $a_{ii} = \deg(v_i)$ for all i.

(iii) For $i \neq j$, a_{ij} equals the number of edges incident to both v_i and v_j.

Proof. The symmetry of A follows directly from the identity

$$A^T = (MM^T)^T = MM^T,$$

which holds for any matrix product of this form.

The diagonal entries satisfy

$$a_{ii} = \sum_{k=1}^{m} m_{ik}^2.$$

Since each entry of the incidence matrix is either 0 or 1, the equality $m_{ik}^2 = m_{ik}$ holds, and therefore

$$a_{ii} = \sum_{k=1}^{m} m_{ik},$$

which counts the number of edges incident to vertex v_i.

For distinct vertices $i \neq j$, the entry

$$a_{ij} = \sum_{k=1}^{m} m_{ik} m_{jk}$$

counts the number of edges shared by v_i and v_j. This establishes the desired interpretation of off-diagonal entries. In the case of simple graphs, the value reduces to either 0 or 1, depending on adjacency.

Thus, the matrix MM^T encodes both degree information and adjacency structure. □

Remark 4.4. *The matrix MM^T provides a unified algebraic representation of network structure, simultaneously capturing vertex importance and connectivity patterns [17, 16].*

Theorem 4.5. *Let $G = (V, E)$ be a graph with adjacency matrix A. Then for any integer $k \geq 1$, the (i, j) entry of A^k is equal to the number of distinct walks of length k from vertex v_i to vertex v_j.*

Proof. The result is a direct consequence of the definition of matrix multiplication together with its combinatorial interpretation in graph theory. Let $A = [a_{ij}]$ be the adjacency matrix of the graph G. By definition, $a_{ij} = 1$ if there exists an edge between vertices v_i and v_j, and 0 otherwise.

Consider first the matrix A^2. Its (i, j) entry is given by

$$(A^2)_{ij} = \sum_{k=1}^{n} a_{ik} a_{kj}.$$

Each term $a_{ik}a_{kj}$ is equal to 1 precisely when there exists an edge from v_i to an intermediate vertex v_k and an edge from v_k to v_j. Thus, every non-zero contribution corresponds to a walk of length two from v_i to v_j passing through v_k. Summing over all possible intermediate vertices v_k yields the total number of distinct two-step walks between v_i and v_j.

This interpretation extends naturally to higher powers of the matrix. For any integer $k \geq 2$, the entry $(A^k)_{ij}$ is obtained through repeated matrix multiplication, which can be expressed as a summation over all possible sequences of intermediate vertices. Specifically, each term in the expansion corresponds to a sequence

$$v_i \to v_{k_1} \to v_{k_2} \to \cdots \to v_{k_{k-1}} \to v_j,$$

where consecutive vertices in the sequence are adjacent in the graph. Such a sequence represents a walk of length k from v_i to v_j.

The structure of matrix multiplication ensures that every valid walk contributes exactly one unit to the sum, while invalid sequences contribute zero. Consequently, the total sum $(A^k)_{ij}$ counts precisely the number of distinct walks of length k between the vertices v_i and v_j.

Hence, the matrix power A^k provides a complete algebraic representation of all walks of length k in the graph, which establishes the desired result. □

Remark 4.6. *This theorem provides a fundamental link between algebraic operations and combinatorial properties of graphs. It is particularly useful in analysing indirect connectivity, route planning, and flow dynamics in transportation systems [7, 20].*

Corollary 4.7. *Let $G = (V, E)$ be a graph with adjacency matrix A of order n. If there exists an integer $k \leq n-1$ such that A^k has no zero entries, then the graph G is connected.*

Proof. If A^k has no zero entries, then for every pair of vertices (v_i, v_j), the entry $(A^k)_{ij} > 0$. By the interpretation of matrix powers, this implies that there exists at least one walk of length k connecting v_i and v_j. Since the existence of a walk guarantees the existence of a path between the vertices, it follows that every pair of vertices in G is connected. Hence, the graph G is connected. □

Remark 4.8. *This corollary provides a simplified criterion for connectivity based solely on a single matrix power, which is useful in computational applications where evaluating A^k is more efficient than constructing the entire reachability matrix.*

Corollary 4.9. *Let $G = (V, E)$ be a graph with adjacency matrix A. If G is disconnected, then for every integer $k \geq 1$, the matrix A^k contains at least one zero entry.*

Proof. If the graph G is disconnected, then there exist at least two vertices v_i and v_j such that no path exists between them. Consequently, there is no walk of any finite length connecting v_i and v_j. Therefore, for all integers $k \geq 1$, the entry $(A^k)_{ij} = 0$. This shows that each matrix power A^k contains at least one zero entry. □

Remark 4.10. *This result highlights that disconnected components in a network can be detected algebraically through persistent zero entries in matrix powers, which is particularly useful in identifying isolated regions in transportation systems.*

5 Application to Route Map

The methodology of matrix operations can be effectively applied to real-world transportation networks to analyse connectivity, accessibility, and structural efficiency. In this section, we demonstrate the use of incidence and adjacency matrices in modelling a regional airline network in Madhya Pradesh [16, 23].

Definition 5.1. *Let $G = (V, E)$ be a graph representing an airline network, where V denotes cities and E represents direct flight routes between them.*

Example 1: Regional Airline Network

We consider a regional airline transportation network consisting of five major cities. Let the vertex set be

$$V = \{v_1, v_2, v_3, v_4, v_5\},$$

where the vertices correspond respectively to Indore (I), Bhopal (B), Jabalpur (J), Gwalior (G), and Khajuraho (K). The graphical representation of this network is shown in Figure 3.

The set of edges representing direct flight connections is given by

$$E = \{e_1, e_2, e_3, e_4, e_5, e_6\},$$

where

$$e_1 : B\text{–}I, \quad e_2 : B\text{–}J, \quad e_3 : I\text{–}J, \quad e_4 : I\text{–}G, \quad e_5 : J\text{–}K, \quad e_6 : G\text{–}K.$$

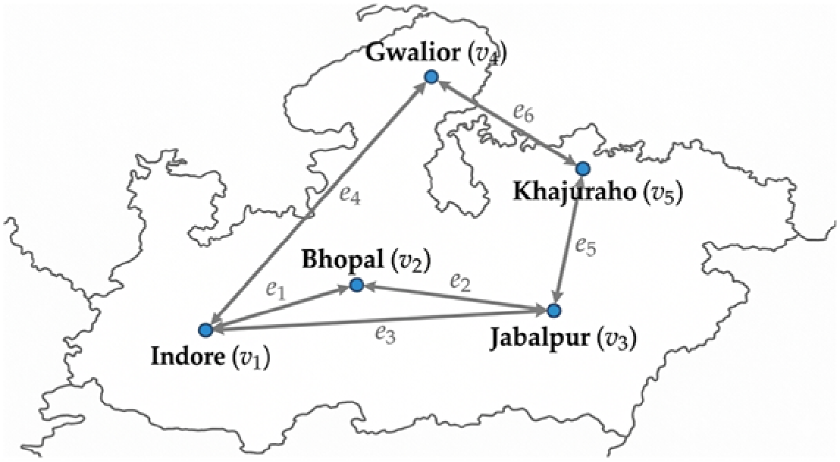

Figure 1: Regional Airline Network

Remark 5.2. *This network models a practical transportation system in which vertices represent cities and edges denote direct flight routes. Such graph-based representations are widely used in analysing connectivity, route optimization, and infrastructure planning [16, 23].*

Incidence Matrix Representation

The incidence matrix M associated with the graph is given by

$$M = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}.$$

Remark 5.3. *Each row corresponds to a vertex and each column corresponds to an edge. The entry $m_{ij} = 1$ indicates that vertex v_i is incident with edge e_j. This representation provides a compact algebraic structure for analysing vertex–edge relationships [1].*

Derived Adjacency Matrix

The derived adjacency matrix is obtained using

$$A = MM^T = \begin{bmatrix} 2 & 1 & 1 & 0 & 0 \\ 1 & 3 & 1 & 1 & 0 \\ 1 & 1 & 3 & 0 & 1 \\ 0 & 1 & 0 & 2 & 1 \\ 0 & 0 & 1 & 1 & 2 \end{bmatrix}.$$

Remark 5.4. *The diagonal entries of A represent vertex degrees, indicating the number of direct connections associated with each city. The off-diagonal entries encode adjacency relations. Thus, the matrix MM^T simultaneously captures both local connectivity and global structural information [17, 16].*

Adjacency Matrix

By eliminating diagonal entries, we obtain the standard adjacency matrix

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}.$$

Remark 5.5. *The adjacency matrix provides a direct representation of connectivity. It is observed that Indore (v_1) and Jabalpur (v_3) have the highest degrees, indicating their roles as central hubs. Such nodes are crucial for maintaining efficient connectivity and ensuring robustness of the transportation network [8].*

Analysis Using Matrix Powers

To examine indirect connectivity, consider

$$A^2 = \begin{bmatrix} 3 & 1 & 1 & 0 & 2 \\ 1 & 2 & 1 & 1 & 1 \\ 1 & 1 & 3 & 2 & 0 \\ 0 & 1 & 2 & 2 & 0 \\ 2 & 1 & 0 & 0 & 2 \end{bmatrix}.$$

Remark 5.6. *The entry (i, j) of A^2 represents the number of distinct two-step paths between vertices v_i and v_j. For example, $(A^2)_{24} = 1$ indicates that there exists exactly one indirect route from Bhopal to Gwalior via an intermediate city. Such analysis reveals hidden connectivity patterns within the network [7].*

Reachability Analysis

Define the reachability matrix

$$S_2 = A + A^2.$$

$$S_2 = \begin{bmatrix} 3 & 2 & 2 & 1 & 2 \\ 2 & 2 & 2 & 1 & 1 \\ 2 & 2 & 3 & 2 & 1 \\ 1 & 1 & 2 & 2 & 1 \\ 2 & 1 & 1 & 1 & 2 \end{bmatrix}$$

Remark 5.7. *Since all entries of S_2 are non-zero, it follows that every pair of cities is connected either directly or through at most one intermediate city. Hence, the network is fully connected within two steps.*

Remark 5.8. *From an application perspective, this indicates a highly efficient transportation system in which travel between any two cities requires at most one stopover. Such properties are desirable in real-world networks, as they enhance accessibility, reduce travel time, and improve system reliability [15, 14].*

Example 2: Extended Network

We now consider an extended transportation network consisting of seven cities. Let the vertex set be

$$V = \{I, B, J, G, R, U, S\},$$

representing Indore (I), Bhopal (B), Jabalpur (J), Gwalior (G), Rewa (R), Ujjain (U), and Sehore (S). The graphical structure of this network is illustrated in Figure 2.

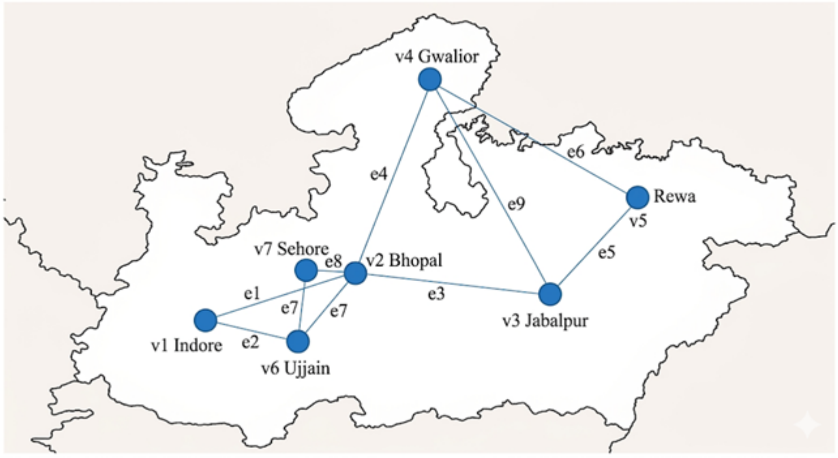

Figure 2: Extended transportation network

Remark 5.9. *This extended network represents a realistic regional transportation system. Such models are useful for analysing connectivity patterns, route optimization, and infrastructure planning in large-scale networks [22, 23].*

Adjacency Matrix Representation

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

Remark 5.10. *The adjacency matrix shows that Bhopal (B) has the highest connectivity, acting as a major hub. In contrast, Rewa (R) has fewer direct connections, indicating a peripheral role. Such heterogeneity is typical in real-world transportation networks [8].*

Higher-Order Connectivity Analysis

$$A^2 = \begin{bmatrix} 2 & 0 & 1 & 1 & 0 & 0 & 2 \\ 0 & 4 & 1 & 1 & 2 & 2 & 0 \\ 1 & 1 & 3 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 3 & 1 & 0 & 1 \\ 0 & 2 & 1 & 1 & 2 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 2 & 0 \\ 2 & 0 & 1 & 1 & 0 & 0 & 2 \end{bmatrix}.$$

Remark 5.11. *The entries of A^2 represent the number of two-step paths between cities. For instance, $(A^2)_{BR} = 2$ indicates that two distinct intermediate routes exist from Bhopal to Rewa.*

Reachability Analysis

$$S_2 = A + A^2 = \begin{bmatrix} 2 & 1 & 1 & 1 & 0 & 1 & 2 \\ 1 & 4 & 2 & 2 & 2 & 2 & 1 \\ 1 & 2 & 3 & 2 & 2 & 0 & 1 \\ 1 & 2 & 2 & 3 & 2 & 0 & 1 \\ 0 & 2 & 2 & 2 & 2 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 & 2 & 1 \\ 2 & 1 & 1 & 1 & 0 & 1 & 2 \end{bmatrix}.$$

Remark 5.12. *The matrix S_2 shows that most cities are connected within two steps, although some zero entries remain, indicating limited accessibility between certain pairs.*

$$S_3 = A + A^2 + A^3$$

Remark 5.13. *The matrix S_3 significantly reduces the number of zero entries, indicating improved connectivity across the network. Most city pairs become reachable within three steps.*

$$S_4 = \begin{bmatrix} 14 & 18 & 19 & 19 & 8 & 12 & 18 \\ 18 & 46 & 26 & 26 & 20 & 14 & 33 \\ 19 & 26 & 41 & 35 & 16 & 10 & 19 \\ 19 & 26 & 35 & 41 & 16 & 10 & 19 \\ 8 & 20 & 16 & 16 & 10 & 4 & 8 \\ 12 & 14 & 10 & 10 & 4 & 10 & 12 \\ 18 & 33 & 19 & 19 & 8 & 12 & 14 \end{bmatrix}.$$

Remark 5.14. *Since S_4 contains no zero entries, the network is fully connected within at most four steps. This ensures complete accessibility across all cities.*

Remark 5.15. *The magnitude of entries in S_4 reflects the number of alternative routes. Cities such as Bhopal and Indore exhibit higher values, confirming their importance as major connectivity hubs in the network [7, 15].*

Example 3: Transportation Network of Major Indian Cities

We consider a large-scale transportation network consisting of ten major Indian cities. Such a model provides a realistic framework for analysing connectivity patterns in national-level transportation systems.

Let the vertex set be

$$V = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_{10}\},$$

where

v_1 : Delhi (D), v_2 : Mumbai (M), v_3 : Kolkata (K), v_4 : Chennai (C),
v_5 : Bengaluru (B), v_6 : Hyderabad (H), v_7 : Ahmedabad (A),
v_8 : Pune (P), v_9 : Jaipur (J), v_{10} : Lucknow (L).

Remark 5.16. *These cities represent major economic and transportation hubs in India. Analysing their interconnections provides valuable insights into large-scale network efficiency and infrastructure planning [22, 19].*

Adjacency Matrix Representation
The adjacency matrix A is given by

$$A = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

Remark 5.17. *The matrix A encodes direct connectivity between cities. It can be observed that Delhi (v_1), Mumbai (v_2), and Ahmedabad (v_7) have relatively higher degrees, indicating their roles as major hubs. Such hub-based structures are typical in real-world transportation networks [8].*

Higher-Order Connectivity
To analyse indirect connections, we consider matrix powers of A.

Remark 5.18. *The matrix A^2 provides information about two-step connectivity. A non-zero entry $(A^2)_{ij}$ indicates that there exists at least one intermediate city connecting v_i and v_j. This helps identify alternative routes and enhances route planning efficiency [7].*

Reachability Analysis
Define the reachability matrix

$$S_k = A + A^2 + A^3 + \cdots + A^k.$$

$$S_2 = \begin{bmatrix} 4 & 2 & 2 & 1 & 1 & 1 & 2 & 1 & 2 & 2 \\ 2 & 4 & 1 & 1 & 1 & 2 & 2 & 2 & 1 & 1 \\ 2 & 1 & 3 & 1 & 1 & 1 & 1 & 0 & 1 & 2 \\ 1 & 1 & 1 & 3 & 2 & 2 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 2 & 3 & 2 & 1 & 2 & 0 & 0 \\ 1 & 2 & 1 & 2 & 2 & 3 & 1 & 1 & 0 & 0 \\ 2 & 2 & 1 & 1 & 1 & 1 & 4 & 2 & 2 & 1 \\ 1 & 2 & 0 & 1 & 2 & 1 & 2 & 3 & 1 & 0 \\ 2 & 1 & 1 & 0 & 0 & 0 & 2 & 1 & 3 & 2 \\ 2 & 1 & 2 & 0 & 0 & 0 & 1 & 0 & 2 & 3 \end{bmatrix}$$

$$S_3 = \begin{bmatrix} 8 & 6 & 5 & 3 & 3 & 3 & 6 & 3 & 6 & 6 \\ 6 & 10 & 4 & 4 & 4 & 6 & 7 & 6 & 4 & 4 \\ 5 & 4 & 7 & 5 & 4 & 4 & 4 & 2 & 4 & 6 \\ 3 & 4 & 5 & 7 & 6 & 6 & 4 & 3 & 2 & 2 \\ 3 & 4 & 4 & 6 & 7 & 6 & 4 & 5 & 2 & 2 \\ 3 & 6 & 4 & 6 & 6 & 7 & 4 & 4 & 2 & 2 \\ 6 & 7 & 4 & 4 & 4 & 4 & 10 & 6 & 6 & 4 \\ 3 & 6 & 2 & 3 & 5 & 4 & 6 & 7 & 3 & 2 \\ 6 & 4 & 4 & 2 & 2 & 2 & 6 & 3 & 7 & 5 \\ 6 & 4 & 6 & 2 & 2 & 2 & 4 & 2 & 5 & 7 \end{bmatrix}$$

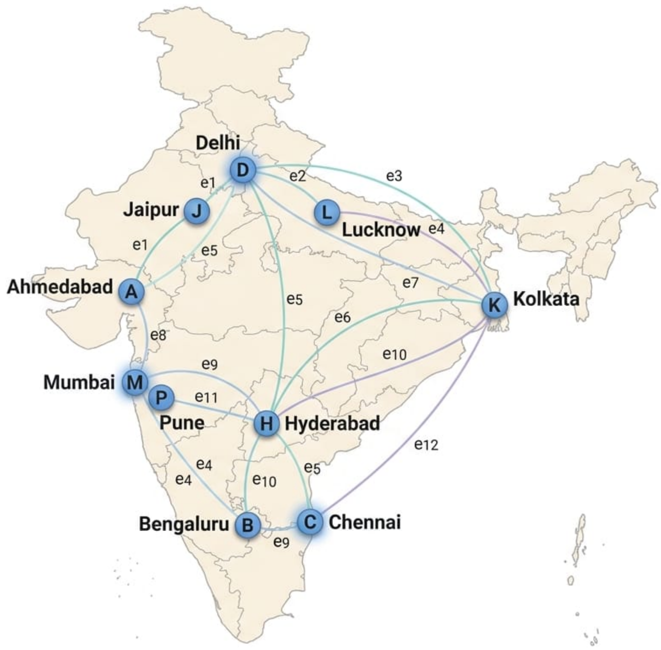

Figure 3: Transportation Network of Major Ten Indian Cities

Remark 5.19. *Since the matrix S_3 contains no zero entries, it follows that the transportation network is fully connected within three steps. Hence, any city can be reached from any other city using at most two intermediate stops.*

This indicates a highly efficient and well-integrated national transportation network, where major hubs such as Delhi, Mumbai, and Ahmedabad facilitate widespread connectivity.

Remark 5.20. *For sufficiently large k (in particular $k \leq n-1$), the matrix S_k captures complete connectivity of the network. In this case, the structure of A ensures that the network is connected, meaning that every city can be reached from any other city through a finite number of intermediate stops.*

Remark 5.21. *From a practical perspective, this implies that the national transportation network is well-integrated. Major hub cities facilitate long-distance travel, while intermediate cities ensure regional accessibility. Such analysis is essential for optimizing transportation efficiency and improving infrastructure planning [15, 14].*

6 Conclusion

In this study, a matrix-based framework for analysing transportation networks has been developed using tools from graph theory and linear algebra. The transformation of incidence matrices into adjacency matrices via the product MM^T provides a unified representation that captures both vertex connectivity and structural properties of the network. The use of matrix powers enables a systematic investigation of indirect connections and path structures, while the reachability matrix offers an effective algebraic criterion for determining network connectivity. The theoretical results establish a strong relationship between matrix operations and graph connectivity, forming a solid mathematical foundation for network analysis.

The proposed methodology has been successfully applied to various transportation network models, including regional and large-scale systems. The analysis demonstrates that central nodes play a crucial role in maintaining connectivity, whereas peripheral nodes require intermediate links to ensure accessibility. The results further indicate that complete connectivity can be achieved within a finite number of steps, highlighting the efficiency of well-structured networks. This approach provides valuable insights for transportation planning, route optimization, and infrastructure development, and can be extended to other complex systems such as communication and logistics networks.

References

[1] R. Khubnani and T. Sharma, "Application of Matrix Operation in Transportation," *International Journal for Research in Engineering Application & Management (IJREAM)*, vol. 5, no. 3, June 2019.

[2] I. Gutman, "The energy of a graph," *Berichte der Mathematisch-Statistischen Sektion im Forschungszentrum Graz*, no. 103, 1978.

[3] E. Ulrychová, "Several Real-World Applications of Linear Algebra Tools," 2010.

[4] P. Erdős and A. Rényi, "On random graphs," *Publicationes Mathematicae*, vol. 6, pp. 290–297, 1959.

[5] D. J. Watts and S. H. Strogatz, "Collective dynamics of small-world networks," *Nature*, vol. 393, no. 6684, pp. 440–442, 1998.

[6] A.-L. Barabási and R. Albert, "Emergence of scaling in random networks," *Science*, vol. 286, no. 5439, pp. 509–512, 1999.

[7] E. Estrada and N. Hatano, "Communicability in complex networks," *Physical Review E*, vol. 77, no. 3, 036111, 2008.

[8] M. E. J. Newman, "The structure and function of complex networks," *SIAM Review*, vol. 45, no. 2, pp. 167–256, 2003.

[9] D. A. Spielman, "Spectral graph theory and its applications," *Proceedings of the International Congress of Mathematicians*, pp. 2933–2950, 2011.

[10] F. Chung, "Laplacians of graphs and Cheeger inequalities," *Annals of Combinatorics*, vol. 1, no. 1, pp. 35–52, 1997.

[11] J. M. Kleinberg, "Navigation in a small world," *Nature*, vol. 406, p. 845, 2000.

[12] V. Latora and M. Marchiori, "Efficient behavior of small-world networks," *Physical Review Letters*, vol. 87, no. 19, 198701, 2001.

[13] E. Estrada, "Quantifying network heterogeneity," *Physical Review E*, vol. 82, no. 6, 066102, 2010.

[14] E. Estrada, "Topological structural analysis of complex networks," *Chaos, Solitons & Fractals*, vol. 145, 110784, 2021.

[15] J. Gao and B. Barzel, "Network robustness and fragility: Percolation on random graphs," *Nature Communications*, vol. 12, no. 1, 2021.

[16] X. Liu, Y. Zhao, and J. Wang, "Matrix-based analysis of complex transportation networks," *Physica A: Statistical Mechanics and its Applications*, vol. 603, 127813, 2022.

[17] H. Zhang and Q. Li, "Spectral properties of adjacency matrices in complex networks," *Applied Mathematics and Computation*, vol. 418, 126834, 2022.

[18] A. Kumar and S. K. Singh, "Graph theoretical modeling of transportation networks using adjacency matrices," *Journal of Applied Mathematics and Computing*, vol. 68, pp. 1235–1252, 2022.

[19] Y. Wang, L. Chen, and Z. Zhang, "Matrix methods in network connectivity and resilience analysis," *IEEE Access*, vol. 11, pp. 45678–45690, 2023.

[20] R. Singh and P. Sharma, "Analysis of graph connectivity using matrix powers and applications," *Mathematics*, vol. 11, no. 5, 2023.

[21] M. Chen and X. Xu, "Applications of linear algebra in large-scale network systems," *Journal of Computational Science*, vol. 70, 102123, 2023.

[22] D. Patel and R. Mehta, "Transportation network optimization using graph matrices," *Operations Research Perspectives*, vol. 11, 100287, 2024.

[23] S. Verma and N. Gupta, "Adjacency matrix-based modelling of smart transportation systems," *Sustainable Computing: Informatics and Systems*, vol. 41, 100925, 2024.

[24] Z. Liu and H. Sun, "Spectral analysis of large-scale complex networks using matrix decomposition," *Applied Network Science*, vol. 9, no. 2, 2024.

[25] J. Wang, X. Li, and Y. Zhao, "Graph neural networks and matrix representations in transportation systems," *IEEE Transactions on Intelligent Transportation Systems*, vol. 25, no. 4, pp. 3456–3468, 2024.

[26] P. Sharma and R. Gupta, "Advanced matrix approaches for smart city transportation modeling," *Sustainable Cities and Society*, vol. 105, 2025.

[27] L. Chen and M. Zhou, "Dynamic network analysis using adjacency and Laplacian matrices," *Chaos, Solitons & Fractals*, vol. 176, 2025.

[28] A. Singh, K. Verma, and S. Tiwari, "Matrix-based optimization techniques in complex network flow problems," *European Journal of Operational Research*, vol. 315, no. 1, pp. 210–225, 2025.