# Original Research Article
# New Rational Contraction on $b$-Metric Spaces

**Abstract**

This paper introduces a new rational contractive principle on $b$-metric spaces, termed the *RMS-rational contraction*, which unifies and strictly extends a broad spectrum of fixed point frameworks. The proposed condition incorporates multi-term interpoint distances in a rational structure and naturally adapts to the geometry of $b$-metrics through the sharp convergence threshold $s\theta < 1$, where $s$ is the $b$-metric coefficient and $\theta$ is determined by the contraction parameters. Within this setting we prove: (i) existence and uniqueness of fixed points, (ii) linear convergence of Picard iterations, (iii) explicit a priori and a posteriori error bounds, and (iv) a quantitative stability result controlling perturbations of fixed points under data variations. A cyclic extension on two closed subsets is also established, guaranteeing convergence to a unique point in their intersection whenever a forward orbit is bounded. Our framework recovers, as special or limiting cases, the classical principles of Banach, Kannan–Chatterjea, Hardy–Rogers, Meir–Keeler, Boyd–Wong, Geraghty, Wardowski $F$-contractions, and integral-type rational contractions. Several illustrative examples in genuine $b$-metric spaces ($s > 1$) demonstrate the sharpness of the threshold and flexibility of the rational structure. Finally, an application to a nonlinear Volterra integral equation in a power-type $b$-metric space is presented, yielding a unique solution along with explicit convergence and stability estimates for the corresponding Picard iteration.

**MSC (2020):** 47H10, 54H25, 54E50. **Keywords:** fixed point theory; $b$-metric spaces; rational contractions; Picard iteration; stability analysis; cyclic mappings.

## 1 Introduction

The Banach contraction principle [1] is one of the cornerstones of nonlinear analysis and fixed point theory. It asserts that a contraction mapping defined on a complete metric

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space admits a unique fixed point, and that the associated Picard iterative sequence converges to this point. This fundamental result has inspired a vast body of research aimed at extending its applicability by relaxing the contractive conditions or by enlarging the underlying space.

One important direction of generalization replaces the classical Lipschitz condition with more flexible inequalities involving multiple distance terms. In this regard, Kannan [2] introduced a contraction condition depending on distances between points and their images, while Chatterjea [3] proposed another variant using symmetric combinations. Hardy and Rogers [4] unified several such contractive conditions into a single framework using convex combinations of distances. Furthermore, Rhoades [5] conducted a systematic comparison of various contractive mappings, highlighting their interrelations and relative strengths.

Another significant line of development involves the use of comparison functions to generalize contraction mappings. Boyd and Wong [6] replaced linear contractions with nonlinear comparison functions, while Meir and Keeler [7] introduced a more flexible condition that ensures convergence without requiring a uniform contraction constant. Later, Geraghty [8] generalized the contraction principle by allowing variable contractive coefficients. Suzuki [9] further refined these ideas by providing a condition that characterizes metric completeness through generalized contractions.

In parallel with these developments, researchers have extended the notion of metric spaces to accommodate broader classes of problems. A notable example is the concept of $b$-metric spaces, introduced by Bakhtin and later formalized by Czerwik [13, 14]. In such spaces, the triangle inequality is replaced by the relaxed condition

$$d(\xi,\zeta) \leq s\big(d(\xi,\eta) + d(\eta,\zeta)\big), \quad s \geq 1,$$

where $s$ is a constant. Although this modification allows a controlled deviation from the classical triangle inequality, many essential properties such as completeness and convergence of sequences remain valid. Consequently, $b$-metric spaces have become an important setting for studying fixed point problems. Several contributions in this direction can be found in [16, 17, 18, 15].

A further extension in fixed point theory is the introduction of functional-type contractions. Wardowski [11] proposed the notion of $F$-contractions, where the contractive condition is governed by a suitable function rather than a constant. This idea was extended by Jleli and Samet [12] through $\alpha$–$\psi$ type contractions, which significantly broaden the scope of applicability. In the context of $b$-metric spaces, the contractive condition often depends on both the contraction parameter and the coefficient $s$, leading to convergence criteria of the form $s\theta < 1$, which ensures the Cauchy property of iterative sequences.

In recent years, considerable attention has been devoted to developing fixed point results in more generalized frameworks. For example, Raji [21] established new fixed point theorems in complete $b$-metric spaces using generalized contractive mappings, thereby weakening classical assumptions. Rao [22] incorporated order structures into $b$-metric spaces, which enhances their applicability in solving nonlinear equations arising in applied mathematics. Zhao et al. [23] extended fixed point results to $G_b$-metric spaces using generalized Ćirić-type contractions, providing a more comprehensive framework for dealing with complex mappings.

Moreover, Moussaoui [24] presented a detailed survey of fuzzy metric spaces, highlighting their importance in modelling uncertainty and imprecision in real-world problems. Wu et al. [25] introduced strict almost $\phi$-contractions combined with binary relations, which significantly extend the classical theory and provide new tools for analysis in abstract spaces.

More recently, Farajzadeh et al. [26] investigated convex $F$-contractions in $b$-metric spaces, offering new insights into the structure and behavior of such mappings. Additionally, Taheri et al. [27] studied fixed point results in $K$-metric type spaces, which represent another important generalization of metric spaces and further enrich the theory.

Motivated by these developments, the present paper introduces a new class of rational-type contractive conditions involving mixed distance terms such as $d(\xi, T\eta)$ and $d(\eta, T\xi)$. This approach allows for greater flexibility compared to standard contraction conditions and weakens the requirement of global Lipschitz continuity. By employing a suitable iterative technique, we establish convergence results based on a linear recursive inequality. The condition $s\theta < 1$ naturally emerges as the key requirement ensuring the convergence of the Picard sequence in the $b$-metric setting.

The results obtained in this paper unify and extend several well-known contraction principles, including those of Banach, Kannan, Chatterjea, Hardy–Rogers, Boyd–Wong, Meir–Keeler, Geraghty, Wardowski, and integral-type contractions [1, 2, 3, 4, 6, 7, 8, 10, 11, 5]. In addition to proving existence and uniqueness of fixed points, we establish stability results and propose a cyclic version of the main theorem.

Finally, we remark that related generalized distance structures, such as partial metric spaces introduced by Matthews [19], can also benefit from rational-type contractive conditions, particularly in settings where self-distance may be nonzero. The paper is organized as follows: Section 2 presents preliminary concepts and definitions, Section 3 contains the main results and their proofs, and the final section provides illustrative examples and concluding remarks.

## 2 Preliminaries and Notation

Throughout this paper, $X$ denotes a nonempty set and $d : X \times X \to [0, \infty)$ represents a distance-type function.

**Definition 2.1** (Bakhtin–Czerwik $b$-metric [13, 14]). *Let $X$ be a nonempty set and let $s \geq 1$ be a real constant. A function $d : X \times X \to [0, \infty)$ is called a $b$-metric on $X$ with coefficient $s$ if, for all $\xi, \eta, \zeta \in X$, the following conditions hold:*

1. $d(\xi, \eta) = 0$ *if and only if* $\xi = \eta$;
2. $d(\xi, \eta) = d(\eta, \xi)$;
3. $d(\xi, \zeta) \leq s\big(d(\xi, \eta) + d(\eta, \zeta)\big)$.

**Definition 2.2** (Open and closed balls [14, 16]). *Let $(X, d)$ be a $b$-metric space, $\xi \in X$, and $r > 0$. The open and closed balls centered at $\xi$ with radius $r$ are defined, respectively, by*

$$B_d(\xi, r) = \{\eta \in X : d(\xi, \eta) < r\}, \qquad \overline{B}_d(\xi, r) = \{\eta \in X : d(\xi, \eta) \leq r\}.$$

*The collection of all open $b$-balls forms a base for a topology on $X$.*

**Definition 2.3** (Convergence and completeness [14, 15]). *Let $(X, d)$ be a $b$-metric space and $(\xi_n)$ a sequence in $X$.*

- *The sequence* $(\xi_n)$ *is said to* converge *(or b-converge) to* $\xi \in X$ *if* $d(\xi_n, \xi) \to 0$ *as* $n \to \infty$.
- *The sequence* $(\xi_n)$ *is called* $b$-Cauchy *if* $d(\xi_m, \xi_n) \to 0$ *as* $m, n \to \infty$.
- *The space* $(X, d)$ *is said to be* complete *if every b-Cauchy sequence is convergent in* $X$.

**Remark 2.1** (Telescoping estimate [14, 16])**.** *Let* $(\xi_n)$ *be a sequence in a b-metric space* $(X, d)$*. Then, for any integers* $m > n$*, we have*

$$d(\xi_m, \xi_n) \leq s \sum_{k=n}^{m-1} d(\xi_{k+1}, \xi_k).$$

*In particular, if the series* $\sum_{k=1}^{\infty} d(\xi_{k+1}, \xi_k)$ *converges (for example, when successive differences decrease geometrically), then* $(\xi_n)$ *is a b-Cauchy sequence.*

**Definition 2.4** (Boundedness concepts [15])**.** *A subset* $\mathcal{A} \subset X$ *is said to be:*

- bounded *if* $\sup_{\xi,\eta\in\mathcal{A}} d(\xi, \eta) < \infty$*;*
- closed *if it contains the limits of all convergent sequences in* $\mathcal{A}$*;*
- totally bounded *if, for every* $\varepsilon > 0$*, it can be covered by finitely many b-balls of radius* $\varepsilon$.

**Definition 2.5** (Continuity and Lipschitz mappings [1, 15])**.** *A mapping* $\Phi : X \to X$ *is said to be* continuous *at* $\xi \in X$ *if* $d(\xi_n, \xi) \to 0$ *implies* $d(\Phi\xi_n, \Phi\xi) \to 0$.

*The mapping* $\Phi$ *is called:*

- Lipschitz *if there exists* $L \geq 0$ *such that*

$$d(\Phi\xi, \Phi\eta) \leq L\, d(\xi, \eta) \quad \textit{for all } \xi, \eta \in X;$$

- nonexpansive *if* $L = 1$*;*
- contractive *if* $L < 1$.

**Definition 2.6** (Picard iteration and fixed points [1, 2, 15])**.** *Let* $\Phi : X \to X$ *and* $\xi_0 \in X$*. The sequence defined by* $\xi_{n+1} = \Phi(\xi_n)$ *is called the* Picard iteration*. A point* $\xi^* \in X$ *is said to be a* fixed point *of* $\Phi$ *if* $\Phi(\xi^*) = \xi^*$.

**Definition 2.7** (Cyclic mappings [15])**.** *Let* $\mathcal{A}, \mathcal{B} \subset X$ *be nonempty sets. A mapping* $\Phi : \mathcal{A} \cup \mathcal{B} \to \mathcal{A} \cup \mathcal{B}$ *is said to be* cyclic *if*

$$\Phi(\mathcal{A}) \subset \mathcal{B} \quad \textit{and} \quad \Phi(\mathcal{B}) \subset \mathcal{A}.$$

**Definition 2.8** (Hausdorff $b$-metric [15, 16])**.** *Let* $\mathcal{CB}(X)$ *denote the family of all nonempty closed and bounded subsets of* $X$*. For* $\mathcal{A}, \mathcal{B} \in \mathcal{CB}(X)$*, define*

$$H_d(\mathcal{A}, \mathcal{B}) = \max\Big\{ \sup_{\xi\in\mathcal{A}} \inf_{\eta\in\mathcal{B}} d(\xi, \eta),\ \sup_{\eta\in\mathcal{B}} \inf_{\xi\in\mathcal{A}} d(\xi, \eta) \Big\}.$$

*Then* $H_d$ *defines a b-metric on* $\mathcal{CB}(X)$ *with the same coefficient* $s$*. Moreover, if* $(X, d)$ *is complete, then* $(\mathcal{CB}(X), H_d)$ *is also complete.*

Now, we present several new and nontrivial examples of $b$-metric spaces. These examples illustrate the flexibility of the $b$-metric structure and demonstrate how nonstandard distance functions can satisfy the relaxed triangle inequality while preserving essential properties of metric spaces.

**Example 2.1.** *Let $X = \mathbb{R}$ and define $d : X \times X \to [0, \infty)$ by*

$$d(\xi, \eta) = \ln\bigl(1 + |\xi - \eta|^p\bigr), \quad p > 1.$$

*Then $(X, d)$ is a $b$-metric space with coefficient $s = 2^{p-1}$.*

*Proof.* We verify the three conditions of a $b$-metric.

**(i) Positivity:** Since $|\xi - \eta|^p \geq 0$, we have

$$d(\xi, \eta) = \ln(1 + |\xi - \eta|^p) \geq 0,$$

and $d(\xi, \eta) = 0$ if and only if $|\xi - \eta| = 0$, i.e., $\xi = \eta$.

**(ii) Symmetry:** Clearly,

$$d(\xi, \eta) = \ln(1 + |\xi - \eta|^p) = \ln(1 + |\eta - \xi|^p) = d(\eta, \xi).$$

**(iii) Relaxed triangle inequality:** Using the inequality

$$|\xi - \zeta|^p \leq 2^{p-1}\bigl(|\xi - \eta|^p + |\eta - \zeta|^p\bigr),$$

we obtain

$$d(\xi, \zeta) = \ln\bigl(1 + |\xi - \zeta|^p\bigr) \leq \ln\Bigl(1 + 2^{p-1}(|\xi - \eta|^p + |\eta - \zeta|^p)\Bigr).$$

Using the inequality $\ln(1 + a + b) \leq \ln(1 + a) + \ln(1 + b)$ for $a, b \geq 0$, we get

$$d(\xi, \zeta) \leq \ln\bigl(1 + 2^{p-1}|\xi - \eta|^p\bigr) + \ln\bigl(1 + 2^{p-1}|\eta - \zeta|^p\bigr).$$

Hence,

$$d(\xi, \zeta) \leq 2^{p-1}\bigl(d(\xi, \eta) + d(\eta, \zeta)\bigr),$$

which shows that $d$ satisfies the $b$-metric inequality with coefficient $s = 2^{p-1}$.

Therefore, $(X, d)$ is a $b$-metric space. □

**Remark 2.2.** *The above construction can be generalized by defining*

$$d(\xi, \eta) = \phi(|\xi - \eta|),$$

*where $\phi(t) = \ln(1 + t^p)$ is an increasing concave function. Such choices generate a new class of $b$-metrics that interpolate between standard metrics and subadditive distance functions.*

**Example 2.2.** *Proof.* We verify the required properties:

**(i) Positivity:** Since the exponential function is strictly increasing, we have

$$d(\xi, \eta) = |e^\xi - e^\eta| \geq 0,$$

and $d(\xi, \eta) = 0$ if and only if $e^\xi = e^\eta$, which implies $\xi = \eta$.

**(ii) Symmetry:** Clearly,

$$d(\xi,\eta) = |e^{\xi} - e^{\eta}| = |e^{\eta} - e^{\xi}| = d(\eta,\xi).$$

**(iii) Triangle inequality:** For any $\xi, \eta, \zeta \in X$, we use the standard triangle inequality in $\mathbb{R}$:

$$|e^{\xi} - e^{\zeta}| \leq |e^{\xi} - e^{\eta}| + |e^{\eta} - e^{\zeta}|.$$

Hence,

$$d(\xi,\zeta) \leq d(\xi,\eta) + d(\eta,\zeta),$$

which shows that $d$ satisfies the $b$-metric condition with $s = 1$.

Thus, $(X, d)$ is a $b$-metric space (in fact, a metric space). $\square$

*Let $X = \mathbb{R}$ and define $d : X \times X \to [0,\infty)$ by*

$$d(\xi,\eta) = |e^{\xi} - e^{\eta}|.$$

*Then $(X, d)$ is a b-metric space with coefficient $s = 1$.*

# 3 Main Results

In this section, we introduce a new class of rational-type contractive mappings and establish the existence, uniqueness, and convergence of fixed points in the setting of $b$-metric spaces. The proposed contraction condition generalizes several well-known contractive mappings and incorporates nonlinear interactions between iterates.

**Definition 3.1** (RMS–Rational Contraction (RMS–RC))**.** *Let $(X, d)$ be a b-metric space with coefficient $s \geq 1$. A mapping $\Phi : X \to X$ is called an* RMS–rational contraction *(briefly, RMS–RC mapping) if there exist constants $\alpha, \beta \in [0, 1)$ and $q \in (0, 1)$ such that, for all $\xi, \eta \in X$,*

$$d(\Phi\xi,\Phi\eta) \;\leq\; \frac{q\,d(\xi,\eta) \;+\; \alpha\,d(\xi,\Phi\xi) \;+\; \beta\,d(\eta,\Phi\eta)}{1 + d(\xi,\Phi\eta) + d(\eta,\Phi\xi)}. \tag{3.1}$$

*Moreover, defining*

$$\theta := \frac{q+\beta}{1-\alpha},$$

*we assume that the threshold condition*

$$s\,\theta < 1 \tag{3.2}$$

*is satisfied.*

**Remark 3.1.** *The above contractive condition is of rational type and involves mixed distance terms such as $d(\xi, \Phi\eta)$ and $d(\eta, \Phi\xi)$. This structure weakens the classical Lipschitz condition while still allowing effective control over successive iterates. The parameter $\theta$ plays a crucial role in determining the convergence rate.*

**Theorem 3.1.** *Let $(X, d)$ be a complete b-metric space with coefficient $s \geq 1$, and let $\Phi : X \to X$ be an RMS–RC mapping satisfying* (3.1) *and* (3.2)*. Then the following statements hold:*

*(i)* Existence and uniqueness: *The mapping* $\Phi$ *admits a unique fixed point* $\xi^* \in X$*, that is,*

$$\Phi\xi^* = \xi^*.$$

*(ii)* Convergence of Picard iteration: *For any initial point* $\xi_0 \in X$*, the iterative sequence defined by*

$$\xi_{n+1} = \Phi\xi_n, \quad n \geq 0,$$

*converges to the fixed point* $\xi^*$*.*

*(iii)* Geometric decay of successive differences: *The sequence of successive differences satisfies the inequality*

$$d(\xi_{n+1}, \xi_n) \;\leq\; \theta\, d(\xi_n, \xi_{n-1}), \quad n \geq 1, \tag{3.3}$$

*which implies a geometric-type decay governed by the parameter* $\theta$*.*

*Proof.* Let $(X, d)$ be a complete $b$-metric space with coefficient $s \geq 1$, and let $\Phi : X \to X$ satisfy the RMS–RC condition (3.1) for some $\alpha, \beta \in [0, 1)$ and $q \in (0, 1)$. Define

$$\theta := \frac{q + \beta}{1 - \alpha}, \qquad \text{and assume } s\theta < 1.$$

Fix an arbitrary initial point $\xi_0 \in X$ and construct the Picard sequence $(\xi_n)$ by $\xi_{n+1} = \Phi\xi_n$ for all $n \geq 0$.

Applying the contractive condition (3.1) to the pair $(\xi_n, \xi_{n-1})$, and using the identities $\xi_{n+1} = \Phi\xi_n$ and $\xi_n = \Phi\xi_{n-1}$, we obtain

$$d(\xi_{n+1}, \xi_n) = d(\Phi\xi_n, \Phi\xi_{n-1}) \leq \frac{q\, d(\xi_n, \xi_{n-1}) + \alpha\, d(\xi_n, \xi_{n+1}) + \beta\, d(\xi_{n-1}, \xi_n)}{1 + d(\xi_n, \xi_n) + d(\xi_{n-1}, \xi_{n+1})}.$$

Since $d(\xi_n, \xi_n) = 0$ and all terms are nonnegative, the denominator is bounded below by 1. Hence,

$$d(\xi_{n+1}, \xi_n) \leq q\, d(\xi_n, \xi_{n-1}) + \alpha\, d(\xi_n, \xi_{n+1}) + \beta\, d(\xi_{n-1}, \xi_n).$$

Rearranging the above inequality yields

$$(1 - \alpha)\, d(\xi_{n+1}, \xi_n) \leq (q + \beta)\, d(\xi_n, \xi_{n-1}),$$

and therefore,

$$d(\xi_{n+1}, \xi_n) \leq \theta\, d(\xi_n, \xi_{n-1}), \quad n \geq 1.$$

By iterating this inequality, it follows by induction that

$$d(\xi_{n+1}, \xi_n) \leq \theta^n\, d(\xi_1, \xi_0) \qquad \text{for all } n \geq 0,$$

which shows that the sequence of successive differences converges to zero as $n \to \infty$, since $\theta < 1$.

Next, let $m > n$. Using the relaxed triangle inequality in a $b$-metric space, we obtain the standard estimate

$$d(\xi_m, \xi_n) \leq s \sum_{k=n}^{m-1} d(\xi_{k+1}, \xi_k).$$

Substituting the geometric bound obtained above, we deduce

$$d(\xi_m, \xi_n) \le s\, d(\xi_1, \xi_0) \sum_{k=n}^{m-1} \theta^k \le s\, d(\xi_1, \xi_0) \sum_{k=n}^{\infty} \theta^k = \frac{s\,\theta^n}{1-\theta}\, d(\xi_1, \xi_0).$$

Since $s\theta < 1$, the right-hand side tends to zero as $n \to \infty$, independently of $m > n$. Thus, $(\xi_n)$ is a Cauchy sequence in $X$. By completeness of $(X, d)$, there exists $\xi^* \in X$ such that $\xi_n \to \xi^*$.

To prove that $\xi^*$ is a fixed point of $\Phi$, we apply (3.1) to the pair $(\xi_n, \xi^*)$:

$$d(\xi_{n+1}, \Phi\xi^*) \le \frac{q\, d(\xi_n, \xi^*) + \alpha\, d(\xi_n, \xi_{n+1}) + \beta\, d(\xi^*, \Phi\xi^*)}{1 + d(\xi_n, \Phi\xi^*) + d(\xi^*, \xi_{n+1})}.$$

Passing to the limit as $n \to \infty$, and using the facts that $d(\xi_n, \xi^*) \to 0$ and $d(\xi_n, \xi_{n+1}) \to 0$, we obtain

$$d(\xi^*, \Phi\xi^*) \le \frac{\beta\, d(\xi^*, \Phi\xi^*)}{1 + d(\xi^*, \Phi\xi^*)}.$$

Rewriting this inequality gives

$$\big(1 + d(\xi^*, \Phi\xi^*) - \beta\big)\, d(\xi^*, \Phi\xi^*) \le 0.$$

Since $\beta < 1$, the coefficient is strictly positive, which implies $d(\xi^*, \Phi\xi^*) = 0$. Hence, $\Phi\xi^* = \xi^*$, proving existence of a fixed point.

To establish uniqueness, suppose that $\eta^* \in X$ is another fixed point, i.e., $\Phi\eta^* = \eta^*$. Applying (3.1) to the pair $(\xi^*, \eta^*)$, we obtain

$$d(\xi^*, \eta^*) = d(\Phi\xi^*, \Phi\eta^*) \le \frac{q\, d(\xi^*, \eta^*) + \alpha\, d(\xi^*, \xi^*) + \beta\, d(\eta^*, \eta^*)}{1 + d(\xi^*, \eta^*) + d(\eta^*, \xi^*)}.$$

Since $d(\xi^*, \xi^*) = d(\eta^*, \eta^*) = 0$ and the denominator is at least 1, it follows that

$$d(\xi^*, \eta^*) \le q\, d(\xi^*, \eta^*).$$

Because $q < 1$, we conclude that $d(\xi^*, \eta^*) = 0$, and hence $\xi^* = \eta^*$.

Finally, since the initial point $\xi_0$ was arbitrary and the limit must coincide with the unique fixed point, it follows that the Picard sequence converges to $\xi^*$ for every choice of $\xi_0 \in X$. This completes the proof. □

**Corollary 3.1.** *Under the assumptions of Theorem 3.1, the following error estimates hold for all $n \ge 1$:*

a posteriori: $d(\xi_n, \xi^*) \le \dfrac{s}{1-\theta}\, d(\xi_n, \xi_{n-1})$, a priori: $d(\xi_n, \xi^*) \le \dfrac{s}{1-\theta}\, \theta^{n-1}\, d(\xi_1, \xi_0)$.

*Proof.* Let $(\xi_n)$ be the Picard sequence defined by $\xi_{n+1} = \Phi\xi_n$, and let $\xi^*$ denote the unique fixed point obtained in Theorem 3.1. Recall that the sequence satisfies

$$d(\xi_{k+1}, \xi_k) \le \theta\, d(\xi_k, \xi_{k-1}) \qquad (k \ge 1), \tag{3.4}$$

and that the $b$-metric telescoping estimate gives

$$d(\xi_m, \xi_n) \le s \sum_{j=n}^{m-1} d(\xi_{j+1}, \xi_j), \qquad m > n. \tag{3.5}$$

Fix $n \geq 1$. Letting $m \to \infty$ in (3.5) and using the convergence $\xi_m \to \xi^*$, we obtain

$$d(\xi_n, \xi^*) \leq s \sum_{j=n}^{\infty} d(\xi_{j+1}, \xi_j).$$

Using (3.4), one obtains for every $\ell \geq 0$ that

$$d(\xi_{n+\ell+1}, \xi_{n+\ell}) \leq \theta^\ell \, d(\xi_{n+1}, \xi_n).$$

Substituting into the above inequality yields

$$d(\xi_n, \xi^*) \leq s \sum_{\ell=0}^{\infty} \theta^\ell \, d(\xi_{n+1}, \xi_n) = \frac{s}{1-\theta} \, d(\xi_{n+1}, \xi_n).$$

Replacing $n$ by $n-1$ gives the a posteriori estimate

$$d(\xi_n, \xi^*) \leq \frac{s}{1-\theta} \, d(\xi_n, \xi_{n-1}), \qquad n \geq 1.$$

Similarly, applying the iterative inequality (3.4) repeatedly from the initial step yields

$$d(\xi_{j+1}, \xi_j) \leq \theta^j \, d(\xi_1, \xi_0), \qquad j \geq 0.$$

Substituting this bound into the telescoping estimate gives

$$d(\xi_n, \xi^*) \leq s \sum_{j=n}^{\infty} \theta^j \, d(\xi_1, \xi_0) = \frac{s}{1-\theta} \, \theta^n \, d(\xi_1, \xi_0).$$

Replacing $n$ by $n-1$ yields the a priori estimate

$$d(\xi_n, \xi^*) \leq \frac{s}{1-\theta} \, \theta^{n-1} \, d(\xi_1, \xi_0), \qquad n \geq 1.$$

Both estimates follow from the geometric decay of successive differences together with the completeness of the $b$-metric space and the convergence of the geometric series ensured by the condition $s\theta < 1$. □

**Remark 3.2.** *The above estimates provide both computable (a posteriori) and predictive (a priori) error bounds, which are useful in numerical implementations of the Picard iteration.*

*Proof.* Let $(X, d)$ be a complete $b$-metric space with coefficient $s \geq 1$, and let $\Phi, \Psi : X \to X$ be RMS–RC mappings with the same parameters $(\alpha, \beta, q)$, so that

$$\theta = \frac{q+\beta}{1-\alpha}, \qquad s\theta < 1.$$

Assume that

$$d(\Phi\xi, \Psi\xi) \leq \varepsilon \qquad (\forall\, \xi \in X). \tag{3.6}$$

By Theorem 3.1, both $\Phi$ and $\Psi$ admit unique fixed points, denoted by $\xi_\Phi$ and $\xi_\Psi$, respectively.

Using the $b$-metric inequality, we write

$$d(\xi_\Phi, \xi_\Psi) = d(\Phi\xi_\Phi, \Psi\xi_\Psi) \le s\big(d(\Phi\xi_\Phi, \Phi\xi_\Psi) + d(\Phi\xi_\Psi, \Psi\xi_\Psi)\big).$$

By (3.6), we have $d(\Phi\xi_\Psi, \Psi\xi_\Psi) \le \varepsilon$, hence

$$d(\xi_\Phi, \xi_\Psi) \le s\, d(\Phi\xi_\Phi, \Phi\xi_\Psi) + s\,\varepsilon.$$

Applying the RMS–RC condition (3.1) to the pair $(\xi_\Phi, \xi_\Psi)$ and using the fact that $\xi_\Phi$ is a fixed point of $\Phi$, we obtain

$$d(\Phi\xi_\Phi, \Phi\xi_\Psi) \le \frac{q\, d(\xi_\Phi, \xi_\Psi) + \beta\, d(\xi_\Psi, \Phi\xi_\Psi)}{1 + d(\xi_\Phi, \Phi\xi_\Psi) + d(\xi_\Psi, \Phi\xi_\Phi)}.$$

Since the denominator is at least 1, it follows that

$$d(\Phi\xi_\Phi, \Phi\xi_\Psi) \le q\, d(\xi_\Phi, \xi_\Psi) + \beta\, d(\xi_\Psi, \Phi\xi_\Psi).$$

Next, using the $b$-metric inequality and (3.6), we estimate

$$d(\xi_\Psi, \Phi\xi_\Psi) \le s\big(d(\xi_\Psi, \Psi\xi_\Psi) + d(\Psi\xi_\Psi, \Phi\xi_\Psi)\big) \le s\,\varepsilon,$$

since $\xi_\Psi = \Psi\xi_\Psi$. Hence,

$$d(\Phi\xi_\Phi, \Phi\xi_\Psi) \le q\, d(\xi_\Phi, \xi_\Psi) + \beta s\,\varepsilon.$$

Substituting into the previous estimate yields

$$d(\xi_\Phi, \xi_\Psi) \le s\big(q\, d(\xi_\Phi, \xi_\Psi) + \beta s\,\varepsilon\big) + s\,\varepsilon,$$

which simplifies to

$$d(\xi_\Phi, \xi_\Psi) \le sq\, d(\xi_\Phi, \xi_\Psi) + s\varepsilon(1 + \beta s).$$

Rearranging terms gives

$$(1 - sq)\, d(\xi_\Phi, \xi_\Psi) \le s\varepsilon(1 + \beta s).$$

Using the relation $\theta = \frac{q+\beta}{1-\alpha}$ and the assumption $s\theta < 1$, one obtains the sharper bound

$$d(\xi_\Phi, \xi_\Psi) \le \frac{s\,\varepsilon}{1 - s\theta}.$$

This completes the proof. □

**Definition 3.2** (Cyclic mapping). *Let $\mathcal{A}, \mathcal{B} \subset X$ be nonempty sets. A mapping $\Phi : \mathcal{A} \cup \mathcal{B} \to \mathcal{A} \cup \mathcal{B}$ is said to be* cyclic *if*

$$\Phi(\mathcal{A}) \subset \mathcal{B} \quad \text{and} \quad \Phi(\mathcal{B}) \subset \mathcal{A}.$$

**Theorem 3.2** (Cyclic RMS–RC on closed pairs). *Let $(X, d)$ be a complete $b$-metric space with coefficient $s \ge 1$, and let $\mathcal{A}, \mathcal{B} \subset X$ be nonempty closed sets. Suppose that $\Phi : \mathcal{A} \cup \mathcal{B} \to \mathcal{A} \cup \mathcal{B}$ is cyclic and satisfies* (3.1) *for all $\xi \in \mathcal{A}$ and $\eta \in \mathcal{B}$ with parameters $(\alpha, \beta, q)$ such that $s\theta < 1$, where $\theta = \frac{q+\beta}{1-\alpha}$. If there exists $\xi_0 \in \mathcal{A}$ such that its orbit $\{\xi_n\}$ is bounded, then:*

*(i) There exists a unique fixed point $\xi^* \in \mathcal{A} \cap \mathcal{B}$ with $\Phi\xi^* = \xi^*$.*

*(ii) For any initial point in $\mathcal{A} \cup \mathcal{B}$, the Picard sequence converges to $\xi^*$ and satisfies* (3.3)*.*

*Proof.* Let $\xi_0 \in \mathcal{A}$ and define the Picard sequence $\xi_{n+1} = \Phi\xi_n$. By cyclicity, it follows that $\xi_{2k} \in \mathcal{A}$ and $\xi_{2k+1} \in \mathcal{B}$ for all $k \in \mathbb{N}$. Since the orbit is assumed to be bounded, both subsequences remain bounded.

Applying the contractive condition (3.1) to consecutive terms $(\xi_n, \xi_{n-1})$, which always lie in $\mathcal{A} \times \mathcal{B}$ or $\mathcal{B} \times \mathcal{A}$, and using $\xi_{n+1} = \Phi\xi_n$, we obtain

$$d(\xi_{n+1}, \xi_n) \le q\, d(\xi_n, \xi_{n-1}) + \alpha\, d(\xi_n, \xi_{n+1}) + \beta\, d(\xi_{n-1}, \xi_n).$$

Rearranging yields

$$(1-\alpha)\, d(\xi_{n+1}, \xi_n) \le (q+\beta)\, d(\xi_n, \xi_{n-1}),$$

and hence

$$d(\xi_{n+1}, \xi_n) \le \theta\, d(\xi_n, \xi_{n-1}), \quad n \ge 1.$$

Iterating this inequality gives $d(\xi_{n+1}, \xi_n) \le \theta^n d(\xi_1, \xi_0) \to 0$. Using the $b$-metric telescoping property, for $m > n$ we obtain

$$d(\xi_m, \xi_n) \le s \sum_{k=n}^{m-1} d(\xi_{k+1}, \xi_k) \le \frac{s\,\theta^n}{1-\theta} d(\xi_1, \xi_0) \to 0,$$

which shows that $(\xi_n)$ is a Cauchy sequence. By completeness, there exists $\xi^* \in X$ such that $\xi_n \to \xi^*$.

Since $\mathcal{A}$ and $\mathcal{B}$ are closed and the subsequences $(\xi_{2k}) \subset \mathcal{A}$ and $(\xi_{2k+1}) \subset \mathcal{B}$ both converge to $\xi^*$, it follows that $\xi^* \in \mathcal{A} \cap \mathcal{B}$.

Applying (3.1) to $(\xi_n, \xi^*)$ and passing to the limit as $n \to \infty$, we obtain

$$d(\xi^*, \Phi\xi^*) \le \frac{\beta\, d(\xi^*, \Phi\xi^*)}{1 + d(\xi^*, \Phi\xi^*)},$$

which implies $d(\xi^*, \Phi\xi^*) = 0$, and hence $\Phi\xi^* = \xi^*$.

To prove uniqueness, let $\eta^* \in \mathcal{A} \cap \mathcal{B}$ be another fixed point. Then

$$d(\xi^*, \eta^*) = d(\Phi\xi^*, \Phi\eta^*) \le \frac{q\, d(\xi^*, \eta^*)}{1 + d(\xi^*, \eta^*) + d(\eta^*, \xi^*)} \le q\, d(\xi^*, \eta^*),$$

which forces $d(\xi^*, \eta^*) = 0$, and hence $\xi^* = \eta^*$.

Finally, for any initial point in $\mathcal{A} \cup \mathcal{B}$, the same argument applies, and the corresponding Picard sequence converges to $\xi^*$ while satisfying the estimate (3.3). This completes the proof. ☐

**Corollary 3.2.** *Within the RMS–RC framework, the following classical contractive conditions are recovered as particular cases:*

*(a) If $\alpha = \beta = 0$, then* (3.1) *reduces to a Banach-type contraction with Lipschitz constant at most $q$, and convergence holds whenever $q < 1/s$.*

*(b) If $q = 0$ and $\alpha, \beta \in (0,1)$ satisfy $s\,\dfrac{\beta}{1-\alpha} < 1$, then Kannan–Chatterjea-type conclusions follow in the b-metric setting.*

(c) *If $\alpha, \beta, q$ depend monotonically on $d(\xi, \eta)$ and the condition $s\,\theta(d) < 1$ holds pointwise, then a Geraghty-type adaptive contraction is obtained.*

*Proof.* Recall that the RMS–RC condition is given by

$$d(\Phi\xi, \Phi\eta) \ \le \ \frac{q\,d(\xi,\eta) + \alpha\,d(\xi,\Phi\xi) + \beta\,d(\eta,\Phi\eta)}{1 + d(\xi,\Phi\eta) + d(\eta,\Phi\xi)}, \qquad \theta = \frac{q+\beta}{1-\alpha}, \quad s\theta < 1.$$

If $\alpha = \beta = 0$, the inequality simplifies to

$$d(\Phi\xi, \Phi\eta) \le \frac{q\,d(\xi,\eta)}{1 + d(\xi,\Phi\eta) + d(\eta,\Phi\xi)} \le q\,d(\xi,\eta),$$

which shows that $\Phi$ is a contraction with constant $q$. In this case, $\theta = q$, and the convergence condition $s\theta < 1$ becomes $q < 1/s$, yielding the Banach-type result.

If $q = 0$, the inequality becomes

$$d(\Phi\xi, \Phi\eta) \le \alpha\,d(\xi,\Phi\xi) + \beta\,d(\eta,\Phi\eta).$$

Here $\theta = \dfrac{\beta}{1-\alpha}$, and the requirement $s\theta < 1$ ensures convergence. This condition corresponds to a Kannan–Chatterjea-type contraction in the $b$-metric framework.

In the adaptive case, suppose that $\alpha, \beta, q$ depend on $r = d(\xi,\eta)$ and define

$$\theta(r) = \frac{q(r) + \beta(r)}{1 - \alpha(r)}.$$

If $s\,\theta(r) < 1$ for all $r > 0$, then along the Picard sequence $(\xi_n)$ one obtains

$$d(\xi_{n+1}, \xi_n) \le \theta(d(\xi_n, \xi_{n-1}))\,d(\xi_n, \xi_{n-1}).$$

Setting $\theta^* = \sup_{r>0} \theta(r) < 1/s$, it follows that

$$d(\xi_{n+1}, \xi_n) \le \theta^* d(\xi_n, \xi_{n-1}),$$

which yields geometric decay of successive differences. The $b$-metric telescoping argument then implies that $(\xi_n)$ is Cauchy and converges to the unique fixed point. This recovers a Geraghty-type adaptive contraction principle.

Thus, in all cases, the desired conclusions follow from Theorem 3.1. □

The rational structure of the RMS–RC condition provides a unified framework that encompasses several classical and modern contraction principles as special cases:

- **Hardy–Rogers type:** The numerator

  $$q\,d(\xi,\eta) + \alpha\,d(\xi,\Phi\xi) + \beta\,d(\eta,\Phi\eta)$$

  represents a convex combination of fundamental distance terms, thereby generalizing Hardy–Rogers-type contractions.

- **Meir–Keeler and Boyd–Wong type:** The denominator

  $$1 + d(\xi,\Phi\eta) + d(\eta,\Phi\xi)$$

  acts as a regulating factor that weakens the contraction for large distances and strengthens it locally, capturing the essence of Meir–Keeler and Boyd–Wong type conditions.

- **Wardowski $F$-contraction:** The rational decay induced by the denominator mimics the behavior of $F$-contractions, where the contractive effect depends on a transformation of the distance rather than a fixed constant.
- **Integral and rational contractions:** The present framework extends integral-type contractions (such as those of Branciari) by incorporating rational expressions, thereby allowing greater flexibility in controlling nonlinear mappings.

Consequently, by appropriate choices of the parameters $(\alpha, \beta, q)$, the RMS–RC framework unifies and extends a wide class of contraction principles, while the condition $s\theta < 1$ provides a sharp convergence criterion adapted to the geometry of $b$-metric spaces.

# 4 Examples

**Example 4.1.** *Let* $X = [-R, R] \subset \mathbb{R}$ *and fix* $\gamma \in (0, 1)$*. Define*

$$d(\xi, \eta) = |\xi - \eta|^\gamma, \qquad s = 2^{1-\gamma} > 1.$$

*Consider the mapping* $\Phi : X \to X$ *given by* $\Phi(\xi) = \lambda\,\xi$*, where* $\lambda \in (0, 1)$*. Choose the parameters*

$$(\alpha, \beta, q) = (0.10,\ 0.10,\ 0.45), \quad \gamma = 0.8, \quad \lambda = 0.30, \quad R = 0.10.$$

*We compute*

$$\theta = \frac{q + \beta}{1 - \alpha} = \frac{0.55}{0.90} \approx 0.6111, \qquad s = 2^{0.2} \approx 1.1487, \qquad s\theta \approx 0.7029 < 1.$$

*Hence, the convergence condition is satisfied.*

*For* $\xi, \eta \in X$*, we have*

$$d(\Phi\xi, \Phi\eta) = |\lambda|^\gamma\, |\xi - \eta|^\gamma = |\lambda|^\gamma\, a, \quad \text{where } a := |\xi - \eta|^\gamma.$$

*Since* $|\xi - \eta| \leq 2R$*, it follows that* $a \leq (2R)^\gamma \approx 0.276$*.*

*Moreover,*

$$d(\xi, \Phi\xi) = (1 - \lambda)^\gamma |\xi|^\gamma, \qquad d(\eta, \Phi\eta) = (1 - \lambda)^\gamma |\eta|^\gamma,$$

*where* $(1 - \lambda)^\gamma = 0.7^{0.8} \approx 0.751$*.*

*Next, we estimate the denominator:*

$$d(\xi, \Phi\eta) = |\xi - \lambda\eta|^\gamma \leq (|\xi| + \lambda|\eta|)^\gamma \leq \big((1 + \lambda)R\big)^\gamma.$$

*Similarly,* $d(\eta, \Phi\xi) \leq ((1 + \lambda)R)^\gamma$*. Hence,*

$$1 + d(\xi, \Phi\eta) + d(\eta, \Phi\xi) \leq 1 + 2\big((1 + \lambda)R\big)^\gamma = 1 + 2(0.13)^{0.8} \approx 1.372 =: D_{\max}.$$

*Using the inequality* $|\xi - \eta|^\gamma \leq |\xi|^\gamma + |\eta|^\gamma$ *(valid for* $\gamma \in (0, 1)$*), we obtain*

$$|\xi|^\gamma + |\eta|^\gamma \geq a.$$

*Thus, the numerator satisfies*

$$q\,a + \alpha\, d(\xi, \Phi\xi) + \beta\, d(\eta, \Phi\eta) \geq \big(q + (\alpha + \beta)(1 - \lambda)^\gamma\big)a.$$

*Therefore, the right-hand side of* (3.1) *can be estimated as*

$$\frac{q\,a+\alpha\,d(\xi,\Phi\xi)+\beta\,d(\eta,\Phi\eta)}{1+d(\xi,\Phi\eta)+d(\eta,\Phi\xi)}\geq\frac{0.45+0.20\cdot0.751}{1.372}\,a\approx0.4374\,a.$$

*On the other hand,*

$$d(\Phi\xi,\Phi\eta)=|\lambda|^{\gamma}a=0.3^{0.8}a\approx0.3818\,a.$$

*Since* $0.3818\,a\leq0.4374\,a$*, the RMS–RC inequality holds for all* $\xi,\eta\in X$*.*

*Hence, all the assumptions of Theorem 3.1 are satisfied. Therefore,* $\Phi$ *admits a unique fixed point, which is clearly* $\xi^*=0$*. Moreover, the Picard iteration converges to* $\xi^*$*, and the estimates*

$$d(\xi_{n+1},\xi_n)\leq\theta\,d(\xi_n,\xi_{n-1}),\qquad d(\xi_n,\xi^*)\leq\frac{s}{1-\theta}\,d(\xi_n,\xi_{n-1})$$

*hold for all* $n\geq1$*.*

**Example 4.2.** *Let* $X=\mathbb{R}^2$ *and define*

$$d(\boldsymbol{\xi},\boldsymbol{\eta})=\|\boldsymbol{\xi}-\boldsymbol{\eta}\|_2^{\gamma},\qquad\gamma=0.7.$$

*Then* $(X,d)$ *is a b-metric space with coefficient*

$$s=2^{1-\gamma}=2^{0.3}\approx1.2311>1.$$

*Let* $U$ *be a fixed orthogonal (rotation) matrix on* $\mathbb{R}^2$*, and define the mapping* $\Phi:X\to X$ *by*

$$\Phi(\boldsymbol{\xi})=\lambda\,U\boldsymbol{\xi},\qquad\lambda=0.35.$$

*Choose the parameters*

$$(\alpha,\beta,q)=(0.20,\,0.20,\,0.25),\qquad\theta=\frac{q+\beta}{1-\alpha}=\frac{0.45}{0.80}=0.5625,\qquad s\theta\approx0.692<1.$$

*Hence, the convergence condition is satisfied.*

*Restrict the mapping to the closed ball*

$$X_R:=\{\boldsymbol{\xi}\in\mathbb{R}^2:\|\boldsymbol{\xi}\|_2\leq R\},\qquad R=0.2.$$

*For* $\boldsymbol{\xi},\boldsymbol{\eta}\in X_R$*, we have*

$$d(\Phi\boldsymbol{\xi},\Phi\boldsymbol{\eta})=\|\lambda U\boldsymbol{\xi}-\lambda U\boldsymbol{\eta}\|_2^{\gamma}=\lambda^{\gamma}\,\|\boldsymbol{\xi}-\boldsymbol{\eta}\|_2^{\gamma}=\lambda^{\gamma}\,d(\boldsymbol{\xi},\boldsymbol{\eta}),$$

*since* $U$ *preserves the Euclidean norm.*

*Moreover,*

$$d(\boldsymbol{\xi},\Phi\boldsymbol{\xi})=\|\boldsymbol{\xi}-\lambda U\boldsymbol{\xi}\|_2^{\gamma}\leq(1+\lambda)^{\gamma}\|\boldsymbol{\xi}\|_2^{\gamma},$$

*and similarly for* $\boldsymbol{\eta}$*. Also,*

$$d(\boldsymbol{\xi},\Phi\boldsymbol{\eta})=\|\boldsymbol{\xi}-\lambda U\boldsymbol{\eta}\|_2^{\gamma}\leq(\|\boldsymbol{\xi}\|_2+\lambda\|\boldsymbol{\eta}\|_2)^{\gamma}\leq\big((1+\lambda)R\big)^{\gamma}.$$

*Hence,*

$$1+d(\boldsymbol{\xi},\Phi\boldsymbol{\eta})+d(\boldsymbol{\eta},\Phi\boldsymbol{\xi})\leq1+2\big((1+\lambda)R\big)^{\gamma}.$$

*Using the subadditivity property of the function* $t \mapsto t^\gamma$ *for* $\gamma \in (0,1)$, *we have*

$$\|\boldsymbol{\xi} - \boldsymbol{\eta}\|_2^\gamma \le \|\boldsymbol{\xi}\|_2^\gamma + \|\boldsymbol{\eta}\|_2^\gamma,$$

*which implies*

$$\|\boldsymbol{\xi}\|_2^\gamma + \|\boldsymbol{\eta}\|_2^\gamma \ge d(\boldsymbol{\xi}, \boldsymbol{\eta}).$$

*Combining these estimates, the numerator of* (3.1) *satisfies a lower bound proportional to* $d(\boldsymbol{\xi}, \boldsymbol{\eta})$, *while the denominator is uniformly bounded above. Consequently, there exists a constant* $C > 0$ *such that*

$$\frac{q\, d(\boldsymbol{\xi}, \boldsymbol{\eta}) + \alpha\, d(\boldsymbol{\xi}, \Phi\boldsymbol{\xi}) + \beta\, d(\boldsymbol{\eta}, \Phi\boldsymbol{\eta})}{1 + d(\boldsymbol{\xi}, \Phi\boldsymbol{\eta}) + d(\boldsymbol{\eta}, \Phi\boldsymbol{\xi})} \ \ge\ C\, d(\boldsymbol{\xi}, \boldsymbol{\eta}),$$

*with* $C > \lambda^\gamma$. *Therefore,*

$$d(\Phi\boldsymbol{\xi}, \Phi\boldsymbol{\eta}) \le \mathit{RHS}\ \mathit{of}\ (3.1),$$

*and the RMS–RC condition holds on* $X_R$.

*Hence, all assumptions of Theorem 3.1 are satisfied. It follows that* $\Phi$ *has a unique fixed point, which is clearly* $\boldsymbol{\xi}^* = \mathbf{0}$. *Moreover, the Picard iteration converges to* $\mathbf{0}$, *and the estimate*

$$d(\boldsymbol{\xi}_{n+1}, \boldsymbol{\xi}_n) \le \theta\, d(\boldsymbol{\xi}_n, \boldsymbol{\xi}_{n-1})$$

*holds for all* $n \ge 1$.

**Example 4.3.** *Let* $X = \mathbb{R}$ *and define*

$$d(\xi, \eta) = |\xi - \eta|^\gamma, \qquad \gamma = 0.75.$$

*Then* $(X, d)$ *is a b-metric space with coefficient*

$$s = 2^{1-\gamma} = 2^{0.25} \approx 1.1892 > 1.$$

*Let* $\mathcal{A} = [-R, 0]$ *and* $\mathcal{B} = [0, R]$, *where* $R = 0.15$, *and define a mapping* $\Phi : \mathcal{A} \cup \mathcal{B} \to \mathcal{A} \cup \mathcal{B}$ *by*

$$\Phi(\xi) = -\lambda\, \xi, \qquad \lambda = 0.40.$$

*Then* $\Phi(\mathcal{A}) \subset \mathcal{B}$ *and* $\Phi(\mathcal{B}) \subset \mathcal{A}$, *so* $\Phi$ *is cyclic.*

*For* $\xi, \eta \in X$, *we have*

$$d(\Phi\xi, \Phi\eta) = |\lambda|^\gamma\, d(\xi, \eta), \qquad d(\xi, \Phi\xi) = |\xi + \lambda\xi|^\gamma = (1+\lambda)^\gamma |\xi|^\gamma.$$

*Choose the parameters*

$$(\alpha, \beta, q) = (0.15,\ 0.15,\ 0.30).$$

*Then*

$$\theta = \frac{q + \beta}{1 - \alpha} = \frac{0.45}{0.85} \approx 0.5294, \qquad s\theta \approx 1.1892 \times 0.5294 \approx 0.629 < 1.$$

*Thus, the convergence condition is satisfied.*

*Now consider* $\xi \in \mathcal{A}$ *and* $\eta \in \mathcal{B}$. *Since* $|\xi|, |\eta| \le R$, *we estimate*

$$d(\xi, \Phi\eta) = |\xi + \lambda\eta|^\gamma \le (|\xi| + \lambda|\eta|)^\gamma \le \big((1+\lambda)R\big)^\gamma,$$

*and similarly for* $d(\eta, \Phi\xi)$*. Hence,*

$$1 + d(\xi, \Phi\eta) + d(\eta, \Phi\xi) \leq 1 + 2\big((1+\lambda)R\big)^{\gamma}.$$

*Using the inequality* $|\xi - \eta|^{\gamma} \leq |\xi|^{\gamma} + |\eta|^{\gamma}$ *for* $\gamma \in (0,1)$*, we obtain*

$$|\xi|^{\gamma} + |\eta|^{\gamma} \geq d(\xi, \eta).$$

*Therefore, the numerator of* (3.1) *satisfies*

$$q\, d(\xi, \eta) + \alpha\, d(\xi, \Phi\xi) + \beta\, d(\eta, \Phi\eta) \geq \big(q + (\alpha+\beta)(1+\lambda)^{\gamma}\big)\, d(\xi, \eta).$$

*Since the denominator is uniformly bounded above on* $\mathcal{A} \times \mathcal{B}$*, it follows that the right-hand side of* (3.1) *dominates*

$$\lambda^{\gamma}\, d(\xi, \eta) = d(\Phi\xi, \Phi\eta),$$

*and hence the RMS–RC condition holds on* $\mathcal{A} \times \mathcal{B}$*.*

*Moreover, for any* $\xi_0 \in \mathcal{A}$*, the Picard sequence* $\xi_{n+1} = \Phi\xi_n$ *satisfies*

$$|\xi_n| \leq \lambda^n |\xi_0| \leq R,$$

*and is therefore bounded. All assumptions of Theorem 3.2 are satisfied.*

*Consequently,* $\Phi$ *admits a unique fixed point in* $\mathcal{A} \cap \mathcal{B} = \{0\}$*, and the Picard iteration converges to* $\xi^* = 0$*. Furthermore, the estimate*

$$d(\xi_{n+1}, \xi_n) \leq \theta\, d(\xi_n, \xi_{n-1})$$

*holds for all* $n \geq 1$*.*

**Example 4.4.** *Let* $X = [-R, R] \subset \mathbb{R}$ *and define*

$$d(\xi, \eta) = |\xi - \eta|^{\gamma}, \qquad \gamma = 0.8.$$

*Then* $(X, d)$ *is a b-metric space with coefficient*

$$s = 2^{1-\gamma} = 2^{0.2} \approx 1.1487 > 1,$$

*and fix* $R = 0.2$*.*

*Let* $\lambda = 0.35$ *and define two mappings* $\Phi, \Psi : X \to X$ *by*

$$\Phi(\xi) = \lambda\,\xi, \qquad \Psi(\xi) = \lambda\,\xi + \delta, \qquad |\delta| \leq 10^{-3}.$$

*Choose the parameters*

$$(\alpha, \beta, q) = (0.20,\ 0.20,\ 0.25).$$

*Then*

$$\theta = \frac{q+\beta}{1-\alpha} = \frac{0.45}{0.80} = 0.5625, \qquad s\theta \approx 1.1487 \times 0.5625 \approx 0.646 < 1.$$

*Hence, both* $\Phi$ *and* $\Psi$ *satisfy the convergence condition of Theorem 3.1.*

*For any* $\xi \in X$*, we compute*

$$d(\Phi\xi, \Psi\xi) = |\lambda\xi - (\lambda\xi + \delta)|^{\gamma} = |\delta|^{\gamma}.$$

*Thus,*

$$d(\Phi\xi, \Psi\xi) \le \varepsilon, \qquad \varepsilon := |\delta|^{\gamma}.$$

*Therefore, all assumptions of Theorem* **??** *are satisfied, and we obtain*

$$d(\xi_\Phi, \xi_\Psi) \le \frac{s\,\varepsilon}{1 - s\theta} = \frac{1.1487\,|\delta|^{\gamma}}{1 - 1.1487 \cdot 0.5625}.$$

*A direct computation gives*

$$\frac{1.1487}{1 - 0.646} \approx 2.43,$$

*and hence*

$$d(\xi_\Phi, \xi_\Psi) \;\le\; 2.43\,|\delta|^{\gamma}.$$

*Observe that* $\xi_\Phi = 0$*, while* $\xi_\Psi$ *is the unique solution of*

$$\xi = \lambda\xi + \delta,$$

*that is,*

$$\xi_\Psi = \frac{\delta}{1 - \lambda}.$$

*Thus, the above estimate provides an explicit bound of the form*

$$|\xi_\Psi|^{\gamma} \le 2.43\,|\delta|^{\gamma},$$

*which quantitatively describes the stability of fixed points under small affine perturbations.*

**Remark 4.1.** *The above example shows that the displacement of fixed points depends continuously on perturbations of the mapping. This highlights the robustness of RMS–RC contractions in applications involving approximation and numerical errors.*

**Example 4.5.** *Let* $X = [-R, R] \subset \mathbb{R}$ *and define*

$$d(\xi, \eta) = |\xi - \eta|^{\gamma}, \qquad \gamma = 0.7.$$

*Then* $(X, d)$ *is a b-metric space with coefficient*

$$s = 2^{\,1-\gamma} = 2^{0.3} \approx 1.2311 > 1.$$

*Define* $\Phi : X \to X$ *by*

$$\Phi(\xi) = \lambda\,\xi^3, \qquad \lambda = 0.6,$$

*and choose* $R = 0.2$*.*

*For* $\xi, \eta \in X$*, using the identity*

$$|\xi^3 - \eta^3| = |\xi - \eta|\,|\xi^2 + \xi\eta + \eta^2|,$$

*and the bound* $|\xi|, |\eta| \le R$*, we obtain*

$$|\xi^3 - \eta^3| \le 3R^2\,|\xi - \eta|.$$

*Therefore,*

$$d(\Phi\xi, \Phi\eta) = |\lambda|^{\gamma}|\xi^3 - \eta^3|^{\gamma} \le \lambda^{\gamma}(3R^2)^{\gamma}|\xi - \eta|^{\gamma} = L\,d(\xi, \eta),$$

*where*

$$L := \lambda^\gamma (3R^2)^\gamma.$$

*With the chosen values,*

$$\lambda^\gamma = 0.6^{0.7} \approx 0.699, \qquad (3R^2)^\gamma = (3 \cdot 0.04)^{0.7} = (0.12)^{0.7} \approx 0.120,$$

*and hence*

$$L \approx 0.699 \times 0.120 \approx 0.0839.$$

*Choose the parameters*

$$(\alpha, \beta, q) = (0.15,\ 0.15,\ 0.25).$$

*Then*

$$\theta = \frac{q+\beta}{1-\alpha} = \frac{0.40}{0.85} \approx 0.4706, \qquad s\theta \approx 1.2311 \times 0.4706 \approx 0.579 < 1.$$

*Thus, the convergence condition is satisfied.*

*Next, we estimate the denominator in* (3.1)*. For* $\xi, \eta \in X$*,*

$$d(\xi, \Phi\eta) = |\xi - \lambda\eta^3|^\gamma \leq (|\xi| + \lambda|\eta|^3)^\gamma \leq \big(R + \lambda R^3\big)^\gamma.$$

*Since* $R = 0.2$*, we have* $R^3 = 0.008$ *and hence*

$$R + \lambda R^3 = 0.2 + 0.6 \cdot 0.008 = 0.2048.$$

*Thus,*

$$1 + d(\xi, \Phi\eta) + d(\eta, \Phi\xi) \leq 1 + 2(0.2048)^{0.7} \approx 1 + 2(0.224) \approx 1.448.$$

*Therefore,*

$$\frac{q\, d(\xi,\eta) + \alpha d(\xi, \Phi\xi) + \beta d(\eta, \Phi\eta)}{1 + d(\xi, \Phi\eta) + d(\eta, \Phi\xi)} \geq \frac{q\, d(\xi,\eta)}{1 + 2(0.2048)^{0.7}} \geq c\, d(\xi, \eta),$$

*where*

$$c := \frac{q}{1 + 2(0.2048)^{0.7}} \approx \frac{0.25}{1.448} \approx 0.1727.$$

*Since* $c > L \approx 0.0839$*, it follows that*

$$d(\Phi\xi, \Phi\eta) \leq \textit{RHS of } (3.1),$$

*and hence the RMS–RC condition holds on* $X$*.*

*Therefore, all the assumptions of Theorem 3.1 are satisfied. It follows that* $\Phi$ *admits a unique fixed point in* $X$*. Solving* $\xi = \lambda\xi^3$ *gives* $\xi^* = 0$*, which is the only solution in* $[-R, R]$*.*

*Moreover, the Picard iteration converges to* $\xi^* = 0$*, and the estimates*

$$d(\xi_{n+1}, \xi_n) \leq \theta\, d(\xi_n, \xi_{n-1}), \qquad d(\xi_n, \xi^*) \leq \frac{s}{1-\theta}\, d(\xi_n, \xi_{n-1})$$

*hold for all* $n \geq 1$*.*

**Remark 4.2.** *The above examples collectively illustrate the applicability and strength of the proposed RMS–RC framework:*

- *Examples 4.1, 4.2, and 4.5 validate Theorem 3.1 and Corollary 3.1 in genuine b-metric spaces with* $s > 1$*, providing explicit values of* $\theta$ *satisfying* $s\theta < 1$*.*
- *Example 4.4 demonstrates the stability result in Theorem* **??** *through a concrete perturbation of the mapping.*
- *Example 4.3 illustrates Theorem 3.2 in the cyclic setting, where the sets* $\mathcal{A}$ *and* $\mathcal{B}$ *are closed, the mapping is cyclic, the orbit is bounded, and the fixed point lies in* $\mathcal{A} \cap \mathcal{B}$*.*

*In each case, the parameters are carefully chosen so that all required inequalities, including the RMS–RC condition and the threshold* $s\theta < 1$*, are rigorously satisfied on the prescribed domains.*

**Example 4.6.** *Let* $X = \mathbb{R}$ *and define*

$$d(\xi, \eta) := |\xi - \eta|^\gamma, \qquad \xi, \eta \in \mathbb{R},$$

*where* $\gamma \in (0, 1)$*. Then* $d$ *is not a metric for* $\gamma < 1$*, but it satisfies*

$$|\xi - \zeta|^\gamma \le (|\xi - \eta| + |\eta - \zeta|)^\gamma \le 2^{1-\gamma}\big(|\xi - \eta|^\gamma + |\eta - \zeta|^\gamma\big),$$

*for all* $\xi, \eta, \zeta \in \mathbb{R}$*. Hence* $(X, d)$ *is a b-metric space with coefficient*

$$s = 2^{1-\gamma} > 1.$$

*Define* $\Phi : X \to X$ *by* $\Phi(\xi) = \lambda\xi$*, where* $|\lambda| < 1$*. Then*

$$d(\Phi\xi, \Phi\eta) = |\lambda|^\gamma \, d(\xi, \eta),$$

*and*

$$d(\xi, \Phi\xi) = |(1 - \lambda)\xi|^\gamma \le (1 + |\lambda|)^\gamma |\xi|^\gamma, \qquad d(\eta, \Phi\eta) \le (1 + |\lambda|)^\gamma |\eta|^\gamma.$$

*Choose* $(\alpha, \beta, q) = \big(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}\big)$*. Then*

$$\theta = \frac{q + \beta}{1 - \alpha} = \frac{1/2}{3/4} = \frac{2}{3}.$$

*The admissibility condition* $s\theta < 1$ *becomes*

$$2^{1-\gamma} \cdot \frac{2}{3} < 1 \quad \Longleftrightarrow \quad \gamma > \log_2\!\left(\frac{3}{2}\right) \approx 0.584962.$$

*We now verify the RMS–RC inequality. Since the denominator in* (3.1) *is at least* 1*, it suffices to show*

$$|\lambda|^\gamma \, d(\xi, \eta) \le q \, d(\xi, \eta) + \alpha \, d(\xi, \Phi\xi) + \beta \, d(\eta, \Phi\eta).$$

*Using the subadditivity* $|\xi - \eta|^\gamma \le |\xi|^\gamma + |\eta|^\gamma$*, we estimate*

$$\alpha \, d(\xi, \Phi\xi) + \beta \, d(\eta, \Phi\eta) \le \tfrac{1}{4}(1 + |\lambda|)^\gamma \big(|\xi|^\gamma + |\eta|^\gamma\big) \le \tfrac{1}{4}(1 + |\lambda|)^\gamma \, d(\xi, \eta).$$

*Thus the right-hand side satisfies*

$$q \, d(\xi, \eta) + \alpha \, d(\xi, \Phi\xi) + \beta \, d(\eta, \Phi\eta) \le \Big(\tfrac{1}{4} + \tfrac{1}{4}(1 + |\lambda|)^\gamma\Big) d(\xi, \eta).$$

*Since $|\lambda| < 1$, one has $(1 + |\lambda|)^\gamma < 2^\gamma < 2$, and hence the coefficient exceeds $|\lambda|^\gamma$. Therefore, the RMS–RC inequality holds on $\mathbb{R}$.*

*Consequently, all assumptions of Theorem 3.1 are satisfied whenever $\gamma > \log_2(3/2)$. The unique fixed point of $\Phi$ is obtained from*

$$\xi^* = \Phi\xi^* \iff \xi^* = \lambda\xi^* \iff \xi^* = 0.$$

*Moreover, for any $\xi_0 \in \mathbb{R}$, the Picard sequence $\xi_{n+1} = \Phi\xi_n$ converges to 0, and the successive differences satisfy*

$$d(\xi_{n+1}, \xi_n) \leq \theta\, d(\xi_n, \xi_{n-1}) = \tfrac{2}{3}\, d(\xi_n, \xi_{n-1}), \qquad n \geq 1.$$

*Hence, the convergence is linear, with rate governed by $(s\theta)^n = (2^{1-\gamma} \cdot \frac{2}{3})^n$, reflecting the influence of the b-metric coefficient $s > 1$.*

# 5 Application

Let $\gamma \in (0, 1)$ and consider the space $X := C([0, 1])$ equipped with the $b$-metric

$$d_\gamma(x, y) := \|x - y\|_\infty^\gamma, \qquad s = 2^{1-\gamma} > 1.$$

Let $g \in C([0, 1])$, and let $k \in C([0, 1]^2)$ be bounded with $\|k\|_\infty \leq K$. Assume that $\varphi : \mathbb{R} \to \mathbb{R}$ is Lipschitz continuous with constant $L_\varphi$ and satisfies $\varphi(0) = 0$. For a parameter $\lambda \in \mathbb{R}$, consider the Volterra integral equation

$$x(t) = g(t) + \lambda \int_0^t k(t, s)\, \varphi(x(s))\, ds, \qquad t \in [0, 1]. \tag{5.1}$$

Define the operator $\Phi : X \to X$ by

$$(\Phi x)(t) := g(t) + \lambda \int_0^t k(t, s)\, \varphi(x(s))\, ds.$$

Assume that

$$2^{1-\gamma}\big(|\lambda| K L_\varphi\big)^\gamma < 1. \tag{5.2}$$

Then the following statements hold:

(i) The equation (5.1) admits a unique solution $x^* \in X$.

(ii) For any initial function $x_0 \in X$, the Picard sequence $x_{n+1} = \Phi x_n$ converges to $x^*$ in $d_\gamma$.

(iii) The convergence satisfies the estimates

$$d_\gamma(x_{n+1}, x_n) \leq \theta\, d_\gamma(x_n, x_{n-1}), \qquad d_\gamma(x_n, x^*) \leq \frac{s}{1-\theta}\, d_\gamma(x_n, x_{n-1}),$$

and

$$d_\gamma(x_n, x^*) \leq \frac{s}{1-\theta}\, \theta^{n-1}\, d_\gamma(x_1, x_0), \quad n \geq 1,$$

where $\theta = (|\lambda| K L_\varphi)^\gamma$.

*Proof.* First, we verify that $(X, d_\gamma)$ is a complete $b$-metric space. For $\gamma \in (0,1)$,

$$\|u+v\|_\infty^\gamma \le (\|u\|_\infty + \|v\|_\infty)^\gamma \le 2^{1-\gamma}\big(\|u\|_\infty^\gamma + \|v\|_\infty^\gamma\big),$$

which shows that $d_\gamma$ satisfies the $b$-metric inequality with coefficient $s = 2^{1-\gamma}$. Moreover, $d_\gamma$-Cauchy sequences are Cauchy in the sup-norm, hence converge uniformly; therefore $(X, d_\gamma)$ is complete.

Next, for any $x, y \in X$ and $t \in [0,1]$, we estimate

$$|(\Phi x)(t) - (\Phi y)(t)| \le |\lambda| \int_0^t |k(t,s)|\, |\varphi(x(s)) - \varphi(y(s))|\, ds \le |\lambda| K L_\varphi \int_0^t |x(s) - y(s)|\, ds.$$

Taking supremum over $t$ yields

$$\|\Phi x - \Phi y\|_\infty \le |\lambda| K L_\varphi \, \|x - y\|_\infty.$$

Thus,

$$d_\gamma(\Phi x, \Phi y) \le (|\lambda| K L_\varphi)^\gamma \, d_\gamma(x,y) = \theta\, d_\gamma(x,y),$$

where $\theta = (|\lambda| K L_\varphi)^\gamma$.

Hence $\Phi$ is a contraction in the sense of the RMS–RC framework with $\alpha = \beta = 0$. The admissibility condition $s\theta < 1$ is exactly (5.2). Therefore, by Theorem 3.1, $\Phi$ admits a unique fixed point $x^* \in X$, and the Picard iteration converges to $x^*$. The fixed point equation $\Phi x^* = x^*$ coincides with (5.1).

Finally, the convergence estimates follow directly from Corollary 3.1 with $\theta = (|\lambda| K L_\varphi)^\gamma$ and $s = 2^{1-\gamma}$. □

**Remark 5.1.** *The contraction constant $\theta = (|\lambda| K L_\varphi)^\gamma$ arises naturally from the Lipschitz estimate of the integral operator. The presence of the $b$-metric coefficient $s = 2^{1-\gamma} > 1$ modifies the classical contraction condition to $s\theta < 1$. For instance, if $\gamma = 0.8$, then $s = 2^{0.2} \approx 1.1892$, and the admissible range becomes*

$$|\lambda| K L_\varphi < 2^{\frac{\gamma-1}{\gamma}} \approx 0.8409,$$

*which is slightly more restrictive than the classical metric case.*

# 6 Conclusion

In this work, we introduced a rational contractive framework tailored to $b$-metric spaces, where the relaxation of the triangle inequality is governed by a coefficient $s \ge 1$. The proposed RMS–Rational Contraction (RMS–RC) integrates mixed distance terms through a rational structure and provides a unified setting that extends several classical contraction principles. Within this framework, we established existence and uniqueness of fixed points, convergence of Picard iterations, and sharp quantitative estimates including both a priori and a posteriori error bounds, along with a stability result under perturbations. Furthermore, a cyclic version of the main theorem was developed for mappings on closed pairs, broadening the applicability of the theory to more general iterative processes.

A key feature of our analysis is that the convergence condition depends on the combined quantity $s\theta$, highlighting the essential influence of the $b$-metric geometry. The results recover, as special or limiting cases, well-known principles such as Banach, Kannan–Chatterjea, Hardy–Rogers, Meir–Keeler, Boyd–Wong, Geraghty, and Wardowski-type contractions. The constructed examples, including genuine $b$-metric spaces with $s > 1$, confirm the effectiveness and sharpness of the approach in both linear and nonlinear contexts. Future work may focus on multivalued extensions via Hausdorff $b$-metrics, development of adaptive or $\psi$-type rational contractions, and applications to fractional differential equations, integral equations, and computational fixed point methods.

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