On Dual Lorentzian Vectors and Angles with Leonardo Number Sequences

Original Research Article

Abstract

This study establishes a formal bridge between number theory and Lorentzian geometry by introducing dual Lorentzian Leonardo vectors. While Leonardo sequences are well-known, their representation in three dimensional dual Lorentzian space. By investigating dual Lorentzian angles, this research characterizes the geometric cases under some constraints. In addition, the rigorous definitions are provide for inner and outer products. The results offer significant insights into the intersection of number theory and dual space kinematics. This framework provides a robust basis for further research into higher-order recurrence relations in kinematic geometry.

Keywords:Dual Lorentzian space; inner and outer products; recursive sequences.

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1 Introduction

The study of recurrence sequences has a long history in number theory, with the Fibonacci and Lucas sequences serving as foundational pillars (Bicknell et al., 1972; Koshy, 2001; Vajda, 1989; Verner and Hoggatt, 1969). Over the decades, these sequences have been generalized into various algebraic structures, such as quaternions (Halıcı, 2012; Horadam, 1963) and complex-valued representations (Halıcı, 2019; Halıcı and Curuk, 2019). In particular, recent developments have focused on the Leonardo sequence, exploring its complex (Karataş, 2022) and dual (Karataş, 2023) variations, as well as its hybrid and split-quaternion forms (Atasoy, 2025).

Beyond pure number theory, these sequences have found significant applications in geometry and kinematics. The use of dual numbers, initially pioneered by Clifford, Study (Study, 1901), and Kotelnikov (Kotelnikov, 1895), provides a robust framework for investigating rigid body motions and spatial displacements (Veldkamp, 1976; Aslan et al., 2020). When embedded into Minkowski 3-space—a setting extensively studied for its unique metric properties (Akutagawa and Nishikawa, 1990; Uğurlu and Çalışkan, 1996)—these sequences allow for a deeper geometric characterization through dual-Lorentzian vectors and dual-angle representations (Babadağ and Atasoy, 2024).

Recent studies have demonstrated that integers can be uniquely represented as sums of Fibonacci and Lucas numbers, revealing deep structural properties within these sequences (Park et al., 2020). Beyond simple integers, the transition to functional representations has been marked by the development of generalized Fibonacci and Lucas polynomials, which provide a continuous framework for discrete recurrence relations (Nallı and Haukkanen, 2009).Furthermore, the integration of these sequences into modern functional analysis has led to the exploration of matrix operators on generalized Fibonacci weighted difference sequence spaces, highlighting their utility in operator theory (Candan, 2022). The versatility of these recurrences is also evident in their generalization into new forms, such as k-Oresme and k-Oresme-Lucas sequences, which extend the classical boundaries of second-order relations (Özkan and Akkuş, 2024). To support the derivation of such complex structures, various combinatorial identities involving the terms of generalized Fibonacci and Lucas sequences have been established, providing the necessary identities for algebraic simplifications in higher-dimensional spaces (Akyüz and Halıcı, 2013)

In pursuit of broader generalizations, many researchers have introduced and studied extensions and analogs of these classical sequences. For instance, Horadam explored a generalized Fibonacci sequence $H_n(a, b; p, q)$, where $H_0 = a$, $H_1 = b$, and

$$H_n = pH_{n-1} + qH_{n-2}, \quad n \geqslant 2$$

(Horadam, 1961, 1963). This formulation allows for the construction of various new integer sequences by adjusting the parameters a, b, p, and q, encompassing numerous well-known recurrences as special cases.

Fibonacci and Lucas sequences are deeply intertwined through their shared recurrence relation but are distinguished by their unique initial conditions.

The Fibonacci sequence $\{F_n\}_{n\geq 0}$ is defined by the second-order linear recurrence relation:

$$F_n = F_{n-1} + F_{n-2}, \quad n \geq 2$$

with the initial seeds $F_0 = 0$ and $F_1 = 1$. These sequences are rich in combinatorial identities (Akyüz and Halıcı, 2013) and have been extended to generalized polynomials (Nallı and Haukkanen, 2009; Abd-Elhameed and Napoli, 2023). The closed-form expression for the n-th Fibonacci number, known as the Binet formula, is given by:

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}$$

where $\alpha = \frac{1+\sqrt{5}}{2}$ is the golden ratio and $\beta = \frac{1-\sqrt{5}}{2} = -1/\alpha$.

The Lucas sequence $\{L_n\}_{n\geq 0}$ follows the same additive recurrence as the Fibonacci numbers:

$$L_n = L_{n-1} + L_{n-2}, \quad n \geq 2$$

however, it is initialized with $L_0 = 2$ and $L_1 = 1$. Its corresponding Binet formula is expressed as:

$$L_n = \alpha^n + \beta^n$$

These sequences are fundamentally linked through several well-known identities that are crucial for derivations in dual-Lorentzian spaces. For example,

$$\begin{gathered} L_n = F_{n-1} + F_{n+1}, \\ 5F_n^2 + 4(-1)^n = L_n^2, \\ F_{2n} = F_n L_n, \\ F_m F_{n+1} - F_{m+1} F_n = (-1)^n F_{m-n}. \end{gathered} \tag{1.1}$$

While these classical sequences provide a robust framework for integer recurrences, this study focuses on their extension to the Leonardo sequence. As will be demonstrated in the following sections, Leonardo numbers exhibit a non-homogeneous recurrence structure, necessitating a more complex embedding into the three-dimensional dual-Lorentzian space.

One such notable sequence is the Leonardo numbers denoted by L_n which are defined by the recurrence relation

$$\mathcal{L}e_n = \mathcal{L}e_{n-1} + \mathcal{L}e_{n-2} + 1, \quad \mathcal{L}e_0 = 1, \mathcal{L}e_1 = 1, \quad n \geq 2. \tag{1.2}$$

This sequence begins as

$$1, 1, 3, 5, 9, 15, 25, 41, 67, 109, 177, \ldots$$

and finds applications in computational theory, particularly in data structure analysis such as smoothsort, where it models the size of heaps in a heap-based sorting algorithm. The recurrence of Leonardo numbers can also be interpreted as a variant of the Fibonacci sequence with an additional increment term, yielding a slower-growing but structurally rich sequence. The Binet's formula for Leonardo number sequence is

$$\mathcal{L}e_n = \frac{2\varphi^{n+1} - 2\psi^{n+1}}{\varphi - \psi} - 1, \quad n \geq 0 \tag{1.3}$$

and

$$\mathcal{L}e_n = 2F_{n+1} - 1, \quad n \geq 0 \tag{1.4}$$

where $\varphi = \frac{1+\sqrt{5}}{2}$ and $\psi = \frac{1-\sqrt{5}}{2}$ (Halıcı, 2019; Catarino and Borges, 2020; Karataş, 2022).

In the context of algebraic structures, dual numbers introduced by Clifford and further developed in geometry and kinematics by Kotelnikov (1895) and Study (1901) offer an intriguing extension to real numbers. A dual number is an expression of the form $\gamma = \gamma_0 + \gamma_1 \varepsilon$, where $\gamma_0, \gamma_1 \in \mathbb{R}$, and ε is a dual unit satisfying $\varepsilon^2 = 0$, $\varepsilon \neq 0$. The set of all such numbers is denoted by

$$\mathbb{D} = \{\gamma = \gamma_0 + \gamma_1 \varepsilon \mid \gamma_0, \gamma_1 \in \mathbb{R}, \varepsilon^2 = 0, \varepsilon \neq 0\}.$$

The square root of a dual number γ is defined as

$$\sqrt{\gamma} = \sqrt{\gamma_0} + \frac{\gamma_1}{2\sqrt{\gamma_0}} \varepsilon. \tag{1.5}$$

Similarly, a dual vector takes the form

$$\vec{\gamma} = \vec{\gamma}_0 + \vec{\gamma}_1\varepsilon,$$

where $\vec{\gamma}_0$ and $\vec{\gamma}_1$ are real vectors. Dual vectors and dual numbers are widely used in representing motions and orientations in Euclidean space, particularly in the study of screws, rotations, and transformations (Veldkamp, 1976; Guggenheimer, 1963).

The objective of this research is twofold: first, to define a new class of dual-Lorentzian vectors based on Leonardo numbers; and second, to investigate their geometric invariants, specifically dual angles and cross-product identities.

In this paper, we explore a dual number extension of the Leonardo number sequence, investigate its algebraic and combinatorial properties, and derive new identities and formulas based on this generalized structure.

1.1 Dual Lorentzian Space $\mathbb{D}_1^3$

Let $\mathbb{R}^3 = \{(a_1, a_2, a_3) \mid a_1, a_2, a_3 \in \mathbb{R}\}$ be a 3-dimensional real vector space given by the vectors $\vec{a} = (a_1, a_2, a_3)$ and $\vec{b} = (b_1, b_2, b_3)$ in $\mathbb{R}^3$. The Lorentzian scalar product of $\vec{a}$ and $\vec{b}$ is defined by

$$\langle \vec{a}, \vec{b} \rangle_{\mathbb{L}} = -a_1b_1 + a_2b_2 + a_3b_3. \tag{1.6}$$

$(\mathbb{R}^3, \langle \cdot, \cdot \rangle)$ is called the 3-dimensional Lorentzian space, or Minkowski 3-space, denoted by $\mathbb{R}_1^3$, and the Lorentzian vector product of $\vec{a}$ and $\vec{b}$ is defined by

$$\begin{aligned} \vec{a} \wedge_{\mathbb{L}} \vec{b} &= \begin{vmatrix} -\vec{e_1} & \vec{e_2} & \vec{e_3} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \\ &= \left(-a_2b_3 + a_3b_2, -a_1b_3 + a_3b_1, a_1b_2 - a_2b_1\right), \end{aligned} \tag{1.7}$$

where $\vec{e_1} \wedge \vec{e_2} = \vec{e_3}$, $\vec{e_2} \wedge \vec{e_3} = -\vec{e_1}$, and $\vec{e_3} \wedge \vec{e_1} = \vec{e_2}$ (Akutagawa and Nishikawa, 1990).

Also, $\vec{a}$ in $\mathbb{R}_1^3$ is called a spacelike vector, a lightlike vector or a timelike vector if

$$\langle \vec{a}, \vec{a} \rangle_{\mathbb{L}} > 0 \text{ or } \vec{a} = \vec{0}, \quad \langle \vec{a}, \vec{a} \rangle_{\mathbb{L}} = 0 \text{ and } \vec{a} \neq \vec{0}, \quad \text{ or } \langle \vec{a}, \vec{a} \rangle_{\mathbb{L}} < 0,$$

respectively.

The Lorentzian scalar product and vector product of dual vectors $\vec{\mathcal{A}} = \vec{a}_0 + \varepsilon\vec{a}_0^*$ and $\vec{\mathcal{B}} = \vec{b}_0 + \varepsilon\vec{b}_0^*$ in $\mathbb{D}_1^3$ are given by

$$\langle \vec{\mathcal{A}}, \vec{\mathcal{B}} \rangle_{\mathbb{L}} = \langle \vec{a_0}, \vec{b_0} \rangle_{\mathbb{L}} + \varepsilon \left(\langle \vec{a_0}, \vec{b}_0^* \rangle_{\mathbb{L}} + \langle \vec{a}_0^*, b_0 \rangle_{\mathbb{L}} \right), \tag{1.8}$$

and

$$\vec{\mathcal{A}} \wedge_{\mathbb{L}} \vec{\mathcal{B}} = \vec{a_0} \wedge_{\mathbb{L}} \vec{b_0} + \varepsilon(\vec{a_0} \wedge_{\mathbb{L}} \vec{b}_0^* + \vec{a}_0^* \wedge_{\mathbb{L}} \vec{b_0}), \tag{1.9}$$

where we call the dual space $\mathbb{D}^3$ together with this Lorentzian inner product as dual Lorentzian space and denote it by $\mathbb{D}_1^3$ (Uğurlu and Çalışkan, 1996).

Definition 1.1. Let $\vec{\mathcal{A}} \in \mathbb{D}_1^3$. The dual vector $\vec{\mathcal{A}}$ is said to be spacelike if the vector $\vec{a}_0$ is spacelike, timelike if the vector $\vec{a}_0$ is timelike, and lightlike (dual null) if the vector $\vec{a}_0$ is lightlike, respectively.

Definition 1.2. The norm of a dual vector $\vec{\mathcal{A}}$ in $\mathbb{D}_1^3$ is given by

$$\left| \vec{\mathcal{A}} \right|_{\mathbb{L}} = \sqrt{\langle \vec{\mathcal{A}}, \vec{\mathcal{A}} \rangle_{\mathbb{L}}} = |\vec{a_0}| + \varepsilon \frac{\langle \vec{a_0}, \vec{a}_0 \rangle}{|\vec{a_0}|}. \tag{1.10}$$

Let $h(x_0 + x_1\varepsilon)$ be a dual function Fike and Alonso (2009). Then the Taylor series expansion of this dual function $x_0 + x_1\varepsilon = a_0 + a_1\varepsilon$ can be given as:

$$h(a_0 + a_1\varepsilon) = h(a_0) + a_1 h'(a_0)\varepsilon, \tag{1.11}$$

where the prime represents differentiation with respect to x, i.e.

$$h'(x) = h'(x + \varepsilon 0) = \frac{d}{dx}h(x). \tag{1.12}$$

In this paper, we introduce the dual Leonardo numbers, which generalize the classical Leonardo number sequence (Karataş, 2022, 2023; Halıcı, 2019; Halıcı and Curuk, 2019; Alp and Koçer, 2021; Catarino and Borges, 2020; Babadağ and Atasoy, 2024; Babadağ et al., 2024; Atasoy, 2025) by employing the concept of dual numbers. We investigate fundamental properties of these numbers and construct new geometric entities, namely, the dual Leonardo vector and dual Leonardo angle within the framework of dual Lorentzian geometry.

2 Dual Leonardo Number Sequences

In this section, the focus is on dual Leonardo number sequences, with an exposition of their fundamental characteristics (Karataş, 2023; Alp and Koçer, 2021; Babadağ and Atasoy, 2024). The following terms are defined as follows: identities and properties.

Definition 2.1. The dual number of the form

$$\mathcal{DL}e_n = \mathcal{L}e_n + \varepsilon\mathcal{L}e_{n+1} \tag{2.1}$$

is called the n^{th} dual Leonardo number *and* $\varepsilon^2 = 0,\ \varepsilon \neq 0$, where $\mathcal{L}e_n$ is n^{th} Leonardo number.

From definition, the following recurrence relation can be prove

$$\mathcal{DL}e_n = \mathcal{DL}e_{n-1} + \mathcal{DL}e_{n-2} + 1 + \varepsilon, \quad n \geq 2.$$

The few dual Leonardo numbers are given as

$$\mathcal{DL}e_1 = 1 + 3\varepsilon,\ \mathcal{DL}e_2 = 3 + 5\varepsilon, ...$$

Theorem 2.1 (Karataş (2023))**.** *The Binet's like formula for dual Leonardo numbers is*

$$\mathcal{DL}e_n = 2\frac{\varphi^{n+1}\underline{\varphi} - \psi^{n+1}\underline{\psi}}{\sqrt{5}} - 1 - \varepsilon \tag{2.2}$$

where

$$\underline{\varphi} = 1 + \varepsilon\varphi \text{ and } \underline{\psi} = 1 + \varepsilon\psi. \tag{2.3}$$

Proof. From (1.3), (2.1) and (2.3), we find that

$$\begin{aligned}
\mathcal{DL}e_n &= \mathcal{L}e_n + \varepsilon\mathcal{L}e_{n+1} \\
&= 2\frac{\varphi^{n+1} - \psi^{n+1}}{\varphi - \psi} - 1 + \varepsilon\left(2\frac{\varphi^{n+2} - \psi^{n+2}}{\varphi - \psi} - 1\right) \\
&= \frac{2\varphi^{n+1}(1 + \varphi\varepsilon) - 2\psi^{n+1}(1 + \psi\varepsilon)}{\varphi - \psi} - 1 - \varepsilon \\
&= 2\frac{\varphi^{n+1}\underline{\varphi} - \psi^{n+1}\underline{\psi}}{\sqrt{5}} - 1 - \varepsilon.
\end{aligned}$$

This is the desired result. □

For example, for $n = 1$,

$$\begin{aligned}\mathcal{DL}e_1 &= 2\frac{\varphi^2\underline{\varphi} - \psi^2\underline{\psi}}{\sqrt{5}} - 1 - \varepsilon\\ &= 2\frac{(\varphi^2 - \psi^2) + \varepsilon(\varphi^3 - \psi^3)}{\sqrt{5}} - 1 - \varepsilon\\ &= 2\frac{\sqrt{5} + 2\varepsilon\sqrt{5}}{\sqrt{5}} - 1 - \varepsilon\\ &= 1 + 3\varepsilon.\end{aligned}$$

Checking result: $\mathcal{L}e_1 = 1$, $\mathcal{L}e_2 = 3$ and $\mathcal{DL}e_1 = \mathcal{L}e_1 + \varepsilon\mathcal{L}e_2 = 1 + 3\varepsilon$.

Theorem 2.2. *Let $\mathcal{DL}e_n$ be the n^{th} dual Leonardo number. Then,*

$$\mathcal{DL}e_n = \mathcal{DL}e_{n-1} + \mathcal{DL}e_{n-2} + (1 + \varepsilon), \tag{2.4}$$

$$\mathcal{DL}e_n - \mathcal{DL}e_{n+1}\varepsilon = \mathcal{L}e_n, \tag{2.5}$$

$$\begin{aligned}\mathcal{DL}e_n\mathcal{DL}e_m + \mathcal{DL}e_{n+1}\mathcal{DL}e_{m+1} &= (\mathcal{L}e_n\mathcal{L}e_m + \mathcal{L}e_{n+1}\mathcal{L}e_{m+1})\\ &\quad + (\mathcal{L}e_n\mathcal{L}e_{m+1} + \mathcal{L}e_{n+1}\mathcal{L}e_m\\ &\quad + \mathcal{L}e_{n+1}\mathcal{L}e_{m+2} + \mathcal{L}e_{n+2}\mathcal{L}e_{m+1})\varepsilon.\end{aligned} \tag{2.6}$$

Proof. (2.4): Using the definition of dual Leonardo numbers, we have

$$\begin{aligned}\mathcal{DL}e_{n-1} + \mathcal{DL}e_{n-2} &= (\mathcal{L}e_{n-1} + \mathcal{L}e_n\varepsilon) + (\mathcal{L}e_{n-2} + \mathcal{L}e_{n-1}\varepsilon)\\ &= (\mathcal{L}e_{n-1} + \mathcal{L}e_{n-2}) + (\mathcal{L}e_n + \mathcal{L}e_{n-1})\varepsilon.\end{aligned}$$

Since Leonardo numbers satisfy

$$\mathcal{L}e_n = \mathcal{L}e_{n-1} + \mathcal{L}e_{n-2} + 1$$

and

$$\mathcal{L}e_{n+1} = \mathcal{L}e_n + \mathcal{L}e_{n-1} + 1$$

we obtain

$$\begin{aligned}\mathcal{DL}e_{n-1} + \mathcal{DL}e_{n-2} &= (\mathcal{L}e_n - 1) + (\mathcal{L}e_{n+1} - 1)\varepsilon\\ &= \mathcal{DL}e_n - (1 + \varepsilon).\end{aligned}$$

Thus,

$$\mathcal{DL}e_n = \mathcal{DL}e_{n-1} + \mathcal{DL}e_{n-2} + (1 + \varepsilon).$$

(2.5): From (3.1),

$$\begin{aligned}\mathcal{DL}e_n - \mathcal{DL}e_{n+1}\varepsilon &= (\mathcal{L}e_n + \mathcal{L}e_{n+1}\varepsilon) - (\mathcal{L}e_{n+1} + \mathcal{L}e_{n+2}\varepsilon)\varepsilon\\ &= \mathcal{L}e_n + \mathcal{L}e_{n+1}\varepsilon - \mathcal{L}e_{n+1}\varepsilon - \mathcal{L}e_{n+2}\varepsilon^2\\ &= \mathcal{L}e_n.\end{aligned}$$

(2.6): From (3.1),

$$\begin{aligned}\mathcal{DL}e_n\mathcal{DL}e_m &= (\mathcal{L}e_n + \mathcal{L}e_{n+1}\varepsilon)(\mathcal{L}e_m + \mathcal{L}e_{m+1}\varepsilon)\\ &= \mathcal{L}e_n\mathcal{L}e_m + (\mathcal{L}e_n\mathcal{L}e_{m+1} + \mathcal{L}e_{n+1}\mathcal{L}e_m)\varepsilon.\end{aligned}$$

Similarly,

$$\begin{aligned}\mathcal{DL}e_{n+1}\mathcal{DL}e_{m+1} &= (\mathcal{L}e_{n+1} + \mathcal{L}e_{n+2}\varepsilon)(\mathcal{L}e_{m+1} + \mathcal{L}e_{m+2}\varepsilon)\\ &= \mathcal{L}e_{n+1}\mathcal{L}e_{m+1} + (\mathcal{L}e_{n+1}\mathcal{L}e_{m+2} + \mathcal{L}e_{n+2}\mathcal{L}e_{m+1})\varepsilon.\end{aligned}$$

Adding both expressions gives

$$
\begin{aligned}
&\mathcal{DL}e_n\mathcal{DL}e_m + \mathcal{DL}e_{n+1}\mathcal{DL}e_{m+1} \\
&= (\mathcal{L}e_n\mathcal{L}e_m + \mathcal{L}e_{n+1}\mathcal{L}e_{m+1}) \\
&+ (\mathcal{L}e_n\mathcal{L}e_{m+1} + \mathcal{L}e_{n+1}\mathcal{L}e_m + L_{n+1}\mathcal{L}e_{m+2} + \mathcal{L}e_{n+2}\mathcal{L}e_{m+1})\,\varepsilon.
\end{aligned}
$$

The proof is completed. □

3 Dual Leonardo Vector and Angle in Dual Lorentzian Space

In this section, we obtain dual Leonardo vector and dual Lorentzian angle by using Leonardo number sequences in dual Lorentzian spaces $\mathbb{D}_1^3$ (see Fig. 1).

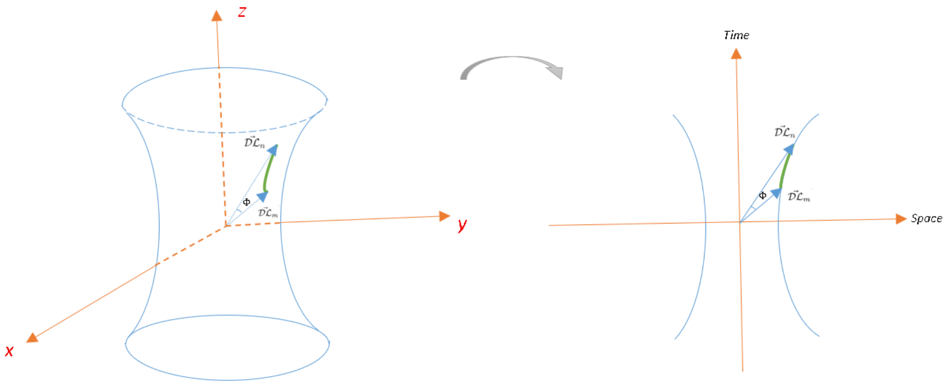

Figure 1: Vector and angle on Lorentzian sphere.

Definition 3.1. The n^{th} dual Leonardo Lorentzian vector $\mathcal{D}\vec{\mathcal{L}}e_n$ is defined as:

$$\mathcal{D}\vec{\mathcal{L}}e_n = \vec{\mathcal{L}}e_n + \varepsilon\vec{\mathcal{L}}e_{n+1}$$

where $\vec{\mathcal{L}}e_n = (\mathcal{L}e_n, \mathcal{L}e_{n+1}, \mathcal{L}e_{n+2})$ is n^{th} Leonardo Lorentzian vector.

Theorem 3.1. *Let $\mathcal{D}\vec{\mathcal{L}}e_n$ and $\mathcal{D}\vec{\mathcal{L}}e_m$ be dual Leonardo Lorentzian vectors in dual Lorentzian spaces $\mathbb{D}_1^3$, the Lorentzian scalar product and the Lorentzian vectoral product are given as:*

$$
\begin{aligned}
\left\langle \mathcal{D}\vec{\mathcal{L}}e_n, \mathcal{D}\vec{\mathcal{L}}e_m \right\rangle_{\mathbb{L}} &= \frac{4}{5}(-1)^{n+1}L_{m-n} + 8L_{m+n+1} + \frac{24}{5}L_{m+n} - 4F_{n+2} - 4F_{m+2} + 1 \\
&+ \left(\frac{4}{5}(-1)^{n+1}L_{m-n} + 16L_{m+n+2} + \frac{48}{5}L_{m+n+1} - 4F_{n+4} - 4F_{m+4} + 2\right)\varepsilon.
\end{aligned} \tag{3.1}
$$

and

$$\mathcal{D}\vec{\mathcal{L}}e_n \wedge_{\mathbb{L}} \mathcal{D}\vec{\mathcal{L}}e_m = \begin{pmatrix} 4(-1)^nF_{m-n} - 2F_{n+1} + 2F_{m+1} \\ 4(-1)^nF_{m-n} - 2F_{n+2} + 2F_{m+2} \\ 4(-1)^nF_{m-n} + 2F_n - 2F_m \end{pmatrix} + \varepsilon \begin{pmatrix} 4(-1)^nF_{m-n} - 2F_{n+3} + 2F_{m+3} \\ 4(-1)^nF_{m-n} - 2F_{n+5} + 2F_{m+5} \\ 4(-1)^nF_{m-n} + 2F_{n+2} - 2F_{m+2} \end{pmatrix}.$$

Proof. Using (1.6) for the Leonardo Lorentzian vectors $\vec{\mathcal{L}e}_n$ and $\vec{\mathcal{L}e}_m$, the Lorentzian scalar product is as follows:

$$\begin{aligned}
\left\langle \vec{\mathcal{L}e}_n, \vec{\mathcal{L}e}_m \right\rangle_{\mathbb{L}} &= -\mathcal{L}e_n\mathcal{L}e_m + \mathcal{L}e_{n+1}\mathcal{L}e_{m+1} + \mathcal{L}e_{n+2}\mathcal{L}e_{m+2} \\
&= -(\frac{2\varphi^{n+1} - 2\psi^{n+1}}{\varphi - \psi} - 1)(\frac{2\varphi^{m+1} - 2\psi^{m+1}}{\varphi - \psi} - 1) \\
&+ (\frac{2\varphi^{n+2} - 2\psi^{n+2}}{\varphi - \psi} - 1)(\frac{2\varphi^{m+2} - 2\psi^{m+2}}{\varphi - \psi} - 1) \\
&+ (\frac{2\varphi^{n+3} - 2\psi^{n+3}}{\varphi - \psi} - 1)(\frac{2\varphi^{m+3} - 2\psi^{m+3}}{\varphi - \psi} - 1) \\
&= \frac{4}{5}(-1)^{n+1}(\varphi^{m-n} + \psi^{m-n}) + \frac{4}{5}\psi^{m+n}(-\psi^2 + \psi^4 + \psi^6) \\
&+ \frac{4}{5}\varphi^{m+n}(-\varphi^2 + \varphi^4 + \varphi^6) + \frac{2}{\sqrt{5}}\psi^m(-\psi + \psi^2 + \psi^3) + \frac{2}{\sqrt{5}}\varphi^m(\varphi - \varphi^2 - \varphi^3) \\
&+ \frac{2}{\sqrt{5}}\psi^n(-\psi + \psi^2 + \psi^3) + \frac{2}{\sqrt{5}}\varphi^n(\varphi - \varphi^2 - \varphi^3) + 1 \\
&= \frac{4}{5}(-1)^{n+1}L_{m-n} + \frac{4}{5}\psi^{m+n}(10\psi + 6) + \frac{4}{5}\varphi^{m+n}(10\varphi + 6) \\
&+ \frac{2}{\sqrt{5}}(\psi^m + \psi^n)(2\psi^2) + \frac{2}{\sqrt{5}}(\varphi^m + \varphi^n)(-2\varphi^2) + 1 \\
&= \frac{4}{5}(-1)^{n+1}L_{m-n} + 8L_{m+n+1} + \frac{24}{5}L_{m+n} - 4F_{n+2} - 4F_{m+2} + 1 \qquad (3.2)
\end{aligned}$$

where L_n is n^{th} Lucas number. Similarly,

$$\begin{aligned}
\left\langle \vec{\mathcal{L}e}_n, \vec{\mathcal{L}e}_{m+1} \right\rangle_{\mathbb{L}} &= \frac{4}{5}(-1)^{n+1}L_{m-n+1} + 8L_{m+n+2} \\
&+ \frac{24}{5}L_{m+n+1} - 4F_{n+2} - 4F_{m+3} + 1 \qquad (3.3)
\end{aligned}$$

and

$$\begin{aligned}
\left\langle \vec{\mathcal{L}e}_{n+1}, \vec{\mathcal{L}e}_m \right\rangle_{\mathbb{L}} &= \frac{4}{5}(-1)^{n+2}L_{m-n-1} + 8L_{m+n+2} \\
&+ \frac{24}{5}L_{m+n+1} - 4F_{n+3} - 4F_{m+2} + 1 \qquad (3.4)
\end{aligned}$$

By using (1.8), (3.2), (3.3) and (3.4), we obtain the dual Lorentzian scalar product as follows:

$$\begin{aligned}
\left\langle \mathcal{D}\vec{\mathcal{L}e}_n, \mathcal{D}\vec{\mathcal{L}e}_m \right\rangle_{\mathbb{L}} &= \left\langle \vec{\mathcal{L}e}_n, \vec{\mathcal{L}e}_m \right\rangle_{\mathbb{L}} + \varepsilon\left(\left\langle \vec{\mathcal{L}e}_n, \vec{\mathcal{L}e}_{m+1} \right\rangle_{\mathbb{L}} + \left\langle \vec{\mathcal{L}e}_{n+1}, \vec{\mathcal{L}e}_m \right\rangle_{\mathbb{L}}\right) \\
&= \frac{4}{5}(-1)^{n+1}L_{m-n} + 8L_{m+n+1} + \frac{24}{5}L_{m+n} - 4F_{n+2} - 4F_{m+2} + 1 \\
&+ \left(\frac{4}{5}(-1)^{n+1}L_{m-n} + 16L_{m+n+2} + \frac{48}{5}L_{m+n+1} - 4F_{n+4} - 4F_{m+4} + 2\right)\varepsilon.
\end{aligned}$$

If we use (1.7) and identity (1.1), the Lorentzian vectoral product is obtained

$$
\begin{aligned}
\vec{\mathcal{L}e}_n \wedge_{\mathbb{L}} \vec{\mathcal{L}e}_m &= \begin{vmatrix} -\vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ \mathcal{L}e_n & \mathcal{L}e_{n+1} & \mathcal{L}e_{n+2} \\ \mathcal{L}e_m & \mathcal{L}e_{m+1} & \mathcal{L}e_{m+2} \end{vmatrix} \\
&= (-\mathcal{L}e_{n+1}\mathcal{L}e_{m+2} + \mathcal{L}e_{n+2}\mathcal{L}e_{m+1}, -\mathcal{L}e_n\mathcal{L}e_{m+2} + \mathcal{L}e_{n+2}\mathcal{L}e_m, \\
&- \mathcal{L}e_{n+1}\mathcal{L}e_m + \mathcal{L}e_{m+1}\mathcal{L}e_n) \\
&= [4(-1)^n F_{m-n} - 2F_{n+1} + 2F_{m+1}]\ \vec{e}_1 \\
&+ [4(-1)^n F_{m-n} - 2F_{n+2} + 2F_{m+2}]\ \vec{e}_2 \\
&+ [4(-1)^n F_{m-n} + 2F_n - 2F_m]\ \vec{e}_3.
\end{aligned}
$$

Similarly

$$
\begin{aligned}
\vec{\mathcal{L}e}_{n+1} \wedge_{\mathbb{L}} \vec{\mathcal{L}e}_m &= [4(-1)^{n+1} F_{m-n-1} - 2F_{n+2} + 2F_{m+1}]\ \vec{e}_1 \\
&+ [4(-1)^{n+1} F_{m-n-1} - 2F_{n+3} + 2F_{m+2}]\ \vec{e}_2 \\
&+ [4(-1)^{n+1} F_{m-n-1} + 2F_{n+1} - 2F_m]\ \vec{e}_3.
\end{aligned}
$$

and

$$
\begin{aligned}
\vec{\mathcal{L}e}_n \wedge_{\mathbb{L}} \vec{\mathcal{L}e}_{m+1} &= [4(-1)^n F_{m-n+1} - 2F_{n+1} + 2F_{m+2}]\ \vec{e}_1 \\
&+ [4(-1)^n F_{m-n+1} - 2F_{n+2} + 2F_{m+3}]\ \vec{e}_2 \\
&+ [4(-1)^n F_{m-n+1} + 2F_n - 2F_{m+1}]\ \vec{e}_3.
\end{aligned}
$$

where $\vec{e}_1,\ \vec{e}_2,\ \vec{e}_3$ are unit direction vectors and

$$
\begin{aligned}
&\mathcal{D}\vec{\mathcal{L}e}_n \wedge_{\mathbb{L}} \mathcal{D}\vec{\mathcal{L}e}_m = \vec{\mathcal{L}e}_n \wedge_{\mathbb{L}} \vec{\mathcal{L}e}_m + \varepsilon(\vec{\mathcal{L}e}_n \wedge_{\mathbb{L}} \vec{\mathcal{L}e}_{m+1} + \vec{\mathcal{L}e}_{n+1} \wedge_{\mathbb{L}} \vec{\mathcal{L}e}_m) \\
&= \begin{pmatrix} 4(-1)^n F_{m-n} - 2F_{n+1} + 2F_{m+1} \\ 4(-1)^n F_{m-n} - 2F_{n+2} + 2F_{m+2} \\ 4(-1)^n F_{m-n} + 2F_n - 2F_m \end{pmatrix} + \varepsilon \begin{pmatrix} 4(-1)^n F_{m-n} - 2F_{n+3} + 2F_{m+3} \\ 4(-1)^n F_{m-n} - 2F_{n+5} + 2F_{m+5} \\ 4(-1)^n F_{m-n} + 2F_{n+2} - 2F_{m+2} \end{pmatrix}.
\end{aligned}
$$

Thus, the proof is completed. □

Corollary 3.2. *The norm of* $\mathcal{D}\vec{\mathcal{L}e}_n$ *is*

$$
\begin{aligned}
\|\mathcal{D}\vec{\mathcal{L}e}_n\|^2 &= \langle \mathcal{D}\vec{\mathcal{L}e}_n, \mathcal{D}\vec{\mathcal{L}e}_n \rangle_{\mathbb{L}} \\
&= \frac{8}{5}(-1)^{n+1} + 8L_{2n+1} + \frac{24}{5}L_{2n} - 8F_{n+2} + 1 \\
&+ \left(\frac{8}{5}(-1)^{n+1} + 16L_{2n+2} + \frac{48}{5}L_{2n+1} - 8F_{n+4} + 2\right)\varepsilon.
\end{aligned}
$$

Proof. The proof is clear from taking $m = n$ in (3.1). □

Proposition 3.1 (Kotelnikov (1895); Study (1901))**.** *Let* $\vec{\mathcal{A}} = \vec{a} + \varepsilon\vec{a}^*$ *be a unit dual Lorentzian vector, then the directed line d that corresponds with* $\vec{\mathcal{A}}$ *has the equation of the form*

$$\vec{r} = \vec{a} \wedge_{\mathbb{L}} \vec{a}^* + \mu\ \vec{a} \tag{3.5}$$

where $0 \le \mu \le 1$.

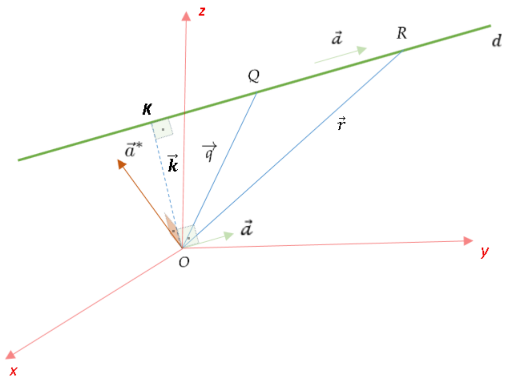

Figure 2: E. Study mapping in $\mathbb{D}^3$, Kotelnikov (1895); Study (1901).

Proof. Let $\vec{\mathcal{A}} = \vec{a} + \varepsilon \vec{a}^*$ be the unit dual vector and if Q and R are points on the corresponding line d, and O is the origin (see Fig. 2), then

$$\vec{OR} = \vec{OQ} + \vec{QR},$$
$$\vec{r} = \vec{q} + \lambda\, \vec{a},$$

where λ is a real parameter. A point R is on the line of vectors $\vec{a}$ and $\vec{a}^*$ if and only if

$$\vec{a} \wedge_{\mathbb{L}} \vec{a}^* = \vec{a} \wedge_{\mathbb{L}} (\vec{q} \wedge_{\mathbb{L}} \vec{a}) = \langle \vec{a}, \vec{a} \rangle_{\mathbb{L}} \vec{q} - \langle \vec{a}, \vec{q} \rangle_{\mathbb{L}} \vec{a}$$

and then

$$\vec{q} = \vec{a} \wedge_{\mathbb{L}} \vec{a}^* + \langle \vec{a}, \vec{q}\, \rangle_{\mathbb{L}}\, \vec{a}$$

and

$$\vec{r} = \vec{a} \wedge_{\mathbb{L}} \vec{a}^* + \langle \vec{a}, \vec{q}\, \rangle_{\mathbb{L}}\, \vec{a} + \lambda\, \vec{a}.$$

By taking $\mu = \langle \vec{a}, \vec{q}\, \rangle_{\mathbb{L}} + \lambda$, we get the result as

$$\vec{r} = \vec{a} \wedge_{\mathbb{L}} \vec{a}^* + \mu\, \vec{a}.$$

□

Theorem 3.3. *Suppose that $\mathcal{D}\vec{\mathcal{L}}e_n$ is a dual Leonardo Lorentzian vector and let*

$$\mathcal{D}\vec{\mathcal{L}}e_n = \vec{\mathcal{L}}e_n + \varepsilon \vec{\mathcal{L}}e_{n+1} = \vec{\mathcal{L}}e_n + \varepsilon \vec{\mathcal{L}}e_n^{\,*} \tag{3.6}$$

be its unitized vector, then the equation of corresponding dual line is

$$\begin{aligned} \vec{r}_n = {} & [4(-1)^n F_{m-n} - 2F_{n+1} + 2F_{m+1} + \mu\, L_n]\, \vec{e}_1 \\ & + [4(-1)^n F_{m-n} - 2F_{n+2} + 2F_{m+2} + \mu\, L_{n+1}]\, \vec{e}_2 \\ & + [4(-1)^n F_{m-n} + 2F_n - 2F_m + \mu\, L_{n+2}]\, \vec{e}_3. \end{aligned}$$

Proof. By using (3.5), we obtain

$$\begin{aligned}
\vec{r_n} &= \vec{\mathcal{L}e}_n \wedge_{\mathbb{L}} \vec{\mathcal{L}e}_{n+1} + \mu\ \vec{\mathcal{L}e}_n, \\
&= \begin{vmatrix} -\vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ \mathcal{L}e_n & \mathcal{L}e_{n+1} & \mathcal{L}e_{n+2} \\ \mathcal{L}e_{n+1} & \mathcal{L}e_{n+2} & \mathcal{L}e_{n+3} \end{vmatrix} + \mu\ (\mathcal{L}e_n \vec{e}_1 + \mathcal{L}e_{n+1}\vec{e}_2 + \mathcal{L}e_{n+2}\vec{e}_3) \\
&= [4(-1)^n F_{m-n} - 2F_{n+1} + 2F_{m+1} + \mu\ \mathcal{L}e_n]\ \vec{e}_1 \\
&+ [4(-1)^n F_{m-n} - 2F_{n+2} + 2F_{m+2} + \mu\ \mathcal{L}e_{n+1}]\ \vec{e}_2 \\
&+ [4(-1)^n F_{m-n} + 2F_n - 2F_m + \mu\ \mathcal{L}e_{n+2}]\ \vec{e}_3.
\end{aligned}$$

□

Definition 3.2. (**Dual center angle**) (Uğurlu and Çalışkan (1996)) Let $\vec{\mathcal{A}} = \vec{a} + \varepsilon\vec{a}^*$ and $\vec{\mathcal{B}} = \vec{b} + \varepsilon\vec{b}^*$ be dual spacelike unit vectors in $\mathbb{D}^3_1$. Given the center angle ϕ and the shortest distance ϕ^*between the directional spacelike lines corresponding to these vectors, the dual number

$$\Phi = \phi + \varepsilon\phi^*$$

is called the dual center angle between these vectors (see Fig. 3).

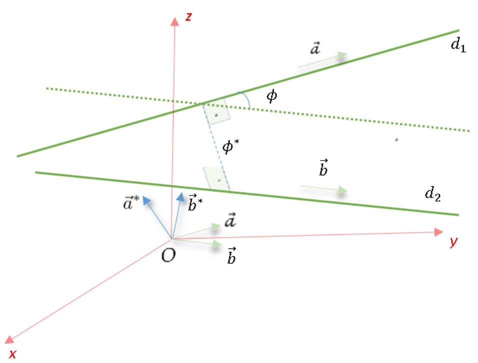

Figure 3: Dual central angle between spacelike lines

Definition 3.3. (Aslan et al., 2020) Let $\vec{\mathcal{A}} = \vec{a} + \varepsilon\vec{a}^*$ and $\vec{\mathcal{B}} = \vec{b} + \varepsilon\vec{b}^*$ be the spacelike dual vectors in $\mathbb{D}^3_1$. The Lorentzian scalar product of these vectors is

$$\langle \vec{\mathcal{A}}, \vec{\mathcal{B}} \rangle_{\mathbb{L}} = \cos\Phi = \cos\phi - \varepsilon\phi^* \sin\phi \tag{3.7}$$

where $\Phi = \phi + \varepsilon\phi^*$ is the dual Lorentzian angle between them.

We will have the following cases for $\vec{\mathcal{A}} = \vec{a} + \varepsilon\vec{a}^* = \mathcal{D}\vec{\mathcal{L}e}_n$ and $\vec{\mathcal{B}} = \vec{b} + \varepsilon\vec{b}^* = \mathcal{D}\vec{\mathcal{L}e}_{\mathrm{m}}$.

Corollary 3.4. *Let $\mathcal{D}\vec{\mathcal{L}e}_n$ and $\mathcal{D}\vec{\mathcal{L}e}_{\mathrm{m}}$ be dual Leonardo vectors in dual Lorentzian spaces, by using* (3.1)*, Definition*(3.2) *and Definition*(3.3)

$$\begin{aligned}
\left\langle \mathcal{D}\vec{\mathcal{L}e}_n, \mathcal{D}\vec{\mathcal{L}e}_m \right\rangle_{\mathbb{L}} &= \frac{4}{5}(-1)^{n+1}L_{m-n} + 8L_{m+n+1} + \frac{24}{5}L_{m+n} - 4F_{n+2} - 4F_{m+2} + 1 \\
&+ \left(\frac{4}{5}(-1)^{n+1}L_{m-n} + 16L_{m+n+2} + \frac{48}{5}L_{m+n+1} - 4F_{n+4} - 4F_{m+4} + 2\right)\varepsilon
\end{aligned}$$

where if we take

$$\cos\phi = \frac{4}{5}(-1)^{n+1}L_{m-n} + 8L_{m+n+1} + \frac{24}{5}L_{m+n} - 4F_{n+2} - 4F_{m+2} + 1$$

and

$$-\phi^* \sin\phi = \frac{4}{5}(-1)^{n+1}L_{m-n} + 16L_{m+n+2} + \frac{48}{5}L_{m+n+1} - 4F_{n+4} - 4F_{m+4} + 2.$$

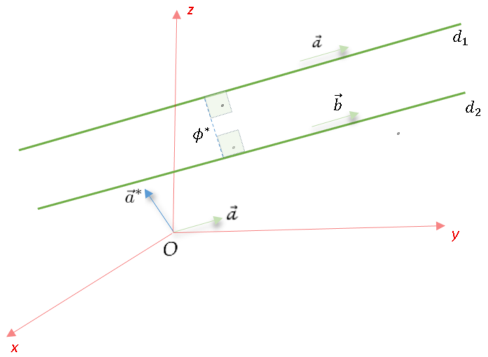

Figure 4: Parallel of dual lines d_1 and d_2

Case 3.5. *If $\phi = 0$ and $\phi^* \neq 0$, then*

$$\left\langle \mathcal{D}\vec{\mathcal{L}}e_n, \mathcal{D}\vec{\mathcal{L}}e_m \right\rangle_{\mathbb{L}} = \cos\Phi = \cos\phi = 1$$

which gives

$$\frac{4}{5}(-1)^{n+1}L_{m-n} + 8L_{m+n+1} + \frac{24}{5}L_{m+n} - 4F_{n+2} - 4F_{m+2} = 0.$$

Thus, corresponding dual Leonardo vectors $\mathcal{D}\vec{\mathcal{L}}e_n$ and $\mathcal{D}\vec{\mathcal{L}}e_{\mathrm{m}}$ are parallel (see Fig. 4).

Case 3.6. *If $\phi^* = 0$ and $\phi \neq 0$, then we obtain*

$$\left\langle \mathcal{D}\vec{\mathcal{L}}e_n, \mathcal{D}\vec{\mathcal{L}}e_m \right\rangle_{\mathbb{L}} = \cos\Phi = \cos\phi$$

which gives

$$\phi = arccos\left(\frac{4}{5}(-1)^{n+1}L_{m-n} + 8L_{m+n+1} + \frac{24}{5}L_{m+n} - 4F_{n+2} - 4F_{m+2} + 1\right)$$

and

$$\frac{4}{5}(-1)^{n+1}L_{m-n} + 16L_{m+n+2} + \frac{48}{5}L_{m+n+1} = 4F_{n+4} + 4F_{m+4} - 2.$$

Thus, corresponding dual Leonardo vectors $\mathcal{D}\vec{\mathcal{L}}e_n$ and $\mathcal{D}\vec{\mathcal{L}}e_{\mathrm{m}}$ intersect each other (see Fig. 5).

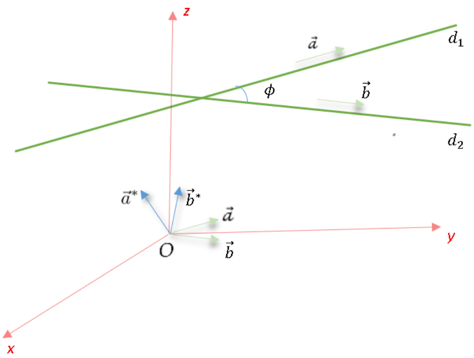

Figure 5: Intersection of dual lines d_1 and d_2

4 Conclusion

This study introduced dual Leonardo sequences, extending classical Leonardo numbers into the dual number system. We established fundamental recurrence relations and algebraic identities. By integrating these sequences into three-dimensional dual Lorentzian space, we defined dual Lorentzian vectors and analyzed their geometric behavior through the formulation of dual Lorentzian angles. The results provide a robust foundation for future research and offer potential applications in kinematics, theoretical physics, and geometric modeling.

This research by providing rigorous definitions for inner and outer products in a dual-Lorentzian space, it establishes a foundation for extending other recursive sequences into non-Euclidean spaces, thereby enriching the literature on number systems.

Despite the theoretical contributions, this study has certain limitations. First, the geometric characterization is restricted to a three-dimensional dual-Lorentzian space ; however, many complex robotic systems require higher-dimensional analysis. Second, while the inner and outer products are well-defined, the physical interpretation of higher-order Leonardo identities in Lorentzian space requires further empirical validation.

Future research will focus on extending these Leonardo vectors to higher-dimensional spaces. The kinematic analysis of mechanisms using dual-Lorentzian angles remain an open and promising field of study.

Disclaimer (Artificial Intelligence)

Author(s) hereby declare that NO generative AI technologies such as Large Language Models (ChatGPT, COPILOT, etc) and text-to-image generators have been used during writing or editing of manuscripts.

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