Original Research Article
A Hierarchical Approach to Nonlinear Rational and φ–Contractive Mappings in Fuzzy Metric Spaces with Applications

Abstract

In this paper, we develop a unified framework of nonlinear contractive mappings in fuzzy metric spaces $(\mathscr{X}, \mathscr{M}, *)$ by integrating classical contraction principles such as Banach-type, Kannan-type, and Reich-type contractions with newly introduced rational and φ–nonlinear contractive conditions. The proposed φ–nonlinear contraction, governed by a control function $\varphi : [0,1] \to [0,1]$ satisfying $\varphi(s) > s$ for all $s \in (0,1)$, provides a significant generalization of standard linear contraction models. Under suitable conditions, we establish existence and uniqueness of fixed points for self-mappings in complete fuzzy metric spaces and develop a hierarchy of contractions that clarifies the structural relationships among the introduced classes. The theoretical results are supported by illustrative examples and applications to nonlinear integral equations and fractional differential equations, demonstrating the effectiveness of the proposed framework in addressing problems arising in nonlinear and applied analysis.

Keywords: Fuzzy metric space; Nonlinear rational contraction; Control function; Fixed point; Integral equations.

MSC (2020): 47H10, 54H25, 46S40, 45G10.

1 Introduction

Fixed point theory constitutes a central and indispensable tool in modern nonlinear analysis, owing to its wide range of applications in differential equations, integral equations, optimization theory, and various branches of applied sciences. The foundation of this theory is the celebrated Banach contraction principle [1], which guarantees the existence and uniqueness of fixed points in complete metric spaces.

Theorem 1.1 (Banach Contraction Principle [1])**.** *Let $(\mathscr{X}, d)$ be a complete metric space and let $\mathscr{T} : \mathscr{X} \to \mathscr{X}$ be a mapping satisfying*

$$d(\mathscr{T}\xi, \mathscr{T}\zeta) \leq \lambda d(\xi, \zeta), \quad \text{for all } \xi, \zeta \in \mathscr{X},$$

where $\lambda \in (0,1)$. Then $\mathscr{T}$ admits a unique fixed point in $\mathscr{X}$.

Over the years, this principle has been extended in several directions to accommodate broader classes of nonlinear mappings. A notable generalization was introduced by Cirić [19], where the contraction condition involves the maximum of several distance expressions.

Theorem 1.2 (Cirić Type Contraction [19])**.** *Let $(\mathscr{X}, d)$ be a complete metric space and let $\mathscr{T} : \mathscr{X} \to \mathscr{X}$ satisfy*

$$d(\mathscr{T}\xi, \mathscr{T}\zeta) \leq \lambda \max\{d(\xi, \zeta), d(\xi, \mathscr{T}\xi), d(\zeta, \mathscr{T}\zeta), d(\xi, \mathscr{T}\zeta), d(\zeta, \mathscr{T}\xi)\},$$

for all $\xi, \zeta \in \mathscr{X}$, where $\lambda \in (0,1)$. Then $\mathscr{T}$ admits a unique fixed point.

Another important extension is the rational contraction introduced by Das and Naik [20], which incorporates nonlinear rational expressions in the contractive condition.

Theorem 1.3 (Rational Contraction [20])**.** *Let $(\mathscr{X}, d)$ be a complete metric space and let $\mathscr{T} : \mathscr{X} \to \mathscr{X}$ satisfy*

$$d(\mathscr{T}\xi, \mathscr{T}\zeta) \leq \alpha \frac{d(\xi, \mathscr{T}\xi) + d(\zeta, \mathscr{T}\zeta)}{1 + d(\xi, \zeta)},$$

for all $\xi, \zeta \in \mathscr{X}$, where $\alpha \in (0,1)$. Then $\mathscr{T}$ admits a unique fixed point.

In order to model uncertainty and imprecision inherent in many real-world problems, the concept of fuzzy metric spaces was introduced. Following the pioneering work of Zadeh [2] on fuzzy sets, Kramosil and Michálek [4] introduced the notion of fuzzy metric spaces, which was later refined and extensively developed by George and Veeramani [5]. These spaces provide a natural generalization of classical metric spaces by incorporating degrees of nearness.

Theorem 1.4 (Grabiec Fixed Point Theorem [7])**.** *Let $(\mathscr{X}, \mathscr{M}, *)$ be a complete fuzzy metric space and let $\mathscr{T} : \mathscr{X} \to \mathscr{X}$ satisfy*

$$\mathscr{M}(\mathscr{T}\xi, \mathscr{T}\zeta, t) \geq \mathscr{M}(\xi, \zeta, kt), \quad \forall \xi, \zeta \in \mathscr{X}, \ t > 0,$$

where $k \in (0,1)$. Then $\mathscr{T}$ admits a unique fixed point in $\mathscr{X}$.

Subsequently, various extensions of fixed point results in fuzzy metric spaces have been established by several authors. In particular, Altun and Türkoğlu [14] and Abbas and Rhoades [15] developed generalized contractive conditions involving the fuzzy metric, significantly broadening the applicability of fixed point theory in fuzzy environments.

Parallel to these developments, generalized metric-type structures such as partial metric spaces and fuzzy partial metric spaces have also attracted considerable attention. The concept of partial metric spaces was introduced by Matthews [34], and later extended to fuzzy settings by Gregori et al. [27] and Sedghi et al. [28]. These frameworks allow a more flexible treatment of self-distance and have important applications in computer science and applied analysis.

Motivated by the above developments, in this paper we investigate a unified class of nonlinear rational contractive mappings in fuzzy metric spaces. The proposed contractive condition involves a nonlinear rational expression defined in terms of the fuzzy metric $\mathscr{M}$ and is designed to strictly generalize several existing contraction types in both classical and fuzzy settings.

Our main objective is to establish existence and uniqueness results for fixed points of self-mappings in complete fuzzy metric spaces under this new framework. The obtained results extend and unify various known theorems in the literature.

Finally, we present an application of our main results to a nonlinear integral equation, demonstrating the effectiveness of the developed theory in solving problems arising in nonlinear analysis [21].

2 Preliminaries

The concept of fuzzy metric spaces provides a natural and effective generalization of classical metric spaces by incorporating uncertainty into the notion of distance. This framework was initially introduced by Kramosil and Michálek [4] in the context of probabilistic metric spaces and was subsequently refined by George and Veeramani [5]. The latter formulation is now widely adopted due to its analytical tractability and suitability for fixed point theory. These developments are fundamentally based on the theory of fuzzy sets introduced by Zadeh [2], where uncertainty is modeled via membership functions.

We begin by recalling the notion of a continuous t–norm, which plays a crucial role in the structure of fuzzy metric spaces.

Definition 2.1 (t–norm [5])**.** A binary operation $* : [0,1] \times [0,1] \to [0,1]$ is called a *continuous t–norm* if for all $a, b, c \in [0,1]$, the following conditions are satisfied:

(T1) $*$ is associative and commutative;

(T2) $a * 1 = a$;

(T3) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$;

(T4) $*$ is continuous on $[0,1] \times [0,1]$.

Standard examples include the product t–norm $a * b = ab$ and the minimum t–norm $a * b = \min\{a, b\}$.

We now recall the definition of a fuzzy metric space in the sense of George and Veeramani.

Definition 2.2 (Fuzzy Metric Space [5, 4])**.** A triple $(\mathscr{X}, \mathscr{M}, *)$ is called a *fuzzy metric space* if $\mathscr{X}$ is a nonempty set, $*$ is a continuous t–norm, and $\mathscr{M} : \mathscr{X} \times \mathscr{X} \times (0, \infty) \to [0, 1]$ is a mapping satisfying, for all $\xi, \zeta, \chi \in \mathscr{X}$ and $t, s > 0$:

(F1) $\mathscr{M}(\xi, \zeta, t) > 0$;

(F2) $\mathscr{M}(\xi, \zeta, t) = 1$ if and only if $\xi = \zeta$;

(F3) $\mathscr{M}(\xi, \zeta, t) = \mathscr{M}(\zeta, \xi, t)$;

(F4) $\mathscr{M}(\xi, \chi, t + s) \geq \mathscr{M}(\xi, \zeta, t) * \mathscr{M}(\zeta, \chi, s)$;

(F5) for each $\xi, \zeta \in \mathscr{X}$, the mapping $t \mapsto \mathscr{M}(\xi, \zeta, t)$ is continuous on $(0, \infty)$.

The value $\mathscr{M}(\xi, \zeta, t)$ represents the degree of nearness between ξ and ζ at time t, where values closer to 1 indicate stronger proximity.

Definition 2.3 (Open Ball and Induced Topology [5])**.** Let $(\mathscr{X}, \mathscr{M}, *)$ be a fuzzy metric space. For $\xi \in \mathscr{X}$, $r \in (0, 1)$, and $t > 0$, the *open ball* centered at ξ is defined by

$$B(\xi, r, t) = \{\zeta \in \mathscr{X} : \mathscr{M}(\xi, \zeta, t) > 1 - r\}.$$

The family of all such open balls generates a topology on $\mathscr{X}$, called the *topology induced by the fuzzy metric.*

Definition 2.4 (Convergence [5])**.** A sequence $\{\xi_n\}$ in $(\mathscr{X}, \mathscr{M}, *)$ is said to *converge* to $\xi \in \mathscr{X}$ if, for every $t > 0$,

$$\lim_{n \to \infty} \mathscr{M}(\xi_n, \xi, t) = 1.$$

In this case, we write $\xi_n \to \xi$.

Definition 2.5 (Cauchy Sequence [5])**.** A sequence $\{\xi_n\}$ in $(\mathscr{X}, \mathscr{M}, *)$ is called a *Cauchy sequence* if for every $\varepsilon \in (0, 1)$ and $t > 0$, there exists $N \in \mathbb{N}$ such that

$$\mathscr{M}(\xi_n, \xi_m, t) > 1 - \varepsilon, \quad \forall m, n \geq N.$$

Definition 2.6 (Completeness [5])**.** A fuzzy metric space $(\mathscr{X}, \mathscr{M}, *)$ is said to be *complete* if every Cauchy sequence converges to a point in $\mathscr{X}$.

We conclude this section with a fundamental property of convergence in fuzzy metric spaces.

Lemma 2.1 (Uniqueness of Limit [5]). In a fuzzy metric space $(\mathscr{X}, \mathscr{M}, *)$, every convergent sequence has a unique limit.

The above notions form the analytical foundation required for establishing fixed point results in fuzzy metric spaces. In particular, convergence and completeness play a crucial role in ensuring the existence and uniqueness of fixed points under generalized contractive conditions.

3 Some New Examples

In this section, we construct illustrative examples of fuzzy metric spaces satisfying the axioms of Kramosil and Michálek [4] as refined by George and Veeramani [5]. These examples demonstrate how classical metric structures can be embedded into the fuzzy framework via suitable transformations involving a time parameter. They also provide insight into convergence behavior and topological equivalence.

Example 3.1. Let $\mathscr{X} = \mathbb{R}$ and define

$$\mathscr{M}(\xi, \zeta, t) = \frac{t}{t + |\xi - \zeta|}, \qquad t > 0,$$

where the t–norm is given by $a * b = ab$.

Proof. We verify that $(\mathscr{X}, \mathscr{M}, *)$ is a fuzzy metric space.

(F1) Since $t + |\xi - \zeta| > 0$, we have

$$\mathscr{M}(\xi, \zeta, t) = \frac{t}{t + |\xi - \zeta|} > 0.$$

(F2) Clearly, $\mathscr{M}(\xi, \zeta, t) = 1$ if and only if $|\xi - \zeta| = 0$, i.e., $\xi = \zeta$.

(F3) Symmetry follows from $|\xi - \zeta| = |\zeta - \xi|$.

(F4) Let $\xi, \eta, \zeta \in \mathscr{X}$ and $t, s > 0$. By the triangle inequality,

$$|\xi - \zeta| \leq |\xi - \eta| + |\eta - \zeta|.$$

Thus,

$$t + s + |\xi - \zeta| \leq t + s + |\xi - \eta| + |\eta - \zeta|.$$

Hence,

$$\mathscr{M}(\xi, \zeta, t + s) = \frac{t + s}{t + s + |\xi - \zeta|} \geq \frac{t + s}{t + s + |\xi - \eta| + |\eta - \zeta|}.$$

On the other hand,

$$\mathscr{M}(\xi, \eta, t)\mathscr{M}(\eta, \zeta, s) = \frac{t}{t + |\xi - \eta|} \cdot \frac{s}{s + |\eta - \zeta|}.$$

Now, using the inequality

$$\frac{t + s}{t + s + a + b} \geq \frac{t}{t + a} \cdot \frac{s}{s + b}, \quad \forall a, b \geq 0,$$

we obtain

$$\mathscr{M}(\xi, \zeta, t + s) \geq \mathscr{M}(\xi, \eta, t) * \mathscr{M}(\eta, \zeta, s).$$

(F5) For fixed $\xi, \zeta \in \mathscr{X}$, the function

$$t \mapsto \frac{t}{t + |\xi - \zeta|}$$

is continuous on $(0, \infty)$.

Therefore, all axioms of a fuzzy metric space are satisfied.

Finally, consider $\xi_n = \frac{1}{n}$. Then for every $t > 0$,

$$\mathscr{M}\left(\tfrac{1}{n}, 0, t\right) = \frac{t}{t + \frac{1}{n}} \to 1,$$

which shows that $\xi_n \to 0$. Hence, the induced topology coincides with the usual topology on $\mathbb{R}$. □

Example 3.2. Let $\mathscr{X} = [0, \infty)$ and define

$$\mathscr{M}(\xi, \zeta, t) = \exp\Big(-\frac{|\xi - \zeta|}{t}\Big), \qquad t > 0,$$

where the t–norm is given by $a * b = ab$.

Proof. We verify that $(\mathscr{X}, \mathscr{M}, *)$ is a fuzzy metric space.

(F1) Since $|\xi - \zeta| \geq 0$ and $t > 0$, we have

$$\mathscr{M}(\xi, \zeta, t) = \exp\Big(-\frac{|\xi - \zeta|}{t}\Big) > 0.$$

(F2) Clearly,

$$\mathscr{M}(\xi, \zeta, t) = 1 \iff |\xi - \zeta| = 0 \iff \xi = \zeta.$$

(F3) Symmetry follows from $|\xi - \zeta| = |\zeta - \xi|$.

(F5) For fixed $\xi, \zeta \in \mathscr{X}$, the function

$$t \mapsto \exp\Big(-\frac{|\xi - \zeta|}{t}\Big)$$

is continuous on $(0, \infty)$.

(F4) Let $\xi, \eta, \zeta \in \mathscr{X}$ and $t, s > 0$. By the triangle inequality,

$$|\xi - \zeta| \leq |\xi - \eta| + |\eta - \zeta|.$$

Hence,

$$\frac{|\xi - \zeta|}{t + s} \leq \frac{|\xi - \eta|}{t + s} + \frac{|\eta - \zeta|}{t + s} \leq \frac{|\xi - \eta|}{t} + \frac{|\eta - \zeta|}{s}.$$

Multiplying by -1 and exponentiating yields

$$\exp\Big(-\frac{|\xi - \zeta|}{t + s}\Big) \geq \exp\Big(-\frac{|\xi - \eta|}{t}\Big) \exp\Big(-\frac{|\eta - \zeta|}{s}\Big).$$

Thus,

$$\mathscr{M}(\xi, \zeta, t + s) \geq \mathscr{M}(\xi, \eta, t) * \mathscr{M}(\eta, \zeta, s).$$

Therefore, all axioms of a fuzzy metric space are satisfied.

Finally, consider the sequence $\xi_n = \frac{1}{n}$. Then for every $t > 0$,

$$\mathscr{M}\Big(\tfrac{1}{n}, 0, t\Big) = \exp\Big(-\frac{1}{nt}\Big) \to 1,$$

which shows that $\xi_n \to 0$ in $(\mathscr{X}, \mathscr{M}, *)$.

Moreover, the sequence $\xi_n = n$ is not Cauchy since

$$\mathscr{M}(n, m, t) = \exp\Big(-\frac{|n - m|}{t}\Big) \to 0 \quad \text{as } |n - m| \to \infty.$$

□

Example 3.3. Let $\mathscr{X} = [0,1]$ and define

$$\mathscr{M}(\xi,\zeta,t) = \frac{t^2}{t^2 + |\xi - \zeta|}, \qquad t > 0,$$

where the t–norm is given by $a * b = ab$.

Proof. We verify that $(\mathscr{X}, \mathscr{M}, *)$ is a fuzzy metric space.

(F1) Since $t^2 + |\xi - \zeta| > 0$, we have

$$\mathscr{M}(\xi,\zeta,t) = \frac{t^2}{t^2 + |\xi - \zeta|} > 0.$$

(F2) Clearly,

$$\mathscr{M}(\xi,\zeta,t) = 1 \iff |\xi - \zeta| = 0 \iff \xi = \zeta.$$

(F3) Symmetry follows from $|\xi - \zeta| = |\zeta - \xi|$.

(F5) For fixed $\xi, \zeta \in \mathscr{X}$, the function

$$t \mapsto \frac{t^2}{t^2 + |\xi - \zeta|}$$

is continuous on $(0, \infty)$.

(F4) Let $\xi, \eta, \zeta \in \mathscr{X}$ and $t, s > 0$. Set

$$a = |\xi - \eta|, \quad b = |\eta - \zeta|.$$

Then, by the triangle inequality,

$$|\xi - \zeta| \leq a + b.$$

Hence,

$$\mathscr{M}(\xi,\zeta,t+s) = \frac{(t+s)^2}{(t+s)^2 + |\xi - \zeta|} \geq \frac{(t+s)^2}{(t+s)^2 + a + b}.$$

On the other hand,

$$\mathscr{M}(\xi,\eta,t)\mathscr{M}(\eta,\zeta,s) = \frac{t^2}{t^2 + a} \cdot \frac{s^2}{s^2 + b}.$$

Now, using the inequality

$$\frac{(t+s)^2}{(t+s)^2 + a + b} \geq \frac{t^2}{t^2 + a} \cdot \frac{s^2}{s^2 + b}, \quad \forall a, b \geq 0,$$

we obtain

$$\mathscr{M}(\xi,\zeta,t+s) \geq \mathscr{M}(\xi,\eta,t) * \mathscr{M}(\eta,\zeta,s).$$

Thus all axioms of a fuzzy metric space are satisfied.

Finally, consider $\xi_n = \frac{n}{n+1}$. Then for every $t > 0$,

$$\mathscr{M}\Big(\tfrac{n}{n+1}, 1, t\Big) = \frac{t^2}{t^2 + \frac{1}{n+1}} \to 1,$$

and hence $\xi_n \to 1$.

Moreover, since $[0,1]$ is complete in the usual metric and the above fuzzy metric is induced by a continuous monotone transformation of $|\xi - \zeta|$, it follows that $(\mathscr{X}, \mathscr{M}, *)$ is complete. $\square$

4 New Contractions

In this section, we introduce a unified framework of nonlinear contractive mappings in fuzzy metric spaces. The aim is to generalize classical contraction principles while maintaining mathematical consistency and ensuring applicability to fixed point theory.

Throughout this section, let $(\mathscr{X}, \mathscr{M}, *)$ be a fuzzy metric space.

Definition 4.1 (Comparison function)**.** A function $\phi : [0,1] \to [0,1]$ is called a comparison function if it satisfies:

1. ϕ is continuous and nondecreasing;
2. $\phi(r) > r$ for all $r \in (0,1)$;
3. $\phi(1) = 1$.

Remark 4.1. In contrast to classical metric contractions, the condition $\phi(r) > r$ reflects the nature of fuzzy metrics, where larger values of $\mathscr{M}$ indicate stronger nearness. Thus, such mappings enhance fuzzy proximity.

Definition 4.2 (Banach-type fuzzy contraction)**.** A mapping $\mathscr{T} : \mathscr{X} \to \mathscr{X}$ is said to be a Banach-type fuzzy contraction if

$$\mathscr{M}(\mathscr{T}\xi, \mathscr{T}\zeta, t) \geq \phi\big(\mathscr{M}(\xi, \zeta, t)\big), \quad \forall \xi, \zeta \in \mathscr{X},\ t > 0.$$

Definition 4.3 (Kannan-type fuzzy contraction)**.** A mapping $\mathscr{T} : \mathscr{X} \to \mathscr{X}$ is said to be a Kannan-type fuzzy contraction if

$$\mathscr{M}(\mathscr{T}\xi, \mathscr{T}\zeta, t) \geq \phi\left(\frac{\mathscr{M}(\xi, \mathscr{T}\xi, t) + \mathscr{M}(\zeta, \mathscr{T}\zeta, t)}{2}\right), \quad \forall \xi, \zeta \in \mathscr{X},\ t > 0.$$

Definition 4.4 (Reich-type fuzzy contraction)**.** Let $a, b, c \geq 0$ with $a + b + c = 1$. A mapping $\mathscr{T} : \mathscr{X} \to \mathscr{X}$ is said to be a Reich-type fuzzy contraction if

$$\mathscr{M}(\mathscr{T}\xi, \mathscr{T}\zeta, t) \geq \phi\big(a\,\mathscr{M}(\xi, \zeta, t) + b\,\mathscr{M}(\xi, \mathscr{T}\xi, t) + c\,\mathscr{M}(\zeta, \mathscr{T}\zeta, t)\big), \quad \forall \xi, \zeta \in \mathscr{X},\ t > 0.$$

Definition 4.5 (Selective generalized fuzzy contraction)**.** A mapping $\mathscr{T} : \mathscr{X} \to \mathscr{X}$ is called a selective generalized fuzzy contraction if it satisfies at least one of the Banach-type, Kannan-type, or Reich-type conditions.

Definition 4.6 (Type I: Nonlinear rational contraction)**.** A mapping $\mathscr{T} : \mathscr{X} \to \mathscr{X}$ is said to satisfy a Type I nonlinear rational contraction if

$$\mathscr{M}(\mathscr{T}\xi, \mathscr{T}\zeta, t) \geq \frac{\kappa\,\mathscr{M}(\xi, \zeta, t)}{1 + \lambda\big(1 - \mathscr{M}(\xi, \mathscr{T}\xi, t)\big) + \mu\big(1 - \mathscr{M}(\zeta, \mathscr{T}\zeta, t)\big)},$$

for all $\xi, \zeta \in \mathscr{X}$ and $t > 0$, where $\kappa \in (0,1)$ and $\lambda, \mu \geq 0$.

Definition 4.7 (Type II: Pure nonlinear rational contraction)**.** A mapping $\mathscr{T} : \mathscr{X} \to \mathscr{X}$ is said to satisfy a Type II nonlinear rational contraction if

$$\mathscr{M}(\mathscr{T}\xi, \mathscr{T}\zeta, t) \geq \frac{\mathscr{M}(\xi, \zeta, t)}{1 + \lambda\big(1 - \mathscr{M}(\xi, \mathscr{T}\xi, t)\big) + \mu\big(1 - \mathscr{M}(\zeta, \mathscr{T}\zeta, t)\big)},$$

for all $\xi, \zeta \in \mathscr{X}$ and $t > 0$, where $\lambda, \mu \geq 0$.

Definition 4.8 (φ–nonlinear contraction)**.** A mapping $\mathscr{T} : \mathscr{X} \to \mathscr{X}$ is called a φ–nonlinear contraction if

$$\mathscr{M}(\mathscr{T}\xi, \mathscr{T}\zeta, t) \geq \varphi\big(\mathscr{M}(\xi, \zeta, t)\big), \quad \forall \xi, \zeta \in \mathscr{X},\ t > 0,$$

where $\varphi : [0, 1] \to [0, 1]$ is a comparison function.

Proposition 4.1. Let $\mathscr{T}$ satisfy any of the above contractive conditions. Then for all $\xi, \zeta \in \mathscr{X}$ and $t > 0$,

$$0 < \mathscr{M}(\mathscr{T}\xi, \mathscr{T}\zeta, t) \leq 1.$$

Proof. Since $(\mathscr{X}, \mathscr{M}, *)$ is a fuzzy metric space, one has $0 < \mathscr{M}(\xi, \zeta, t) \leq 1$. Each contractive condition involves operations (averages, products, rational expressions, or functions ϕ, φ) that preserve the interval $[0, 1]$. Hence the result follows. □

Proposition 4.2. Let $\mathscr{T}$ be a φ–nonlinear contraction. Then for all $\xi \neq \zeta$,

$$\mathscr{M}(\mathscr{T}\xi, \mathscr{T}\zeta, t) > \mathscr{M}(\xi, \zeta, t).$$

Proof. By definition,

$$\mathscr{M}(\mathscr{T}\xi, \mathscr{T}\zeta, t) \geq \varphi(\mathscr{M}(\xi, \zeta, t)).$$

Since $\varphi(s) > s$ for all $s \in (0, 1)$, the conclusion follows. □

Remark 4.2. This property shows that φ–nonlinear contractions strictly increase fuzzy proximity, which is the fundamental mechanism ensuring convergence of iterative sequences.

Proposition 4.3. Let $\{\xi_n\}$ be the Picard sequence defined by $\xi_{n+1} = \mathscr{T}\xi_n$. If $\mathscr{T}$ is a φ–nonlinear contraction, then

$$\mathscr{M}(\xi_{n+1}, \xi_n, t) \geq \varphi\big(\mathscr{M}(\xi_n, \xi_{n-1}, t)\big), \quad \forall n \geq 1.$$

Proof. Apply the φ–contractive condition to the pair (ξ_n, ξ_{n-1}) and use $\xi_{n+1} = \mathscr{T}\xi_n$. □

Remark 4.3. By induction, one obtains

$$\mathscr{M}(\xi_{n+1}, \xi_n, t) \geq \varphi^n(\mathscr{M}(\xi_1, \xi_0, t)),$$

which provides a quantitative description of the convergence behavior.

Proposition 4.4. For Type I and Type II contractions, the denominator satisfies

$$1 \leq 1 + \lambda(1 - \mathscr{M}) + \mu(1 - \mathscr{M}) \leq 1 + \lambda + \mu.$$

Proof. Since $0 < \mathscr{M} \leq 1$, we have $0 \leq 1 - \mathscr{M} \leq 1$. Substituting into the expression yields the desired bounds. □

Remark 4.4. The above estimate guarantees that the rational expressions are well-defined and uniformly bounded.

Example 1. Let $\mathscr{X} = \mathbb{R}$ and define

$$\mathscr{M}(\xi, \zeta, t) = \frac{t}{t + |\xi - \zeta|}, \quad t > 0.$$

Define $\mathscr{T}\xi = \frac{\xi}{2}$.

Then

$$|\mathscr{T}\xi - \mathscr{T}\zeta| = \frac{1}{2}|\xi - \zeta|.$$

Hence

$$\mathscr{M}(\mathscr{T}\xi, \mathscr{T}\zeta, t) = \frac{t}{t + \frac{1}{2}|\xi - \zeta|} = \frac{2t}{2t + |\xi - \zeta|}.$$

Let $s = \mathscr{M}(\xi, \zeta, t) = \frac{t}{t+|\xi-\zeta|}$. Then

$$|\xi - \zeta| = \frac{t(1-s)}{s}.$$

Substituting, we obtain

$$\mathscr{M}(\mathscr{T}\xi, \mathscr{T}\zeta, t) = \frac{2s}{1+s}.$$

Define $\phi(s) = \frac{2s}{1+s}$. Since $\phi(s) > s$ for all $s \in (0, 1)$ and $\phi(1) = 1$, we have

$$\mathscr{M}(\mathscr{T}\xi, \mathscr{T}\zeta, t) = \phi(\mathscr{M}(\xi, \zeta, t)).$$

Thus $\mathscr{T}$ is a Banach-type fuzzy contraction.

Example 2. Let $\mathscr{X} = [0, 1]$ and define

$$\mathscr{M}(\xi, \zeta, t) = \exp\left(-\frac{|\xi - \zeta|}{t}\right), \quad t > 0.$$

Define $\mathscr{T}\xi = \frac{\xi+1}{2}$.

Then

$$|\mathscr{T}\xi - \mathscr{T}\zeta| = \frac{1}{2}|\xi - \zeta|.$$

Hence

$$\mathscr{M}(\mathscr{T}\xi, \mathscr{T}\zeta, t) = \exp\left(-\frac{|\xi - \zeta|}{2t}\right) = \sqrt{\mathscr{M}(\xi, \zeta, t)}.$$

Define $\varphi(s) = \sqrt{s}$. Then φ is continuous, increasing, satisfies $\varphi(1) = 1$, and

$$\varphi(s) > s \quad \text{for all } s \in (0, 1).$$

Thus

$$\mathscr{M}(\mathscr{T}\xi, \mathscr{T}\zeta, t) = \varphi(\mathscr{M}(\xi, \zeta, t)),$$

and hence $\mathscr{T}$ is a φ–nonlinear contraction.

Example 3. Let $\mathscr{X} = [0,1]$ and define

$$\mathscr{M}(\xi,\zeta,t) = \frac{t}{t+|\xi-\zeta|}, \quad t>0.$$

Define $\mathscr{T}\xi = \frac{\xi}{3}$.
Then

$$|\mathscr{T}\xi - \mathscr{T}\zeta| = \frac{1}{3}|\xi-\zeta|.$$

Hence

$$\mathscr{M}(\mathscr{T}\xi,\mathscr{T}\zeta,t) = \frac{t}{t+\frac{1}{3}|\xi-\zeta|} = \frac{3t}{3t+|\xi-\zeta|}.$$

Let $s = \mathscr{M}(\xi,\zeta,t) = \frac{t}{t+|\xi-\zeta|}$. Then

$$|\xi-\zeta| = \frac{t(1-s)}{s}.$$

Substituting, we obtain

$$\mathscr{M}(\mathscr{T}\xi,\mathscr{T}\zeta,t) = \frac{3s}{2s+1}.$$

Now observe that for all $s \in (0,1)$,

$$\frac{3s}{2s+1} > s.$$

Moreover, since $0 < s \le 1$, we have

$$\frac{3s}{2s+1} \ge \frac{\kappa s}{1+\lambda(1-s)},$$

for suitable constants $\kappa \in (0,1)$ and $\lambda > 0$.
Thus

$$\mathscr{M}(\mathscr{T}\xi,\mathscr{T}\zeta,t) \ge \frac{\kappa\,\mathscr{M}(\xi,\zeta,t)}{1+\lambda\big(1-\mathscr{M}(\xi,\zeta,t)\big)},$$

which shows that $\mathscr{T}$ satisfies a Type I nonlinear rational contraction.

Example 4. Let $\mathscr{X} = [0,1]$ be endowed with the fuzzy metric

$$\mathscr{M}(\xi,\zeta,t) = \frac{t}{t+|\xi-\zeta|}, \quad t>0,$$

and define $\mathscr{T} : \mathscr{X} \to \mathscr{X}$ by

$$\mathscr{T}\xi = \frac{\xi+1}{3}.$$

Then

$$|\mathscr{T}\xi - \mathscr{T}\zeta| = \frac{1}{3}|\xi-\zeta|.$$

Hence

$$\mathscr{M}(\mathscr{T}\xi,\mathscr{T}\zeta,t) = \frac{t}{t+\frac{1}{3}|\xi-\zeta|} = \frac{3t}{3t+|\xi-\zeta|}.$$

Now,

$$|\xi - \mathscr{T}\xi| = \left|\xi - \frac{\xi+1}{3}\right| = \frac{|2\xi - 1|}{3} \leq \frac{1}{3},$$

and similarly,

$$|\zeta - \mathscr{T}\zeta| \leq \frac{1}{3}.$$

Therefore,

$$\mathscr{M}(\xi, \mathscr{T}\xi, t) = \frac{t}{t + |\xi - \mathscr{T}\xi|} \geq \frac{t}{t + \frac{1}{3}},$$

and

$$\mathscr{M}(\zeta, \mathscr{T}\zeta, t) \geq \frac{t}{t + \frac{1}{3}}.$$

Hence,

$$\frac{\mathscr{M}(\xi, \mathscr{T}\xi, t) + \mathscr{M}(\zeta, \mathscr{T}\zeta, t)}{2} \geq \frac{t}{t + \frac{1}{3}}.$$

Let $s = \frac{t}{t+\frac{1}{3}} \in (0, 1)$. Then define

$$\phi(s) = \frac{3s}{2s + 1}.$$

It is easy to verify that ϕ is continuous, increasing, satisfies $\phi(1) = 1$, and $\phi(s) > s$ for all $s \in (0, 1)$.

Moreover, since

$$\mathscr{M}(\mathscr{T}\xi, \mathscr{T}\zeta, t) = \frac{3t}{3t + |\xi - \zeta|} \geq \phi\left(\frac{\mathscr{M}(\xi, \mathscr{T}\xi, t) + \mathscr{M}(\zeta, \mathscr{T}\zeta, t)}{2}\right),$$

it follows that $\mathscr{T}$ satisfies the Kannan-type fuzzy contraction.

Thus, $\mathscr{T}$ is a Kannan-type contraction and hence belongs to the class of selective generalized fuzzy contractions.

5 Main Results

In this section, we establish the structural relationships among the classes of nonlinear contractions introduced earlier and derive the corresponding fixed point results. In particular, we demonstrate that stronger rational contractive conditions imply weaker ones, and we show that the φ–nonlinear framework ensures convergence of the Picard iteration intrinsically.

Throughout this section, let $(\mathscr{X}, \mathscr{M}, *)$ be a fuzzy metric space and let $\mathscr{T} : \mathscr{X} \to \mathscr{X}$ be a self-mapping.

Theorem 5.1. *Every Type II nonlinear rational contraction is a Type I nonlinear rational contraction, i.e.,*

$$\text{Type II} \Rightarrow \text{Type I}.$$

Proof. Assume that $\mathscr{T}$ satisfies the Type II condition:

$$\mathscr{M}(\mathscr{T}\xi, \mathscr{T}\zeta, t) \geq \frac{\mathscr{M}(\xi, \zeta, t)}{1 + \lambda(1 - \mathscr{M}(\xi, \mathscr{T}\xi, t)) + \mu(1 - \mathscr{M}(\zeta, \mathscr{T}\zeta, t))}.$$

Let $\kappa \in (0,1)$. Since $0 < \mathscr{M}(\xi,\zeta,t) \le 1$, we have

$$\kappa \mathscr{M}(\xi,\zeta,t) \le \mathscr{M}(\xi,\zeta,t).$$

Since the denominator is strictly positive, it follows that

$$\mathscr{M}(\mathscr{T}\xi,\mathscr{T}\zeta,t) \ge \frac{\kappa \mathscr{M}(\xi,\zeta,t)}{1+\lambda(1-\mathscr{M}(\xi,\mathscr{T}\xi,t))+\mu(1-\mathscr{M}(\zeta,\mathscr{T}\zeta,t))}.$$

Thus $\mathscr{T}$ satisfies the Type I condition. □

Remark 5.1. The Type II condition is strictly stronger than the Type I condition, as it provides a sharper lower bound without the scaling factor κ.

Theorem 5.2. *Every Banach-type fuzzy contraction is a φ–nonlinear contraction.*

Proof. Let $(\mathscr{X},\mathscr{M},*)$ be a fuzzy metric space and let $\mathscr{T}:\mathscr{X}\to\mathscr{X}$ be a Banach-type fuzzy contraction. Then there exists a comparison function $\phi:[0,1]\to[0,1]$ such that

$$\mathscr{M}(\mathscr{T}\xi,\mathscr{T}\zeta,t) \ge \phi\big(\mathscr{M}(\xi,\zeta,t)\big), \quad \forall \xi,\zeta\in\mathscr{X},\ t>0.$$

Define $\varphi:[0,1]\to[0,1]$ by $\varphi(s)=\phi(s)$ for all $s\in[0,1]$. Since ϕ is continuous and nondecreasing, it follows that φ is also continuous and nondecreasing. Moreover, for every $s\in(0,1)$, one has $\varphi(s)=\phi(s)>s$, and clearly $\varphi(1)=\phi(1)=1$. Hence φ satisfies all the conditions of a comparison function.

Substituting $\varphi=\phi$ into the given contractive condition, we obtain

$$\mathscr{M}(\mathscr{T}\xi,\mathscr{T}\zeta,t) \ge \varphi\big(\mathscr{M}(\xi,\zeta,t)\big), \quad \forall \xi,\zeta\in\mathscr{X},\ t>0.$$

Therefore, $\mathscr{T}$ satisfies the definition of a φ–nonlinear contraction. This completes the proof. □

Remark 5.2. Thus the class of φ–nonlinear contractions properly generalizes the Banach-type fuzzy contractions.

Theorem 5.3. *Let $(\mathscr{X},\mathscr{M},*)$ be a fuzzy metric space and let $\mathscr{T}:\mathscr{X}\to\mathscr{X}$ be a φ–nonlinear contraction satisfying*

$$\lim_{n\to\infty}\varphi^n(s)=1 \quad \textit{for all } s\in(0,1).$$

Then for any $\xi_0\in\mathscr{X}$, the Picard sequence $\xi_{n+1}=\mathscr{T}\xi_n$ is Cauchy.

Proof. Let $\xi_{n+1}=\mathscr{T}\xi_n$. Then, by the φ–contractive condition,

$$\mathscr{M}(\xi_{n+1},\xi_n,t)=\mathscr{M}(\mathscr{T}\xi_n,\mathscr{T}\xi_{n-1},t)\ge\varphi\big(\mathscr{M}(\xi_n,\xi_{n-1},t)\big).$$

By induction, it follows that

$$\mathscr{M}(\xi_{n+1},\xi_n,t)\ge\varphi^n\big(\mathscr{M}(\xi_1,\xi_0,t)\big).$$

Let $s_0=\mathscr{M}(\xi_1,\xi_0,t)\in(0,1]$. By hypothesis,

$$\varphi^n(s_0)\to 1 \quad \text{as } n\to\infty,$$

and hence

$$\mathscr{M}(\xi_{n+1}, \xi_n, t) \to 1.$$

Now let $\varepsilon \in (0, 1)$ and fix $t > 0$. Since

$$\mathscr{M}(\xi_k, \xi_{k+1}, t) \to 1,$$

there exists $N \in \mathbb{N}$ such that

$$\mathscr{M}(\xi_k, \xi_{k+1}, t) > 1 - \varepsilon \quad \text{for all } k \geq N.$$

Let $m > n \geq N$. Using the triangular property (F4) of the fuzzy metric, we obtain

$$\mathscr{M}(\xi_n, \xi_m, t) \geq \mathscr{M}\left(\xi_n, \xi_{n+1}, \frac{t}{m-n}\right) * \mathscr{M}\left(\xi_{n+1}, \xi_{n+2}, \frac{t}{m-n}\right) * \cdots * \mathscr{M}\left(\xi_{m-1}, \xi_m, \frac{t}{m-n}\right).$$

Since $\mathscr{M}(\xi_k, \xi_{k+1}, t)$ is nondecreasing in t, it follows that

$$\mathscr{M}\left(\xi_k, \xi_{k+1}, \frac{t}{m-n}\right) \geq \mathscr{M}(\xi_k, \xi_{k+1}, t) > 1 - \varepsilon \quad \text{for all } k \geq N.$$

Therefore,

$$\mathscr{M}(\xi_n, \xi_m, t) > (1 - \varepsilon)^{m-n}.$$

Since ε is arbitrary, this implies that for every $\delta \in (0, 1)$ there exists N such that

$$\mathscr{M}(\xi_n, \xi_m, t) > 1 - \delta \quad \text{for all } m, n \geq N.$$

Hence $\{\xi_n\}$ is a Cauchy sequence in $(\mathscr{X}, \mathscr{M}, *)$. □

Theorem 5.4. *Let $(\mathscr{X}, \mathscr{M}, *)$ be a complete fuzzy metric space and let $\mathscr{T} : \mathscr{X} \to \mathscr{X}$ be a φ–nonlinear contraction satisfying*

$$\lim_{n\to\infty} \varphi^n(s) = 1 \quad \text{for all } s \in (0, 1).$$

Then $\mathscr{T}$ admits a unique fixed point in $\mathscr{X}$.

Proof. Let $\xi_{n+1} = \mathscr{T}\xi_n$. By the previous theorem, $\{\xi_n\}$ is a Cauchy sequence. Since $(\mathscr{X}, \mathscr{M}, *)$ is complete, there exists $\xi^* \in \mathscr{X}$ such that $\xi_n \to \xi^*$.

We first show that ξ^* is a fixed point of $\mathscr{T}$. For any $t > 0$, using the triangular property (F4), we have

$$\mathscr{M}(\mathscr{T}\xi^*, \xi^*, t) \geq \mathscr{M}(\mathscr{T}\xi^*, \mathscr{T}\xi_n, t/2) * \mathscr{M}(\mathscr{T}\xi_n, \xi^*, t/2).$$

Now, by the contractive condition,

$$\mathscr{M}(\mathscr{T}\xi^*, \mathscr{T}\xi_n, t/2) \geq \varphi\big(\mathscr{M}(\xi^*, \xi_n, t/2)\big).$$

Also, since $\xi_{n+1} = \mathscr{T}\xi_n$ and $\xi_n \to \xi^*$, we have

$$\mathscr{M}(\mathscr{T}\xi_n, \xi^*, t/2) = \mathscr{M}(\xi_{n+1}, \xi^*, t/2) \to 1.$$

Further, since $\xi_n \to \xi^*$, we obtain

$$\mathscr{M}(\xi^*, \xi_n, t/2) \to 1,$$

and hence

$$\varphi\big(\mathscr{M}(\xi^*, \xi_n, t/2)\big) \to 1.$$

Combining these, we get

$$\mathscr{M}(\mathscr{T}\xi^*, \xi^*, t) = 1.$$

Therefore, $\mathscr{T}\xi^* = \xi^*$, and hence ξ^* is a fixed point.

To prove uniqueness, suppose that η^* is another fixed point of $\mathscr{T}$. Then

$$\mathscr{M}(\xi^*, \eta^*, t) = \mathscr{M}(\mathscr{T}\xi^*, \mathscr{T}\eta^*, t) \geq \varphi\big(\mathscr{M}(\xi^*, \eta^*, t)\big).$$

If $\xi^* \neq \eta^*$, then $\mathscr{M}(\xi^*, \eta^*, t) \in (0, 1)$ and hence

$$\varphi\big(\mathscr{M}(\xi^*, \eta^*, t)\big) > \mathscr{M}(\xi^*, \eta^*, t),$$

which is a contradiction. Therefore, $\xi^* = \eta^*$.

Thus, $\mathscr{T}$ admits a unique fixed point in $\mathscr{X}$. □

Theorem 5.5. *The following inclusions hold:*

Type II $\subseteq$ *Type I,* *Banach-type* $\subseteq$ *φ–nonlinear,* *Kannan-type, Reich-type* $\subseteq$ *Selective generalized.*

Proof. The inclusion Type II $\subseteq$ Type I follows from Theorem **??**, where it was shown that any mapping satisfying the Type II nonlinear rational contractive condition also satisfies the Type I condition by introducing a suitable constant $\kappa \in (0, 1)$.

The inclusion Banach-type $\subseteq$ φ–nonlinear follows from the fact that every Banach-type fuzzy contraction is governed by a comparison function ϕ satisfying $\phi(s) > s$ for $s \in (0, 1)$. Defining $\varphi = \phi$, the Banach-type condition immediately yields the φ–nonlinear contractive condition.

Finally, by definition, a selective generalized fuzzy contraction is a mapping that satisfies at least one of the Banach-type, Kannan-type, or Reich-type contractive conditions. Hence every Kannan-type or Reich-type contraction belongs to the class of selective generalized contractions.

Therefore, all stated inclusions hold. □

6 Illustrative Examples

In this section, we present examples illustrating the applicability, sharpness, and strictness of the theoretical results established earlier.

Example 6.1 (Strict inclusion: φ–nonlinear $\not\subseteq$ Banach-type). Let $\mathscr{X} = [0, 1]$ and define the fuzzy metric

$$\mathscr{M}(\xi, \zeta, t) = \frac{t}{t + |\xi - \zeta|}, \quad t > 0.$$

Define $\mathscr{T} : \mathscr{X} \to \mathscr{X}$ by

$$\mathscr{T}\xi = \sqrt{\xi}.$$

Then

$$|\mathscr{T}\xi - \mathscr{T}\zeta| = |\sqrt{\xi} - \sqrt{\zeta}| \leq \sqrt{|\xi - \zeta|}.$$

Hence

$$\mathscr{M}(\mathscr{T}\xi, \mathscr{T}\zeta, t) \geq \frac{t}{t + \sqrt{|\xi - \zeta|}}.$$

Let $s = \mathscr{M}(\xi, \zeta, t) = \frac{t}{t+|\xi-\zeta|}$. Then

$$|\xi - \zeta| = \frac{t(1-s)}{s}.$$

Thus

$$\mathscr{M}(\mathscr{T}\xi, \mathscr{T}\zeta, t) \geq \frac{t}{t + \sqrt{\frac{t(1-s)}{s}}}.$$

Define

$$\varphi(s) = \frac{t}{t + \sqrt{\frac{t(1-s)}{s}}}.$$

Then φ is continuous, increasing, $\varphi(1) = 1$, and $\varphi(s) > s$ for all $s \in (0, 1)$. Hence $\mathscr{T}$ is a φ–nonlinear contraction.

However, $\mathscr{T}$ is not Banach-type. Indeed, near $\xi = 0$,

$$|\sqrt{\xi} - \sqrt{\zeta}| \approx \frac{1}{2\sqrt{\xi}}|\xi - \zeta|,$$

which prevents any uniform comparison function of Banach-type from existing.

Therefore,

$$\text{Banach-type} \subsetneq \varphi\text{–nonlinear}.$$

Example 6.2. Let $\mathscr{X} = [0, 1]$ and define

$$\mathscr{M}(\xi, \zeta, t) = \frac{t}{t + |\xi - \zeta|}, \quad t > 0.$$

Define $\mathscr{T}\xi = \frac{\xi}{3}$. Then

$$|\mathscr{T}\xi - \mathscr{T}\zeta| = \frac{1}{3}|\xi - \zeta|.$$

Hence

$$\mathscr{M}(\mathscr{T}\xi, \mathscr{T}\zeta, t) = \frac{3t}{3t + |\xi - \zeta|}.$$

Let $s = \frac{t}{t+|\xi-\zeta|}$. Then

$$\mathscr{M}(\mathscr{T}\xi, \mathscr{T}\zeta, t) = \frac{3s}{2s + 1}.$$

Since $\frac{3s}{2s+1} > s$ for all $s \in (0, 1)$, one can choose $\lambda > 0$ such that

$$\frac{3s}{2s + 1} \geq \frac{s}{1 + \lambda(1 - s)}.$$

Thus $\mathscr{T}$ satisfies the Type II nonlinear rational contraction.

The unique fixed point is $\xi^* = 0$.

Example 6.3. Let $\mathscr{X} = [0, 1]$ and define

$$\mathscr{T}\xi = \frac{\xi + 1}{2}.$$

Then

$$|\mathscr{T}\xi - \mathscr{T}\zeta| = \frac{1}{2}|\xi - \zeta|.$$

Hence

$$\mathscr{M}(\mathscr{T}\xi, \mathscr{T}\zeta, t) = \frac{t}{t + \frac{1}{2}|\xi - \zeta|}.$$

Let $s = \mathscr{M}(\xi, \zeta, t)$. Then

$$\mathscr{M}(\mathscr{T}\xi, \mathscr{T}\zeta, t) = \frac{2s}{1+s}.$$

Define $\phi(s) = \frac{2s}{1+s}$. Then $\phi(s) > s$ for all $s \in (0,1)$ and $\phi(1) = 1$. Hence $\mathscr{T}$ is a Banach-type fuzzy contraction.

The unique fixed point is $\xi^* = 1$.

Example 6.4. Let $C([0,1])$ be equipped with the fuzzy metric

$$\mathscr{M}(\phi, \psi, t) = \frac{t}{t + \|\phi - \psi\|_\infty}.$$

Define

$$(\mathscr{T}\phi)(x) = \int_0^1 \kappa(x, y, \phi(y))\, dy,$$

where

$$|\kappa(x, y, \xi) - \kappa(x, y, \zeta)| \leq \frac{1}{4}|\xi - \zeta|.$$

Then

$$\|\mathscr{T}\phi - \mathscr{T}\psi\|_\infty \leq \frac{1}{4}\|\phi - \psi\|_\infty.$$

Let $s = \mathscr{M}(\phi, \psi, t)$. Then

$$\mathscr{M}(\mathscr{T}\phi, \mathscr{T}\psi, t) = \frac{4s}{1+3s}.$$

Define

$$\varphi(s) = \frac{4s}{1+3s}.$$

Then $\varphi(s) > s$ for all $s \in (0,1)$ and $\varphi(1) = 1$. Hence $\mathscr{T}$ is a φ–nonlinear contraction. Therefore, the integral equation admits a unique solution.

7 Applications

In this section, we demonstrate the applicability of the main fixed point theorem for φ–nonlinear contractions by deriving existence and uniqueness results for nonlinear integral equations and fractional differential equations.

Theorem 7.1 (Existence and uniqueness for a nonlinear integral equation)**.** *Let* $\kappa : [0,1] \times [0,1] \times [0,1] \to [0,1]$ *be continuous and satisfy*

$$|\kappa(x, y, \xi) - \kappa(x, y, \zeta)| \leq \frac{1}{4}|\xi - \zeta|, \qquad \forall\, x, y \in [0,1],\ \xi, \zeta \in [0,1].$$

Then the nonlinear integral equation

$$\phi(x) = \int_0^1 \kappa(x, y, \phi(y))\, dy$$

admits a unique solution $\phi^* \in C([0,1])$*.*

Proof. Let $\mathscr{X} = C([0,1])$ endowed with the supremum norm $\|\cdot\|_\infty$, and define the operator $\mathscr{T} : \mathscr{X} \to \mathscr{X}$ by

$$(\mathscr{T}\phi)(x) = \int_0^1 \kappa(x, y, \phi(y))\, dy.$$

Since κ is continuous, the mapping $\mathscr{T}$ is well defined.

Consider on $\mathscr{X}$ the fuzzy metric

$$\mathscr{M}(\phi, \psi, t) = \frac{t}{t + \|\phi - \psi\|_\infty}, \qquad t > 0,$$

with the product t–norm $a * b = ab$. Then $(\mathscr{X}, \mathscr{M}, *)$ is a complete fuzzy metric space.

Now let $\phi, \psi \in \mathscr{X}$. For each $x \in [0,1]$, we have

$$|(\mathscr{T}\phi)(x) - (\mathscr{T}\psi)(x)| = \left| \int_0^1 \big(\kappa(x, y, \phi(y)) - \kappa(x, y, \psi(y))\big)\, dy \right|.$$

Using the given Lipschitz condition on κ, it follows that

$$|(\mathscr{T}\phi)(x) - (\mathscr{T}\psi)(x)| \le \int_0^1 \frac{1}{4}|\phi(y) - \psi(y)|\, dy \le \frac{1}{4}\|\phi - \psi\|_\infty.$$

Taking the supremum over $x \in [0,1]$, we obtain

$$\|\mathscr{T}\phi - \mathscr{T}\psi\|_\infty \le \frac{1}{4}\|\phi - \psi\|_\infty.$$

Consequently,

$$\mathscr{M}(\mathscr{T}\phi, \mathscr{T}\psi, t) = \frac{t}{t + \|\mathscr{T}\phi - \mathscr{T}\psi\|_\infty} \ge \frac{t}{t + \frac{1}{4}\|\phi - \psi\|_\infty}.$$

Let

$$s = \mathscr{M}(\phi, \psi, t) = \frac{t}{t + \|\phi - \psi\|_\infty}.$$

Then

$$\|\phi - \psi\|_\infty = \frac{t(1-s)}{s},$$

and hence

$$\mathscr{M}(\mathscr{T}\phi, \mathscr{T}\psi, t) \ge \frac{t}{t + \frac{1}{4} \cdot \frac{t(1-s)}{s}} = \frac{4s}{1 + 3s}.$$

Define

$$\varphi(s) = \frac{4s}{1 + 3s}, \qquad s \in [0,1].$$

Then φ is continuous, nondecreasing, satisfies $\varphi(1) = 1$, and

$$\varphi(s) - s = \frac{3s(1-s)}{1 + 3s} > 0 \qquad \text{for all } s \in (0,1).$$

Therefore,

$$\mathscr{M}(\mathscr{T}\phi, \mathscr{T}\psi, t) \ge \varphi\big(\mathscr{M}(\phi, \psi, t)\big),$$

showing that $\mathscr{T}$ is a φ–nonlinear contraction.

Hence, by the main fixed point theorem, $\mathscr{T}$ admits a unique fixed point $\phi^* \in C([0,1])$. This fixed point is precisely the unique solution of the given nonlinear integral equation. □

Theorem 7.2 (Fractional differential equation)**.** *Let $F : [0,1] \times \mathbb{R} \to \mathbb{R}$ be continuous and satisfy*

$$|F(x,\xi) - F(x,\zeta)| \le L|\xi - \zeta|, \qquad \forall\, x \in [0,1],\ \xi,\zeta \in \mathbb{R},$$

where

$$0 < L < \Gamma(\alpha+1), \qquad 0 < \alpha < 1.$$

Then the Caputo fractional differential equation

$${}^{C}D^{\alpha}\phi(x) = F(x,\phi(x)), \qquad 0 < \alpha < 1, \qquad \phi(0) = \phi_0,$$

admits a unique solution $\phi^* \in C([0,1])$.

Proof. It is well known that the above Caputo fractional differential equation is equivalent to the Volterra integral equation

$$\phi(x) = \phi_0 + \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} F(t,\phi(t))\, dt.$$

Define the operator $\mathscr{T} : C([0,1]) \to C([0,1])$ by

$$(\mathscr{T}\phi)(x) = \phi_0 + \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} F(t,\phi(t))\, dt.$$

Since F is continuous, the mapping $\mathscr{T}$ is well defined on $C([0,1])$.

Equip $C([0,1])$ with the fuzzy metric

$$\mathscr{M}(\phi,\psi,t) = \frac{t}{t + \|\phi-\psi\|_\infty}, \qquad t > 0,$$

together with the product t–norm. Then $(C([0,1]),\mathscr{M},*)$ is complete.

Now let $\phi,\psi \in C([0,1])$. For each $x \in [0,1]$, we obtain

$$|(\mathscr{T}\phi)(x) - (\mathscr{T}\psi)(x)| \le \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} |F(t,\phi(t)) - F(t,\psi(t))|\, dt.$$

Using the Lipschitz condition on F, this yields

$$|(\mathscr{T}\phi)(x) - (\mathscr{T}\psi)(x)| \le \frac{L}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} |\phi(t)-\psi(t)|\, dt \le \frac{L}{\Gamma(\alpha)} \|\phi-\psi\|_\infty \int_0^x (x-t)^{\alpha-1}\, dt.$$

Since

$$\int_0^x (x-t)^{\alpha-1}\, dt = \frac{x^\alpha}{\alpha} \le \frac{1}{\alpha},$$

it follows that

$$|(\mathscr{T}\phi)(x) - (\mathscr{T}\psi)(x)| \le \frac{L}{\Gamma(\alpha+1)} \|\phi-\psi\|_\infty.$$

Taking the supremum over $x \in [0,1]$, we obtain

$$\|\mathscr{T}\phi - \mathscr{T}\psi\|_\infty \le \frac{L}{\Gamma(\alpha+1)} \|\phi-\psi\|_\infty.$$

Set

$$q = \frac{L}{\Gamma(\alpha+1)}.$$

By assumption, $0 < q < 1$.

Hence

$$\mathscr{M}(\mathscr{T}\phi, \mathscr{T}\psi, t) = \frac{t}{t + \|\mathscr{T}\phi - \mathscr{T}\psi\|_\infty} \geq \frac{t}{t + q\|\phi - \psi\|_\infty}.$$

Let

$$s = \mathscr{M}(\phi, \psi, t) = \frac{t}{t + \|\phi - \psi\|_\infty}.$$

Then

$$\|\phi - \psi\|_\infty = \frac{t(1-s)}{s},$$

and therefore

$$\mathscr{M}(\mathscr{T}\phi, \mathscr{T}\psi, t) \geq \frac{t}{t + q\frac{t(1-s)}{s}} = \frac{s}{q + (1-q)s}.$$

Define

$$\varphi(s) = \frac{s}{q + (1-q)s}, \qquad s \in [0, 1].$$

Since $0 < q < 1$, the function φ is continuous, nondecreasing, satisfies $\varphi(1) = 1$, and

$$\varphi(s) - s = \frac{s(1-q)(1-s)}{q + (1-q)s} > 0 \qquad \text{for all } s \in (0, 1).$$

Thus,

$$\mathscr{M}(\mathscr{T}\phi, \mathscr{T}\psi, t) \geq \varphi\big(\mathscr{M}(\phi, \psi, t)\big),$$

which shows that $\mathscr{T}$ is a φ–nonlinear contraction.

Therefore, by the main fixed point theorem, $\mathscr{T}$ has a unique fixed point $\phi^* \in C([0, 1])$. This fixed point is precisely the unique solution of the fractional differential equation. □

Remark 7.1. The above applications show that φ–nonlinear contractions provide a unified and effective framework for studying both nonlinear integral equations and fractional differential equations. In particular, the nonlinear control functions

$$\varphi(s) = \frac{4s}{1 + 3s} \qquad \text{and} \qquad \varphi(s) = \frac{s}{q + (1-q)s}$$

with $0 < q < 1$ satisfy

$$\varphi(1) = 1 \qquad \text{and} \qquad \varphi(s) > s \quad \text{for all } s \in (0, 1),$$

and therefore fit naturally into the abstract fixed point framework developed in this paper.

8 Conclusion

In this paper, we have developed a unified framework of nonlinear contractive mappings in fuzzy metric spaces by systematically integrating classical contraction principles, including Banach-type, Kannan-type, and Reich-type contractions, with newly introduced rational and φ–nonlinear contractive conditions. The proposed φ–nonlinear contraction serves as a flexible and powerful generalization that captures both linear and nonlinear contractive behaviors within a single analytical structure. The main fixed point theorem ensures the existence and uniqueness of fixed points for self-mappings satisfying these conditions in complete fuzzy metric spaces. Furthermore, the established hierarchy of contractions provides a clear structural relationship among various contractive classes, thereby strengthening the theoretical foundation and coherence of the framework.

The validity and applicability of the developed theory are supported by illustrative examples on real intervals, as well as by applications to nonlinear integral equations and fractional differential equations, demonstrating its effectiveness in addressing problems arising in applied analysis. In particular, the nonlinear control function

$$\varphi(s) = \frac{4s}{1 + 3s}$$

highlights the capability of the framework to handle non-classical contractive behavior beyond standard linear models. The results obtained in this work open several promising directions for future research, including extensions to multivalued mappings, hybrid rational-type contractions, and more general settings such as fuzzy normed spaces, intuitionistic fuzzy metric spaces, and probabilistic metric spaces. These extensions are expected to further broaden both the theoretical scope and practical applicability of the proposed framework.

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