Original Research Article
A Hierarchical Approach to Nonlinear Rational and φ–Contractive Mappings in Fuzzy Metric Spaces with Applications

Abstract

In this paper, we develop a unified framework of nonlinear contractive mappings in fuzzy metric spaces $(\mathscr{X}, \mathscr{M}, *)$ by integrating classical contraction principles such as Banach-type, Kannan-type, and Reich-type contractions with newly introduced rational and φ–nonlinear contractive conditions. The proposed φ–nonlinear contraction, governed by a control function $\varphi : [0,1] \to [0,1]$ satisfying $\varphi(s) > s$ for all $s \in (0,1)$, provides a significant generalization of standard linear contraction models. Under suitable conditions, we establish existence and uniqueness of fixed points for self-mappings in complete fuzzy metric spaces and develop a hierarchy of contractions that clarifies the structural relationships among the introduced classes. The theoretical results are supported by illustrative examples and applications to nonlinear integral equations and fractional differential equations, demonstrating the effectiveness of the proposed framework in addressing problems arising in nonlinear and applied analysis.

Keywords: Fuzzy metric space; Nonlinear rational contraction; Control function; Fixed point; Integral equations.

MSC (2020): 47H10, 54H25, 46S40, 45G10.

1 Introduction

Fixed point theory constitutes a central and indispensable tool in modern nonlinear analysis, owing to its wide range of applications in differential equations, integral equations, optimization theory, and various branches of applied sciences. The foundation of this theory is the celebrated Banach contraction principle [1], which guarantees the existence and uniqueness of fixed points in complete metric spaces.

Theorem 1.1 (Banach Contraction Principle [1]). *Let $(\mathscr{X}, d)$ be a complete metric space and let $\mathscr{T} : \mathscr{X} \to \mathscr{X}$ be a mapping satisfying*

$$d(\mathscr{T}\xi, \mathscr{T}\zeta) \leq \lambda d(\xi, \zeta), \quad \text{for all } \xi, \zeta \in \mathscr{X},$$

where $\lambda \in (0, 1)$. Then $\mathscr{T}$ admits a unique fixed point in $\mathscr{X}$.

Over the years, this principle has been extended in several directions to accommodate broader classes of nonlinear mappings. A notable generalization was introduced by Cirić [19], where the contraction condition involves the maximum of several distance expressions.

Theorem 1.2 (Cirić Type Contraction [19]). *Let $(\mathscr{X}, d)$ be a complete metric space and let $\mathscr{T} : \mathscr{X} \to \mathscr{X}$ satisfy*

$$d(\mathscr{T}\xi, \mathscr{T}\zeta) \leq \lambda \max\{d(\xi, \zeta), d(\xi, \mathscr{T}\xi), d(\zeta, \mathscr{T}\zeta), d(\xi, \mathscr{T}\zeta), d(\zeta, \mathscr{T}\xi)\},$$

for all $\xi, \zeta \in \mathscr{X}$, where $\lambda \in (0, 1)$. Then $\mathscr{T}$ admits a unique fixed point.

Another important extension is the rational contraction introduced by Das and Naik [20], which incorporates nonlinear rational expressions in the contractive condition.

Theorem 1.3 (Rational Contraction [20]). *Let $(\mathscr{X}, d)$ be a complete metric space and let $\mathscr{T} : \mathscr{X} \to \mathscr{X}$ satisfy*

$$d(\mathscr{T}\xi, \mathscr{T}\zeta) \leq \alpha \frac{d(\xi, \mathscr{T}\xi) + d(\zeta, \mathscr{T}\zeta)}{1 + d(\xi, \zeta)},$$

for all $\xi, \zeta \in \mathscr{X}$, where $\alpha \in (0, 1)$. Then $\mathscr{T}$ admits a unique fixed point.

In order to model uncertainty and imprecision inherent in many real-world problems, the concept of fuzzy metric spaces was introduced. Following the pioneering work of Zadeh [2] on fuzzy sets, Kramosil and Michálek [4] introduced the notion of fuzzy metric spaces, which was later refined and extensively developed by George and Veeramani [5]. These spaces provide a natural generalization of classical metric spaces by incorporating degrees of nearness.

Theorem 1.4 (Grabiec Fixed Point Theorem [7]). *Let $(\mathscr{X}, \mathscr{M}, *)$ be a complete fuzzy metric space and let $\mathscr{T} : \mathscr{X} \to \mathscr{X}$ satisfy*

$$\mathscr{M}(\mathscr{T}\xi, \mathscr{T}\zeta, t) \geq \mathscr{M}(\xi, \zeta, kt), \quad \forall \xi, \zeta \in \mathscr{X},\ t > 0,$$

where $k \in (0, 1)$. Then $\mathscr{T}$ admits a unique fixed point in $\mathscr{X}$.

Subsequently, various extensions of fixed point results in fuzzy metric spaces have been established by several authors. In particular, Altun and Türkoğlu [14] and Abbas and Rhoades [15] developed generalized contractive conditions involving the fuzzy metric, significantly broadening the applicability of fixed point theory in fuzzy environments.

Parallel to these developments, generalized metric-type structures such as partial metric spaces and fuzzy partial metric spaces have also attracted considerable attention. The concept of partial metric spaces was introduced by Matthews [34], and later extended to fuzzy settings by Gregori et al. [27] and Sedghi et al. [28]. These frameworks allow a more flexible treatment of self-distance and have important applications in computer science and applied analysis.

Motivated by the above developments, in this paper we investigate a unified class of nonlinear rational contractive mappings in fuzzy metric spaces. The proposed contractive condition involves a nonlinear rational expression defined in terms of the fuzzy metric $\mathscr{M}$ and is designed to strictly generalize several existing contraction types in both classical and fuzzy settings.

Our main objective is to establish existence and uniqueness results for fixed points of self-mappings in complete fuzzy metric spaces under this new framework. The obtained results extend and unify various known theorems in the literature.

Finally, we present an application of our main results to a nonlinear integral equation, demonstrating the effectiveness of the developed theory in solving problems arising in nonlinear analysis [21].

2 Preliminaries

The concept of fuzzy metric spaces provides a natural and effective generalization of classical metric spaces by incorporating uncertainty into the notion of distance. This framework was initially introduced by Kramosil and Michálek [4] in the context of probabilistic metric spaces and was subsequently refined by George and Veeramani [5]. The latter formulation is now widely adopted due to its analytical tractability and suitability for fixed point theory. These developments are fundamentally based on the theory of fuzzy sets introduced by Zadeh [2], where uncertainty is modeled via membership functions.

We begin by recalling the notion of a continuous t–norm, which plays a crucial role in the structure of fuzzy metric spaces.

Definition 2.1 (t–norm [5])**.** A binary operation $* : [0,1] \times [0,1] \to [0,1]$ is called a *continuous t–norm* if for all $a, b, c \in [0,1]$, the following conditions are satisfied:

(T1) $*$ is associative and commutative;

(T2) $a * 1 = a$;

(T3) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$;

(T4) $*$ is continuous on $[0,1] \times [0,1]$.

Standard examples include the product t–norm $a * b = ab$ and the minimum t–norm $a * b = \min\{a, b\}$.

We now recall the definition of a fuzzy metric space in the sense of George and Veeramani.

Definition 2.2 (Fuzzy Metric Space [5, 4])**.** A triple $(\mathscr{X}, \mathscr{M}, *)$ is called a *fuzzy metric space* if $\mathscr{X}$ is a nonempty set, $*$ is a continuous t–norm, and $\mathscr{M} : \mathscr{X} \times \mathscr{X} \times (0, \infty) \to [0, 1]$ is a mapping satisfying, for all $\xi, \zeta, \chi \in \mathscr{X}$ and $t, s > 0$:

(F1) $\mathscr{M}(\xi, \zeta, t) > 0$;

(F2) $\mathscr{M}(\xi, \zeta, t) = 1$ if and only if $\xi = \zeta$;

(F3) $\mathscr{M}(\xi, \zeta, t) = \mathscr{M}(\zeta, \xi, t)$;

(F4) $\mathscr{M}(\xi, \chi, t + s) \geq \mathscr{M}(\xi, \zeta, t) * \mathscr{M}(\zeta, \chi, s)$;

(F5) for each $\xi, \zeta \in \mathscr{X}$, the mapping $t \mapsto \mathscr{M}(\xi, \zeta, t)$ is continuous on $(0, \infty)$.

The value $\mathscr{M}(\xi, \zeta, t)$ represents the degree of nearness between ξ and ζ at time t, where values closer to 1 indicate stronger proximity.

Definition 2.3 (Open Ball and Induced Topology [5])**.** Let $(\mathscr{X}, \mathscr{M}, *)$ be a fuzzy metric space. For $\xi \in \mathscr{X}$, $r \in (0, 1)$, and $t > 0$, the *open ball* centered at ξ is defined by

$$B(\xi, r, t) = \{\zeta \in \mathscr{X} : \mathscr{M}(\xi, \zeta, t) > 1 - r\}.$$

The family of all such open balls generates a topology on $\mathscr{X}$, called the *topology induced by the fuzzy metric.*

Definition 2.4 (Convergence [5])**.** A sequence $\{\xi_n\}$ in $(\mathscr{X}, \mathscr{M}, *)$ is said to *converge* to $\xi \in \mathscr{X}$ if, for every $t > 0$,

$$\lim_{n \to \infty} \mathscr{M}(\xi_n, \xi, t) = 1.$$

In this case, we write $\xi_n \to \xi$.

Definition 2.5 (Cauchy Sequence [5])**.** A sequence $\{\xi_n\}$ in $(\mathscr{X}, \mathscr{M}, *)$ is called a *Cauchy sequence* if for every $\varepsilon \in (0, 1)$ and $t > 0$, there exists $N \in \mathbb{N}$ such that

$$\mathscr{M}(\xi_n, \xi_m, t) > 1 - \varepsilon, \quad \forall m, n \geq N.$$

Definition 2.6 (Completeness [5])**.** A fuzzy metric space $(\mathscr{X}, \mathscr{M}, *)$ is said to be *complete* if every Cauchy sequence converges to a point in $\mathscr{X}$.

We conclude this section with a fundamental property of convergence in fuzzy metric spaces.

Lemma 2.1 (Uniqueness of Limit [5]). In a fuzzy metric space $(\mathscr{X}, \mathscr{M}, *)$, every convergent sequence has a unique limit.

The above notions form the analytical foundation required for establishing fixed point results in fuzzy metric spaces. In particular, convergence and completeness play a crucial role in ensuring the existence and uniqueness of fixed points under generalized contractive conditions.

3 Some New Examples

In this section, we construct illustrative examples of fuzzy metric spaces satisfying the axioms of Kramosil and Michálek [4] as refined by George and Veeramani [5]. These examples demonstrate how classical metric structures can be embedded into the fuzzy framework via suitable transformations involving a time parameter. They also provide insight into convergence behavior and topological equivalence.

Example 3.1. Let $\mathscr{X} = \mathbb{R}$ and define

$$\mathscr{M}(\xi, \zeta, t) = \frac{t}{t + |\xi - \zeta|}, \qquad t > 0,$$

where the t–norm is given by $a * b = ab$.

Proof. We verify that $(\mathscr{X}, \mathscr{M}, *)$ is a fuzzy metric space.

(F1) Since $t + |\xi - \zeta| > 0$, we have

$$\mathscr{M}(\xi, \zeta, t) = \frac{t}{t + |\xi - \zeta|} > 0.$$

(F2) Clearly, $\mathscr{M}(\xi, \zeta, t) = 1$ if and only if $|\xi - \zeta| = 0$, i.e., $\xi = \zeta$.

(F3) Symmetry follows from $|\xi - \zeta| = |\zeta - \xi|$.

(F4) Let $\xi, \eta, \zeta \in \mathscr{X}$ and $t, s > 0$. By the triangle inequality,

$$|\xi - \zeta| \leq |\xi - \eta| + |\eta - \zeta|.$$

Thus,

$$t + s + |\xi - \zeta| \leq t + s + |\xi - \eta| + |\eta - \zeta|.$$

Hence,

$$\mathscr{M}(\xi, \zeta, t + s) = \frac{t + s}{t + s + |\xi - \zeta|} \geq \frac{t + s}{t + s + |\xi - \eta| + |\eta - \zeta|}.$$

On the other hand,

$$\mathscr{M}(\xi, \eta, t)\mathscr{M}(\eta, \zeta, s) = \frac{t}{t + |\xi - \eta|} \cdot \frac{s}{s + |\eta - \zeta|}.$$

Now, using the inequality

$$\frac{t + s}{t + s + a + b} \geq \frac{t}{t + a} \cdot \frac{s}{s + b}, \quad \forall a, b \geq 0,$$

we obtain

$$\mathscr{M}(\xi, \zeta, t + s) \geq \mathscr{M}(\xi, \eta, t) * \mathscr{M}(\eta, \zeta, s).$$

(F5) For fixed $\xi, \zeta \in \mathscr{X}$, the function

$$t \mapsto \frac{t}{t + |\xi - \zeta|}$$

is continuous on $(0, \infty)$.

Therefore, all axioms of a fuzzy metric space are satisfied.

Finally, consider $\xi_n = \frac{1}{n}$. Then for every $t > 0$,

$$\mathscr{M}\left(\tfrac{1}{n}, 0, t\right) = \frac{t}{t + \frac{1}{n}} \to 1,$$

which shows that $\xi_n \to 0$. Hence, the induced topology coincides with the usual topology on $\mathbb{R}$. □

Example 3.2. Let $\mathscr{X} = [0, \infty)$ and define

$$\mathscr{M}(\xi, \zeta, t) = \exp\Big(-\frac{|\xi - \zeta|}{t}\Big), \qquad t > 0,$$

where the t–norm is given by $a * b = ab$.

Proof. We verify that $(\mathscr{X}, \mathscr{M}, *)$ is a fuzzy metric space.

(F1) Since $|\xi - \zeta| \geq 0$ and $t > 0$, we have

$$\mathscr{M}(\xi, \zeta, t) = \exp\Big(-\frac{|\xi - \zeta|}{t}\Big) > 0.$$

(F2) Clearly,

$$\mathscr{M}(\xi, \zeta, t) = 1 \iff |\xi - \zeta| = 0 \iff \xi = \zeta.$$

(F3) Symmetry follows from $|\xi - \zeta| = |\zeta - \xi|$.

(F5) For fixed $\xi, \zeta \in \mathscr{X}$, the function

$$t \mapsto \exp\Big(-\frac{|\xi - \zeta|}{t}\Big)$$

is continuous on $(0, \infty)$.

(F4) Let $\xi, \eta, \zeta \in \mathscr{X}$ and $t, s > 0$. By the triangle inequality,

$$|\xi - \zeta| \leq |\xi - \eta| + |\eta - \zeta|.$$

Hence,

$$\frac{|\xi - \zeta|}{t + s} \leq \frac{|\xi - \eta|}{t + s} + \frac{|\eta - \zeta|}{t + s} \leq \frac{|\xi - \eta|}{t} + \frac{|\eta - \zeta|}{s}.$$

Multiplying by -1 and exponentiating yields

$$\exp\Big(-\frac{|\xi - \zeta|}{t + s}\Big) \geq \exp\Big(-\frac{|\xi - \eta|}{t}\Big)\exp\Big(-\frac{|\eta - \zeta|}{s}\Big).$$

Thus,

$$\mathscr{M}(\xi, \zeta, t + s) \geq \mathscr{M}(\xi, \eta, t) * \mathscr{M}(\eta, \zeta, s).$$

Therefore, all axioms of a fuzzy metric space are satisfied.

Finally, consider the sequence $\xi_n = \frac{1}{n}$. Then for every $t > 0$,

$$\mathscr{M}\Big(\tfrac{1}{n}, 0, t\Big) = \exp\Big(-\frac{1}{nt}\Big) \to 1,$$

which shows that $\xi_n \to 0$ in $(\mathscr{X}, \mathscr{M}, *)$.

Moreover, the sequence $\xi_n = n$ is not Cauchy since

$$\mathscr{M}(n, m, t) = \exp\Big(-\frac{|n - m|}{t}\Big) \to 0 \quad \text{as } |n - m| \to \infty.$$

□

Example 3.3. Let $\mathscr{X} = [0, 1]$ and define

$$\mathscr{M}(\xi, \zeta, t) = \frac{t^2}{t^2 + |\xi - \zeta|}, \qquad t > 0,$$

where the t–norm is given by $a * b = ab$.

Proof. We verify that $(\mathscr{X}, \mathscr{M}, *)$ is a fuzzy metric space.

(F1) Since $t^2 + |\xi - \zeta| > 0$, we have

$$\mathscr{M}(\xi, \zeta, t) = \frac{t^2}{t^2 + |\xi - \zeta|} > 0.$$

(F2) Clearly,

$$\mathscr{M}(\xi, \zeta, t) = 1 \iff |\xi - \zeta| = 0 \iff \xi = \zeta.$$

(F3) Symmetry follows from $|\xi - \zeta| = |\zeta - \xi|$.

(F5) For fixed $\xi, \zeta \in \mathscr{X}$, the function

$$t \mapsto \frac{t^2}{t^2 + |\xi - \zeta|}$$

is continuous on $(0, \infty)$.

(F4) Let $\xi, \eta, \zeta \in \mathscr{X}$ and $t, s > 0$. Set

$$a = |\xi - \eta|, \quad b = |\eta - \zeta|.$$

Then, by the triangle inequality,

$$|\xi - \zeta| \leq a + b.$$

Hence,

$$\mathscr{M}(\xi, \zeta, t + s) = \frac{(t + s)^2}{(t + s)^2 + |\xi - \zeta|} \geq \frac{(t + s)^2}{(t + s)^2 + a + b}.$$

On the other hand,

$$\mathscr{M}(\xi, \eta, t)\mathscr{M}(\eta, \zeta, s) = \frac{t^2}{t^2 + a} \cdot \frac{s^2}{s^2 + b}.$$

Now, using the inequality

$$\frac{(t + s)^2}{(t + s)^2 + a + b} \geq \frac{t^2}{t^2 + a} \cdot \frac{s^2}{s^2 + b}, \quad \forall a, b \geq 0,$$

we obtain

$$\mathscr{M}(\xi, \zeta, t + s) \geq \mathscr{M}(\xi, \eta, t) * \mathscr{M}(\eta, \zeta, s).$$

Thus all axioms of a fuzzy metric space are satisfied.

Finally, consider $\xi_n = \frac{n}{n+1}$. Then for every $t > 0$,

$$\mathscr{M}\Big(\tfrac{n}{n+1}, 1, t\Big) = \frac{t^2}{t^2 + \frac{1}{n+1}} \to 1,$$

and hence $\xi_n \to 1$.

Moreover, since $[0, 1]$ is complete in the usual metric and the above fuzzy metric is induced by a continuous monotone transformation of $|\xi - \zeta|$, it follows that $(\mathscr{X}, \mathscr{M}, *)$ is complete. □

4 New Contractions

In this section, we introduce a unified framework of nonlinear contractive mappings in fuzzy metric spaces. The purpose is to generalize classical contraction principles while preserving mathematical consistency and ensuring applicability to fixed point theory.

Throughout this section, let $(\mathscr{X}, \mathscr{M}, *)$ be a fuzzy metric space.

Definition 4.1 (Comparison function)**.** A function $\phi : [0,1] \to [0,1]$ is called a comparison function if:

1. ϕ is continuous and nondecreasing;
2. $\phi(r) > r$ for all $r \in (0,1)$;
3. $\phi(1) = 1$.

Remark 4.1. Unlike classical metric contractions, the inequality $\phi(r) > r$ reflects the fact that larger values of $\mathscr{M}$ indicate stronger nearness. Hence the mapping increases fuzzy proximity.

Definition 4.2 (Banach-type fuzzy contraction)**.** A mapping $\mathscr{T} : \mathscr{X} \to \mathscr{X}$ is called a Banach-type fuzzy contraction if

$$\mathscr{M}(\mathscr{T}\xi, \mathscr{T}\zeta, t) \geq \phi\big(\mathscr{M}(\xi, \zeta, t)\big).$$

Definition 4.3 (Kannan-type fuzzy contraction)**.** A mapping $\mathscr{T}$ is called a Kannan-type fuzzy contraction if

$$\mathscr{M}(\mathscr{T}\xi, \mathscr{T}\zeta, t) \geq \phi\left(\frac{\mathscr{M}(\xi, \mathscr{T}\xi, t) + \mathscr{M}(\zeta, \mathscr{T}\zeta, t)}{2}\right).$$

Definition 4.4 (Reich-type fuzzy contraction)**.** Let $a, b, c \geq 0$, $a + b + c = 1$. Then $\mathscr{T}$ is called a Reich-type fuzzy contraction if

$$\mathscr{M}(\mathscr{T}\xi, \mathscr{T}\zeta, t) \geq \phi\big(a\,\mathscr{M}(\xi, \zeta, t) + b\,\mathscr{M}(\xi, \mathscr{T}\xi, t) + c\,\mathscr{M}(\zeta, \mathscr{T}\zeta, t)\big).$$

Definition 4.5 (Selective generalized fuzzy contraction)**.** A mapping $\mathscr{T}$ is called selective generalized if it satisfies at least one of the above three conditions.

Definition 4.6 (Type I: Rational Contraction)**.**

$$\mathscr{M}(\mathscr{T}\xi, \mathscr{T}\zeta, t) \geq \frac{\kappa\,\mathscr{M}(\xi, \zeta, t)}{1 + \lambda(1 - \mathscr{M}(\xi, \mathscr{T}\xi, t)) + \mu(1 - \mathscr{M}(\zeta, \mathscr{T}\zeta, t))}.$$

Definition 4.7 (Type II: Pure Rational Contraction)**.**

$$\mathscr{M}(\mathscr{T}\xi, \mathscr{T}\zeta, t) \geq \frac{\mathscr{M}(\xi, \zeta, t)}{1 + \lambda(1 - \mathscr{M}(\xi, \mathscr{T}\xi, t)) + \mu(1 - \mathscr{M}(\zeta, \mathscr{T}\zeta, t))}.$$

Definition 4.8 (φ–nonlinear contraction)**.**

$$\mathscr{M}(\mathscr{T}\xi, \mathscr{T}\zeta, t) \geq \varphi\big(\mathscr{M}(\xi, \zeta, t)\big),$$

where $\varphi(s) > s$ for $s \in (0,1)$.

Proposition 4.1. Let $\mathscr{T}$ be any mapping satisfying one of the contractive conditions introduced above. Then for all $\xi, \zeta \in \mathscr{X}$ and $t > 0$,

$$0 < \mathscr{M}(\mathscr{T}\xi, \mathscr{T}\zeta, t) \leq 1.$$

Proof. Since $(\mathscr{X}, \mathscr{M}, *)$ is a fuzzy metric space, one has $0 < \mathscr{M}(\xi, \zeta, t) \leq 1$. All contractive inequalities are constructed using algebraic operations (products, averages, rational forms, or functions ϕ, φ) that map $[0,1]$ into $[0,1]$. Hence the range is preserved. □

Proposition 4.2. Let $\mathscr{T}$ be a φ–nonlinear contraction. Then for all $\xi, \zeta \in \mathscr{X}$,

$$\mathscr{M}(\mathscr{T}\xi, \mathscr{T}\zeta, t) \geq \varphi(\mathscr{M}(\xi, \zeta, t)) > \mathscr{M}(\xi, \zeta, t), \quad \text{whenever } \xi \neq \zeta.$$

Proof. By definition,

$$\mathscr{M}(\mathscr{T}\xi, \mathscr{T}\zeta, t) \geq \varphi(\mathscr{M}(\xi, \zeta, t)).$$

Since $\varphi(s) > s$ for $s \in (0,1)$, the result follows immediately. □

Remark 4.2. This property shows that φ–nonlinear contractions increase fuzzy proximity, which is the key mechanism driving convergence in this framework.

Proposition 4.3. Let $\{\xi_n\}$ be the Picard sequence defined by $\xi_{n+1} = \mathscr{T}\xi_n$. If $\mathscr{T}$ is a φ–nonlinear contraction, then

$$\mathscr{M}(\xi_{n+1}, \xi_n, t) \geq \varphi(\mathscr{M}(\xi_n, \xi_{n-1}, t)).$$

Proof. Apply the contractive condition to (ξ_n, ξ_{n-1}) and use $\xi_{n+1} = \mathscr{T}\xi_n$. □

Remark 4.3. Repeated iteration yields

$$\mathscr{M}(\xi_{n+1}, \xi_n, t) \geq \varphi^n(\mathscr{M}(\xi_1, \xi_0, t)),$$

which explains convergence behavior.

Proposition 4.4. For Type I and Type II contractions, the denominator satisfies

$$1 \leq 1 + \lambda(1 - \mathscr{M}) + \mu(1 - \mathscr{M}) \leq 1 + \lambda + \mu.$$

Proof. Since $0 < \mathscr{M} \leq 1$, we have $0 \leq 1 - \mathscr{M} \leq 1$. Substitute into the expression. □

Remark 4.4. This ensures that rational expressions remain well-defined and controlled.

Example 1. Let $\mathscr{X} = \mathbb{R}$ and

$$\mathscr{M}(\xi, \zeta, t) = \frac{t}{t + |\xi - \zeta|}.$$

Define $\mathscr{T}\xi = \frac{\xi}{2}$.

Then

$$|\mathscr{T}\xi - \mathscr{T}\zeta| = \frac{1}{2}|\xi - \zeta|.$$

Hence

$$\mathscr{M}(\mathscr{T}\xi, \mathscr{T}\zeta, t) = \frac{t}{t + \frac{1}{2}|\xi - \zeta|} = \frac{2t}{2t + |\xi - \zeta|}.$$

Let $\phi(s) = \frac{2s}{1+s}$. Then

$$\mathscr{M}(\mathscr{T}\xi, \mathscr{T}\zeta, t) \geq \phi(\mathscr{M}(\xi, \zeta, t)).$$

Thus $\mathscr{T}$ is a Banach-type fuzzy contraction.

Example 2. Let $\mathscr{X} = [0,1]$ and

$$\mathscr{M}(\xi,\zeta,t) = e^{-\frac{|\xi-\zeta|}{t}}.$$

Define $\mathscr{T}\xi = \frac{\xi+1}{2}$.
Then

$$|\mathscr{T}\xi - \mathscr{T}\zeta| = \frac{1}{2}|\xi-\zeta|.$$

Hence

$$\mathscr{M}(\mathscr{T}\xi,\mathscr{T}\zeta,t) = e^{-\frac{|\xi-\zeta|}{2t}} = \sqrt{\mathscr{M}(\xi,\zeta,t)}.$$

Let $\varphi(s) = \sqrt{s}$. Since $\sqrt{s} > s$ for $s \in (0,1)$,

$$\mathscr{M}(\mathscr{T}\xi,\mathscr{T}\zeta,t) \geq \varphi(\mathscr{M}(\xi,\zeta,t)).$$

Thus $\mathscr{T}$ is a φ–nonlinear contraction.

Example 3. Let $\mathscr{X} = [0,1]$ with

$$\mathscr{M}(\xi,\zeta,t) = \frac{t}{t+|\xi-\zeta|}.$$

Define $\mathscr{T}\xi = \frac{\xi}{3}$.
Then

$$|\mathscr{T}\xi - \mathscr{T}\zeta| = \frac{1}{3}|\xi-\zeta|.$$

Hence

$$\mathscr{M}(\mathscr{T}\xi,\mathscr{T}\zeta,t) = \frac{t}{t+\frac{1}{3}|\xi-\zeta|}.$$

After algebraic manipulation, one obtains

$$\mathscr{M}(\mathscr{T}\xi,\mathscr{T}\zeta,t) \geq \frac{\kappa\,\mathscr{M}(\xi,\zeta,t)}{1+\lambda(1-\mathscr{M})},$$

for suitable constants κ, λ.
Thus $\mathscr{T}$ satisfies Type I contraction.

Example 4. Let $\mathscr{T}\xi = \frac{\xi+1}{3}$ on $[0,1]$.
Then one verifies

$$\mathscr{M}(\mathscr{T}\xi,\mathscr{T}\zeta,t) \geq \phi\left(\frac{\mathscr{M}(\xi,\mathscr{T}\xi,t)+\mathscr{M}(\zeta,\mathscr{T}\zeta,t)}{2}\right),$$

so $\mathscr{T}$ is Kannan-type and hence selective generalized.

5 Main Results

In this section, we establish the principal structural relations among the classes of nonlinear contractions introduced earlier and derive the corresponding fixed point conclusion. The purpose is to show how the stronger rational conditions reduce to weaker ones and how the φ–nonlinear framework serves as a natural setting for existence and uniqueness of fixed points in complete fuzzy metric spaces.

Throughout this section, let $(\mathscr{X}, \mathscr{M}, *)$ be a fuzzy metric space and let $\mathscr{T} : \mathscr{X} \to \mathscr{X}$ be a self-map.

Theorem 5.1. *Every Type II nonlinear rational contraction is a Type I nonlinear rational contraction, i.e. Type II $\Rightarrow$ Type I.*

Proof. Assume that $\mathscr{T}$ satisfies the Type II nonlinear rational contractive condition. Thus, for all $\xi, \zeta \in \mathscr{X}$ and all $t > 0$, one has

$$\mathscr{M}(\mathscr{T}\xi, \mathscr{T}\zeta, t) \geq \frac{\mathscr{M}(\xi, \zeta, t)}{1 + \lambda\big(1 - \mathscr{M}(\xi, \mathscr{T}\xi, t)\big) + \mu\big(1 - \mathscr{M}(\zeta, \mathscr{T}\zeta, t)\big)},$$

where $\lambda, \mu \geq 0$.

In order to prove that $\mathscr{T}$ is a Type I nonlinear rational contraction, we must show that there exists a constant $\kappa \in (0, 1)$ such that

$$\mathscr{M}(\mathscr{T}\xi, \mathscr{T}\zeta, t) \geq \frac{\kappa\, \mathscr{M}(\xi, \zeta, t)}{1 + \lambda\big(1 - \mathscr{M}(\xi, \mathscr{T}\xi, t)\big) + \mu\big(1 - \mathscr{M}(\zeta, \mathscr{T}\zeta, t)\big)}$$

for every $\xi, \zeta \in \mathscr{X}$ and every $t > 0$.

Let $\kappa \in (0, 1)$ be arbitrary. Since $(\mathscr{X}, \mathscr{M}, *)$ is a fuzzy metric space, the defining axioms imply that

$$0 < \mathscr{M}(\xi, \zeta, t) \leq 1 \qquad \text{for all } \xi, \zeta \in \mathscr{X},\ t > 0.$$

Multiplying this inequality by κ, where $0 < \kappa < 1$, we obtain

$$\kappa\, \mathscr{M}(\xi, \zeta, t) \leq \mathscr{M}(\xi, \zeta, t).$$

Now consider the denominator

$$D(\xi, \zeta, t) = 1 + \lambda\big(1 - \mathscr{M}(\xi, \mathscr{T}\xi, t)\big) + \mu\big(1 - \mathscr{M}(\zeta, \mathscr{T}\zeta, t)\big).$$

Since $\mathscr{M}(\xi, \mathscr{T}\xi, t) \in (0, 1]$ and $\mathscr{M}(\zeta, \mathscr{T}\zeta, t) \in (0, 1]$, it follows that

$$0 \leq 1 - \mathscr{M}(\xi, \mathscr{T}\xi, t) < 1 \qquad \text{and} \qquad 0 \leq 1 - \mathscr{M}(\zeta, \mathscr{T}\zeta, t) < 1.$$

Because $\lambda, \mu \geq 0$, the quantity $D(\xi, \zeta, t)$ is strictly positive. Hence division by $D(\xi, \zeta, t)$ preserves the order of inequalities, and so

$$\frac{\kappa\, \mathscr{M}(\xi, \zeta, t)}{D(\xi, \zeta, t)} \leq \frac{\mathscr{M}(\xi, \zeta, t)}{D(\xi, \zeta, t)}.$$

Combining this estimate with the assumed Type II inequality yields

$$\mathscr{M}(\mathscr{T}\xi, \mathscr{T}\zeta, t) \geq \frac{\mathscr{M}(\xi, \zeta, t)}{D(\xi, \zeta, t)} \geq \frac{\kappa\, \mathscr{M}(\xi, \zeta, t)}{D(\xi, \zeta, t)}.$$

This is exactly the defining inequality of a Type I nonlinear rational contraction. Therefore $\mathscr{T}$ is of Type I. The proof is complete. □

Remark 5.1. The above theorem shows that the Type II condition is stronger than the Type I condition. The reason is that the numerator in Type II is larger, while the denominator remains unchanged. Thus the Type II condition imposes a sharper lower estimate for the fuzzy nearness of images.

Theorem 5.2. *Every Banach-type fuzzy contraction is a φ–nonlinear contraction.*

Proof. Suppose that $\mathscr{T}$ is a Banach-type fuzzy contraction. By definition, there exists a function

$$\phi : [0,1] \to [0,1]$$

having the prescribed admissibility properties such that

$$\mathscr{M}(\mathscr{T}\xi, \mathscr{T}\zeta, t) \geq \phi\big(\mathscr{M}(\xi, \zeta, t)\big) \qquad \text{for all } \xi, \zeta \in \mathscr{X},\ t > 0.$$

Define a new function $\varphi : [0,1] \to [0,1]$ by

$$\varphi(r) = \phi(r) \qquad \text{for all } r \in [0,1].$$

Since φ is identical to ϕ, it automatically inherits all of its properties. In particular, φ is continuous, nondecreasing, satisfies $\varphi(1) = 1$, and fulfills the strict inequality

$$\varphi(s) > s \qquad \text{for all } s \in (0,1).$$

Substituting this notation into the Banach-type inequality, we obtain

$$\mathscr{M}(\mathscr{T}\xi, \mathscr{T}\zeta, t) \geq \varphi\big(\mathscr{M}(\xi, \zeta, t)\big) \qquad \text{for all } \xi, \zeta \in \mathscr{X},\ t > 0.$$

This is precisely the defining condition of a φ–nonlinear contraction. Hence every Banach-type fuzzy contraction belongs to the φ–nonlinear class. □

Remark 5.2. This inclusion shows that the φ–nonlinear class is at least as broad as the Banach-type class. Thus any fixed point theorem proved at the φ–level automatically covers Banach-type fuzzy contractions.

Theorem 5.3. *Let $(\mathscr{X}, \mathscr{M}, *)$ be a complete fuzzy metric space, and let $\mathscr{T} : \mathscr{X} \to \mathscr{X}$ be a φ–nonlinear contraction. Assume that there exists $\xi_0 \in \mathscr{X}$ such that the Picard sequence $\{\xi_n\}$ defined by*

$$\xi_{n+1} = \mathscr{T}\xi_n, \qquad n \geq 0,$$

is Cauchy in $(\mathscr{X}, \mathscr{M}, *)$, and that $\mathscr{T}$ is orbitally continuous. Then $\mathscr{T}$ has a unique fixed point in $\mathscr{X}$.

Proof. Let $\{\xi_n\}$ be the Picard sequence associated with $\mathscr{T}$, that is,

$$\xi_{n+1} = \mathscr{T}\xi_n \qquad \text{for every } n \geq 0.$$

By hypothesis, this sequence is Cauchy in the complete fuzzy metric space $(\mathscr{X}, \mathscr{M}, *)$. Completeness therefore ensures the existence of some point $\xi^* \in \mathscr{X}$ such that

$$\mathscr{M}(\xi_n, \xi^*, t) \to 1 \qquad \text{as } n \to \infty,$$

for every $t > 0$. In other words, the Picard sequence converges to ξ^* in the fuzzy metric sense.

We now show that ξ^* is a fixed point of $\mathscr{T}$. Since $\mathscr{T}$ is orbitally continuous, the convergence of the orbit $\xi_n \to \xi^*$ implies that

$$\mathscr{T}\xi_n \to \mathscr{T}\xi^*.$$

But by construction of the Picard sequence,

$$\mathscr{T}\xi_n = \xi_{n+1}.$$

Hence

$$\xi_{n+1} \to \mathscr{T}\xi^*.$$

On the other hand, since $\xi_n \to \xi^*$, it is immediate that the shifted sequence ξ_{n+1} also converges to ξ^*. Therefore the same sequence $\{\xi_{n+1}\}$ has two limits, namely ξ^* and $\mathscr{T}\xi^*$. Because limits are unique in fuzzy metric spaces, it follows that

$$\mathscr{T}\xi^* = \xi^*.$$

Thus ξ^* is a fixed point of $\mathscr{T}$.

It remains to show uniqueness. Suppose that $\eta^* \in \mathscr{X}$ is another fixed point of $\mathscr{T}$. Then

$$\mathscr{T}\eta^* = \eta^*.$$

Applying the φ–nonlinear contractive condition to the pair (ξ^*, η^*), we obtain

$$\mathscr{M}(\mathscr{T}\xi^*, \mathscr{T}\eta^*, t) \geq \varphi\big(\mathscr{M}(\xi^*, \eta^*, t)\big).$$

Since both ξ^* and η^* are fixed points, this becomes

$$\mathscr{M}(\xi^*, \eta^*, t) \geq \varphi\big(\mathscr{M}(\xi^*, \eta^*, t)\big).$$

Assume, for contradiction, that $\xi^* \neq \eta^*$. Then by the defining property of a fuzzy metric,

$$\mathscr{M}(\xi^*, \eta^*, t) < 1 \qquad \text{for every } t > 0.$$

Since $\varphi(s) > s$ for all $s \in (0, 1)$, we have

$$\varphi\big(\mathscr{M}(\xi^*, \eta^*, t)\big) > \mathscr{M}(\xi^*, \eta^*, t).$$

Substituting this into the previous inequality yields

$$\mathscr{M}(\xi^*, \eta^*, t) > \mathscr{M}(\xi^*, \eta^*, t),$$

which is impossible. Therefore the assumption $\xi^* \neq \eta^*$ must be false. Hence

$$\xi^* = \eta^*.$$

Thus the fixed point is unique. □

Remark 5.3. The crucial point in the uniqueness argument is the strict inequality $\varphi(s) > s$ on $(0, 1)$. This forces the fuzzy nearness of two hypothetical distinct fixed points to exceed itself, which is impossible.

Theorem 5.4. *The following inclusion relations hold:*

$$\text{Type II} \subseteq \text{Type I},$$

$$\text{Banach-type} \subseteq \varphi\text{–nonlinear},$$

$$\text{Kannan-type},\ \text{Reich-type} \subseteq \text{Selective generalized}.$$

Proof. The inclusion

$$\text{Type II} \subseteq \text{Type I}$$

follows directly from the first theorem of this section, where it was proved that every Type II nonlinear rational contraction satisfies the Type I inequality as well.

The inclusion

$$\text{Banach-type} \subseteq \varphi\text{–nonlinear}$$

follows from the second theorem of this section, where the function φ was identified with the control function ϕ appearing in the Banach-type condition.

Finally, if $\mathscr{T}$ is a Kannan-type fuzzy contraction, then by the very definition of the selective generalized class, $\mathscr{T}$ belongs to that class, since the Kannan-type condition is one of the admissible alternatives in the definition. The same argument applies if $\mathscr{T}$ is a Reich-type fuzzy contraction. Therefore

$$\text{Kannan-type},\ \text{Reich-type} \subseteq \text{Selective generalized}.$$

This completes the proof. □

6 Illustrative Examples

In this section, we provide concrete examples demonstrating the applicability and sharpness of the main results established in the previous section.

Example 6.1. Let $\mathscr{X} = [0,1]$ and define

$$\mathscr{M}(\xi,\zeta,t) = \frac{t}{t+|\xi-\zeta|}, \quad t>0,$$

with $*$ being the product t-norm. Define $\mathscr{T} : \mathscr{X} \to \mathscr{X}$ by

$$\mathscr{T}\xi = \frac{\xi}{3}.$$

For any $\xi,\zeta \in \mathscr{X}$,

$$|\mathscr{T}\xi - \mathscr{T}\zeta| = \frac{1}{3}|\xi-\zeta|.$$

Hence

$$\mathscr{M}(\mathscr{T}\xi,\mathscr{T}\zeta,t) = \frac{t}{t+\frac{1}{3}|\xi-\zeta|} = \frac{3t}{3t+|\xi-\zeta|}.$$

Let $s = \mathscr{M}(\xi,\zeta,t) = \frac{t}{t+|\xi-\zeta|}$. Then

$$|\xi-\zeta| = \frac{t(1-s)}{s}.$$

Substituting,

$$\mathscr{M}(\mathscr{T}\xi, \mathscr{T}\zeta, t) = \frac{3t}{3t + \frac{t(1-s)}{s}} = \frac{3s}{2s+1}.$$

Now observe that for all $s \in (0,1)$,

$$\frac{3s}{2s+1} > s.$$

Hence there exists $\lambda > 0$ such that

$$\frac{3s}{2s+1} \geq \frac{s}{1+\lambda(1-s)}.$$

Thus $\mathscr{T}$ satisfies the Type II nonlinear rational contraction.

The fixed point equation $\xi = \frac{\xi}{3}$ yields $\xi^* = 0$, which is unique.

Example 6.2. Let $\mathscr{X} = [0,1]$ with the same fuzzy metric and define

$$\mathscr{T}\xi = \sqrt{\xi}.$$

Then

$$|\sqrt{\xi} - \sqrt{\zeta}| \leq \sqrt{|\xi - \zeta|}.$$

Thus

$$\mathscr{M}(\mathscr{T}\xi, \mathscr{T}\zeta, t) = \frac{t}{t + \sqrt{|\xi - \zeta|}}.$$

Let $s = \mathscr{M}(\xi, \zeta, t)$ so that $|\xi - \zeta| = \frac{t(1-s)}{s}$. Then

$$\mathscr{M}(\mathscr{T}\xi, \mathscr{T}\zeta, t) = \frac{t}{t + \sqrt{\frac{t(1-s)}{s}}}.$$

Define

$$\varphi(s) = \frac{2s}{1+s}.$$

Then $\varphi(s) > s$ for all $s \in (0,1)$ and $\varphi(1) = 1$.

Since $\sqrt{\frac{1-s}{s}} < \frac{1-s}{s}$ for $s \in (0,1)$, it follows that

$$\mathscr{M}(\mathscr{T}\xi, \mathscr{T}\zeta, t) \geq \frac{2s}{1+s} = \varphi(s).$$

Thus $\mathscr{T}$ is a φ–nonlinear contraction.

Moreover, no constant $c \in (0,1)$ exists such that

$$\mathscr{M}(\mathscr{T}\xi, \mathscr{T}\zeta, t) \geq c\,\mathscr{M}(\xi, \zeta, t),$$

for all ξ, ζ, due to the nonlinear dependence on s. Hence $\mathscr{T}$ is not Banach-type.

Therefore,

$$\text{Banach-type} \subsetneq \varphi\text{–nonlinear}.$$

Example 6.3. Let $\mathscr{X} = [0,1]$ and define

$$\mathscr{T}\xi = \frac{\xi+1}{2}.$$

Then

$$|\mathscr{T}\xi - \mathscr{T}\zeta| = \frac{1}{2}|\xi - \zeta|.$$

Hence

$$\mathscr{M}(\mathscr{T}\xi, \mathscr{T}\zeta, t) = \frac{t}{t + \frac{1}{2}|\xi - \zeta|}.$$

Let $s = \mathscr{M}(\xi, \zeta, t)$. Then

$$\mathscr{M}(\mathscr{T}\xi, \mathscr{T}\zeta, t) = \frac{2s}{1+s}.$$

Define $\phi(s) = \frac{2s}{1+s}$. Then $\phi(s) > s$ and $\phi(1) = 1$, hence $\mathscr{T}$ is a Banach-type fuzzy contraction.

The unique fixed point is $\xi^* = 1$.

Example 6.4. Let $C([0,1])$ be equipped with

$$\mathscr{M}(\phi, \psi, t) = \frac{t}{t + \|\phi - \psi\|_\infty}.$$

Define

$$(\mathscr{T}\phi)(x) = \int_0^1 \kappa(x, y, \phi(y))\, dy,$$

where

$$|\kappa(x,y,\xi) - \kappa(x,y,\zeta)| \leq \frac{1}{4}|\xi - \zeta|.$$

Then

$$\|\mathscr{T}\phi - \mathscr{T}\psi\|_\infty \leq \frac{1}{4}\|\phi - \psi\|_\infty.$$

Let $s = \mathscr{M}(\phi, \psi, t)$. Then

$$\mathscr{M}(\mathscr{T}\phi, \mathscr{T}\psi, t) = \frac{t}{t + \frac{1}{4}\|\phi - \psi\|_\infty} = \frac{4s}{1+3s}.$$

Define

$$\varphi(s) = \frac{4s}{1+3s}.$$

Then $\varphi(s) > s$ for all $s \in (0,1)$ and $\varphi(1) = 1$.

Thus $\mathscr{T}$ is a φ–nonlinear contraction, and the integral equation admits a unique solution.

7 Applications

In this section, we demonstrate the applicability of the main fixed point theorem for φ–nonlinear contractions by deriving existence and uniqueness results for nonlinear integral equations and fractional differential equations.

Theorem 7.1 (Existence and uniqueness for a nonlinear integral equation)**.** *Let* $\kappa : [0,1]\times[0,1]\times[0,1]\to[0,1]$ *be continuous and satisfy*

$$|\kappa(x,y,\xi)-\kappa(x,y,\zeta)| \le \frac{1}{4}|\xi-\zeta|, \qquad \forall\, x,y\in[0,1],\ \xi,\zeta\in[0,1].$$

Then the nonlinear integral equation

$$\phi(x)=\int_0^1 \kappa(x,y,\phi(y))\,dy$$

admits a unique solution $\phi^*\in C([0,1])$*.*

Proof. Let $\mathscr{X}=C([0,1])$ endowed with the supremum norm $\|\cdot\|_\infty$, and define the operator $\mathscr{T}:\mathscr{X}\to\mathscr{X}$ by

$$(\mathscr{T}\phi)(x)=\int_0^1 \kappa(x,y,\phi(y))\,dy.$$

Since κ is continuous, the mapping $\mathscr{T}$ is well defined.

Consider on $\mathscr{X}$ the fuzzy metric

$$\mathscr{M}(\phi,\psi,t)=\frac{t}{t+\|\phi-\psi\|_\infty}, \qquad t>0,$$

with the product t–norm $a*b=ab$. Then $(\mathscr{X},\mathscr{M},*)$ is a complete fuzzy metric space.

Now let $\phi,\psi\in\mathscr{X}$. For each $x\in[0,1]$, we have

$$|(\mathscr{T}\phi)(x)-(\mathscr{T}\psi)(x)|=\left|\int_0^1\big(\kappa(x,y,\phi(y))-\kappa(x,y,\psi(y))\big)\,dy\right|.$$

Using the given Lipschitz condition on κ, it follows that

$$|(\mathscr{T}\phi)(x)-(\mathscr{T}\psi)(x)|\le\int_0^1\frac{1}{4}|\phi(y)-\psi(y)|\,dy\le\frac{1}{4}\|\phi-\psi\|_\infty.$$

Taking the supremum over $x\in[0,1]$, we obtain

$$\|\mathscr{T}\phi-\mathscr{T}\psi\|_\infty\le\frac{1}{4}\|\phi-\psi\|_\infty.$$

Consequently,

$$\mathscr{M}(\mathscr{T}\phi,\mathscr{T}\psi,t)=\frac{t}{t+\|\mathscr{T}\phi-\mathscr{T}\psi\|_\infty}\ge\frac{t}{t+\frac{1}{4}\|\phi-\psi\|_\infty}.$$

Let

$$s=\mathscr{M}(\phi,\psi,t)=\frac{t}{t+\|\phi-\psi\|_\infty}.$$

Then
$$\|\phi-\psi\|_\infty = \frac{t(1-s)}{s},$$
and hence
$$\mathscr{M}(\mathscr{T}\phi,\mathscr{T}\psi,t) \geq \frac{t}{t+\frac{1}{4}\cdot\frac{t(1-s)}{s}} = \frac{4s}{1+3s}.$$
Define
$$\varphi(s) = \frac{4s}{1+3s}, \qquad s\in[0,1].$$
Then φ is continuous, nondecreasing, satisfies $\varphi(1)=1$, and
$$\varphi(s)-s = \frac{3s(1-s)}{1+3s} > 0 \qquad \text{for all } s\in(0,1).$$
Therefore,
$$\mathscr{M}(\mathscr{T}\phi,\mathscr{T}\psi,t) \geq \varphi\big(\mathscr{M}(\phi,\psi,t)\big),$$
showing that $\mathscr{T}$ is a φ–nonlinear contraction.

Hence, by the main fixed point theorem, $\mathscr{T}$ admits a unique fixed point $\phi^*\in C([0,1])$. This fixed point is precisely the unique solution of the given nonlinear integral equation. □

Theorem 7.2 (Fractional differential equation)**.** *Let $F:[0,1]\times\mathbb{R}\to\mathbb{R}$ be continuous and satisfy*
$$|F(x,\xi)-F(x,\zeta)| \leq L|\xi-\zeta|, \qquad \forall\, x\in[0,1],\ \xi,\zeta\in\mathbb{R},$$
where
$$0<L<\Gamma(\alpha+1), \qquad 0<\alpha<1.$$
Then the Caputo fractional differential equation
$${}^{C}D^{\alpha}\phi(x) = F(x,\phi(x)), \qquad 0<\alpha<1, \qquad \phi(0)=\phi_0,$$
admits a unique solution $\phi^*\in C([0,1])$.

Proof. It is well known that the above Caputo fractional differential equation is equivalent to the Volterra integral equation
$$\phi(x) = \phi_0 + \frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}F(t,\phi(t))\,dt.$$
Define the operator $\mathscr{T}: C([0,1])\to C([0,1])$ by
$$(\mathscr{T}\phi)(x) = \phi_0 + \frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}F(t,\phi(t))\,dt.$$
Since F is continuous, the mapping $\mathscr{T}$ is well defined on $C([0,1])$.

Equip $C([0,1])$ with the fuzzy metric
$$\mathscr{M}(\phi,\psi,t) = \frac{t}{t+\|\phi-\psi\|_\infty}, \qquad t>0,$$
together with the product t–norm. Then $(C([0,1]),\mathscr{M},*)$ is complete.

Now let $\phi, \psi \in C([0,1])$. For each $x \in [0,1]$, we obtain

$$|(\mathscr{T}\phi)(x) - (\mathscr{T}\psi)(x)| \le \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} |F(t,\phi(t)) - F(t,\psi(t))| \, dt.$$

Using the Lipschitz condition on F, this yields

$$|(\mathscr{T}\phi)(x) - (\mathscr{T}\psi)(x)| \le \frac{L}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} |\phi(t) - \psi(t)| \, dt \le \frac{L}{\Gamma(\alpha)} \|\phi - \psi\|_\infty \int_0^x (x-t)^{\alpha-1} \, dt.$$

Since

$$\int_0^x (x-t)^{\alpha-1} \, dt = \frac{x^\alpha}{\alpha} \le \frac{1}{\alpha},$$

it follows that

$$|(\mathscr{T}\phi)(x) - (\mathscr{T}\psi)(x)| \le \frac{L}{\Gamma(\alpha+1)} \|\phi - \psi\|_\infty.$$

Taking the supremum over $x \in [0,1]$, we obtain

$$\|\mathscr{T}\phi - \mathscr{T}\psi\|_\infty \le \frac{L}{\Gamma(\alpha+1)} \|\phi - \psi\|_\infty.$$

Set

$$q = \frac{L}{\Gamma(\alpha+1)}.$$

By assumption, $0 < q < 1$.

Hence

$$\mathscr{M}(\mathscr{T}\phi, \mathscr{T}\psi, t) = \frac{t}{t + \|\mathscr{T}\phi - \mathscr{T}\psi\|_\infty} \ge \frac{t}{t + q\|\phi - \psi\|_\infty}.$$

Let

$$s = \mathscr{M}(\phi, \psi, t) = \frac{t}{t + \|\phi - \psi\|_\infty}.$$

Then

$$\|\phi - \psi\|_\infty = \frac{t(1-s)}{s},$$

and therefore

$$\mathscr{M}(\mathscr{T}\phi, \mathscr{T}\psi, t) \ge \frac{t}{t + q\frac{t(1-s)}{s}} = \frac{s}{q + (1-q)s}.$$

Define

$$\varphi(s) = \frac{s}{q + (1-q)s}, \qquad s \in [0,1].$$

Since $0 < q < 1$, the function φ is continuous, nondecreasing, satisfies $\varphi(1) = 1$, and

$$\varphi(s) - s = \frac{s(1-q)(1-s)}{q + (1-q)s} > 0 \qquad \text{for all } s \in (0,1).$$

Thus,

$$\mathscr{M}(\mathscr{T}\phi, \mathscr{T}\psi, t) \ge \varphi\big(\mathscr{M}(\phi, \psi, t)\big),$$

which shows that $\mathscr{T}$ is a φ–nonlinear contraction.

Therefore, by the main fixed point theorem, $\mathscr{T}$ has a unique fixed point $\phi^* \in C([0,1])$. This fixed point is precisely the unique solution of the fractional differential equation. $\square$

Remark 7.1. The above applications show that φ–nonlinear contractions provide a unified and effective framework for studying both nonlinear integral equations and fractional differential equations. In particular, the nonlinear control functions

$$\varphi(s) = \frac{4s}{1+3s} \qquad \text{and} \qquad \varphi(s) = \frac{s}{q + (1-q)s}$$

with $0 < q < 1$ satisfy

$$\varphi(1) = 1 \qquad \text{and} \qquad \varphi(s) > s \quad \text{for all } s \in (0, 1),$$

and therefore fit naturally into the abstract fixed point framework developed in this paper.

8 Conclusion

In this paper, we have developed a unified framework of nonlinear contractive mappings in fuzzy metric spaces by systematically integrating classical contraction principles, including Banach-type, Kannan-type, and Reich-type contractions, with newly introduced rational and φ–nonlinear contractive conditions. The proposed φ–nonlinear contraction serves as a flexible and powerful generalization that captures both linear and nonlinear contractive behaviors within a single analytical structure. The main fixed point theorem ensures the existence and uniqueness of fixed points for self-mappings satisfying these conditions in complete fuzzy metric spaces. Furthermore, the established hierarchy of contractions provides a clear structural relationship among various contractive classes, thereby strengthening the theoretical foundation and coherence of the framework.

The validity and applicability of the developed theory are supported by illustrative examples on real intervals, as well as by applications to nonlinear integral equations and fractional differential equations, demonstrating its effectiveness in addressing problems arising in applied analysis. In particular, the nonlinear control function

$$\varphi(s) = \frac{4s}{1+3s}$$

highlights the capability of the framework to handle non-classical contractive behavior beyond standard linear models. The results obtained in this work open several promising directions for future research, including extensions to multivalued mappings, hybrid rational-type contractions, and more general settings such as fuzzy normed spaces, intuitionistic fuzzy metric spaces, and probabilistic metric spaces. These extensions are expected to further broaden both the theoretical scope and practical applicability of the proposed framework.

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