Stability of Solutions to Thermoradiative Models with Temperature-Dependent Viscosity

Abstract. This paper focuses on a class of radiation hydrodynamics models where the transport coefficients depend on temperature, investigating in detail the existence of global strong solutions for the initial-boundary value problem. A local existence theory for solutions is established for a fluid model that incorporates radiation effects, with viscosity $\mu(\theta) = \theta^\alpha$ and thermal conductivity $\kappa(\theta) = \widetilde{\kappa}(1 + \theta^\beta)$, under specific initial conditions. Compared with the work of Wei, Zhang, and Zhu the results of the present work have two distinct advantages: first, our proof is time-uniform; second, it does not require higher integrability conditions on the solutions.

Keywords. radiation hydrodynamics model, Temperature-dependent transport coefficients, large initial data, time-uniform
Math Subject Classification. 35Q35, 76N10

1 Introduction

The core objective of radiation hydrodynamics is to reveal the basic mechanisms underlying the interaction between compressible viscous gases and radiation, and such models hold considerable application potential in various fields. Through the introduction of Lagrangian coordinates, it is possible to construct a radiation hydrodynamics model that takes both viscosity and thermal conductivity effects into account [1–5].

$$v_t = u_x, \tag{1.1}$$

$$u_t + P_x = \left(\mu \frac{u_x}{v}\right)_x, \tag{1.2}$$

$$\left(e + \frac{1}{2}u^2\right)_t + (Pu)_x + q_x = \left(\frac{\kappa\theta_x + \mu u u_x}{v}\right)_x, \tag{1.3}$$

$$-\left(\frac{q_x}{v}\right)_x + avq + b(\theta^4)_x = 0, \tag{1.4}$$

Here, $t > 0$ denotes the time variable, and $x \in \Omega$ denotes the Lagrangian mass coordinate. The unknown functions, namely $v(x,t) > 0, u(x,t) > 0, \theta(x,t) > 0, e(x,t) > 0, q(x,t)$ and $P(x,t)$, correspond to the fluid's specific volume, velocity, absolute temperature, internal energy, radiative heat flux, and pressure, respectively. The positive constants a and b depend solely on the inherent properties of the gas. Additionally, the pressure P and internal energy e satisfy the equations of state for ideal polytropic fluids:

$$P = \frac{R\theta}{v}, \quad e = c_v\theta, \tag{1.5}$$

where $R > 0$ is the specific gas constant and $c_v > 0$ is the heat capacity at constant volume. The transport coefficients μ and κ, both functions of θ, are given by:

$$\mu = \tilde{\mu}\theta^{\alpha}, \quad \kappa = \tilde{\kappa}\left(\theta^{\beta} + 1\right), \tag{1.6}$$

where $\tilde{\mu}, \tilde{\kappa} > 0$ and $\alpha, \beta \geq 0$ [6–8]. Without loss of generality, $\Omega \triangleq (0,1)$is set. The primary goal of this work is to prove the global existence and stability of solutions for the initial-boundary value problem associated with (1.1)–(1.6) with the given initial data:

$$(v, u, \theta)(x, 0) = (v_0, u_0, \theta_0)(x), \quad x \in (0, 1), \tag{1.7}$$

and the boundary conditions:

$$u(0,t) = u(1,t) = 0, \quad q(0,1) = q(1,t) = 0, \quad \theta_x(0,t) = \theta_x(1,t) = 0, \quad t \geq 0. \tag{1.8}$$

In the absence of radiation effects, the model under consideration reduces to the classical compressible Navier-Stokes equations. Kazhikhov et al. [9, 10] first exploited the structural properties of the one-dimensional system.In the case where the viscosity coefficient μ is constant and the heat conductivity κ may depend on temperature, they successfully derived positive upper and lower bounds for the specific volume v and the temperature θ by constructing an explicit expression for the specific volume v of ideal polytropic gases and combining it with the maximum principle. Consequently, they established the global existence and uniqueness of solutions to the one-dimensional viscous and heat-conducting ideal gas equations in bounded domains under large initial data. This method has also been widely extended and applied in subsequent studies [11–14]. For positive constant transport coefficients μ and κ, Li and Liang [15] obtained uniform a priori estimates and investigated the large-time asymptotic behavior of solutions on unbounded domains.Wang and Zhao [16], inspired by natural physical processes, investigated the density- and temperature-dependent regime; that is

$$\mu = \widetilde{\mu} g(v)\theta^{\alpha} \quad \text{and} \quad \kappa = \widetilde{k} g(v)\theta^{\alpha},$$

where $\widetilde{\mu}, \widetilde{\kappa}$ are positive constants, and $g(\cdot) \in C^3(0, \infty)$ satisfies

$$v^{l_1} + v^{-l_2} \leq Cg(v), \quad g'(v)^2 v \leq Cg(v)^3, \quad \forall\ v \in (0, \infty),$$

for some positive numbers $l_1, l_2 \geq 1$ and $C > 0$.

By virtue of the specific construction of the viscosity coefficient μ and heat conductivity coefficient κ mentioned above, Wang and Zhao [16] controlled the nonlinear terms and thus established the global existence and uniqueness of non-vacuum solutions. Moreover, the function $g(\cdot)$ involved in the viscosity term satisfies the corresponding control conditions. This technique is mathematically nontrivial and allows the authors to obtain the upper and lower bounds of the specific volume via Kanel's method (cf. [17]). Sun, Zhang and Zhao [18] investigated the one-dimensional viscous heat-conducting ideal gas model in bounded domains with temperature-dependent transport coefficients under higher-order nonlinear assumptions. They established the global existence, uniqueness and large-time asymptotic behavior of solutions, where the viscosity and thermal conductivity coefficients take the forms $\mu = \theta^\alpha(\alpha > 0$ sufficiently small) and $\kappa = \theta^\beta$ $(0 < \beta < \infty$ arbitrary), respectively. Recently, Wei, Zhang and Zhu [19] established the global existence and uniqueness of solutions to the radiative heat-conducting model with constant viscosity in unbounded domains.

Without loss of generality, we assume that $\tilde{\mu} = R = c_v = 1$. The main result of this paper can be formulated as follows.

Theorem 1.1. *Let M_0 and V_0 be fixed positive constants, and suppose that*

$$\inf_{x\in[0,1]} v_0(x) \geq V_0, \quad \inf_{x\in[0,1]} \theta_0(x) \geq V_0, \quad \|v_0\|_{H^2} + \|(u_0,\theta_0)\|_{H^1} \leq M_0. \tag{1.9}$$

Then there exists a positive constant $\epsilon_0 > 0$, depending only on M_0, V_0, $\tilde{\kappa}$ and β, such that for parameters

$$0 \leq \alpha \leq \epsilon_0, \quad \widetilde{\kappa} \geq \frac{10b}{a}, \quad \beta \geq 6,$$

the problem (1.1)–(1.8) *admits a unique global strong solution* $(v, u, \theta, q)(x, t)$ *on* $[0, 1] \times [0, \infty)$*, satisfying*

$$\inf_{(x,t)\in[0,1]\times[0,\infty)} \{v(x,t), \theta(x,t)\} > 0, \qquad \sup_{(x,t)\in[0,1]\times[0,\infty)} \{v(x,t), \theta(x,t)\} < \infty, \tag{1.10}$$

and for any $s > 0$

$$\begin{cases} v - 1 \in C(0,\infty; H^2) \cap L^2(0,\infty; \dot{H}^1 \cap \dot{H}^2), \\ (u, \theta - 1) \in C(0,\infty; H^2) \cap L^2(0,\infty; \dot{H}^1 \cap \dot{H}^3), \\ (u_t, \theta_t) \in C(0,\infty; L^2) \cap L^2(0,\infty; H^1), \\ q \in C(0,\infty; H^3) \cap L^2(0,\infty; H^3), \\ q_t \in C(s,\infty; H^1) \cap L^2(0,\infty; H^2). \end{cases} . \tag{1.11}$$

Moreover, the solution converges to the non-vacuum equilibrium state $(1, 0, 1, 0)$ *in the sense that*

$$\|(v - 1, u, \theta - 1, q)(t)\|_{L^\infty} \to 0 \quad as \quad t \to \infty., \tag{1.12}$$

A review of the analysis in this paper is presented, and the key ideas leading to the main result (Theorem 1.1) are outlined. A crucial step is to derive positive upper and lower bounds for the specific volume and temperature. To this end, we first adopt the analytical framework developed by Kazhikhov [10] and Nishida [20] to obtain a local representation of the specific volume. In this expression, the nonlinear difficulty arising from the fact that the viscosity is a power-law function of temperature is handled by assuming that the parameter α is sufficiently small, which allows us to establish bounds for the specific volume. Nevertheless, the presence of the radiative heat flux term creates substantial challenges in obtaining a lower bound for the temperature. To overcome this difficulty, more refined estimates for both the temperature and the radiative heat flux are performed, using Young's inequality together with suitable a priori assumptions. Finally, relying on the arguments developed in Section 2, the global well-posedness and stability of strong solutions are established.

The rest of this paper is organized as follows. In Section 2, the necessary a priori estimates are derived. Based on these estimates and the continuation method, Section 3 first establishes uniform positive upper and lower bounds for the temperature $\theta(x, t)$ and the specific volume $v(x, t)$. The local solution is then extended to a global-in-time solution, and the existence, uniqueness, and large-time asymptotic behavior of strong solutions are established, thereby completing the proof of 1.1.

2 A Priori Estimates

For positive numbers m_1, m_2, N and T, define

$$\begin{aligned} &X(0,T;m_1,m_2,N)\\ &\triangleq \Big\{(v,u,\theta,q):(v-1,u,\theta-1)\in C(0,T;H^2),\ (u_x,\theta_x)\in L^2(0,T;H^2),\\ &\qquad q\in C(0,T;H^3)\cap L^2(0,T;H^3), v_x\in L^\infty(0,T;L^2)\cap L^2(0,T;H^1),\\ &\qquad \mathcal{E}(0,T)\le N^2,\ v(x,t)\ge m_1,\ \theta(x,t)\ge m_2,\ \forall\ (x,t)\in[0,1]\times[0,T]\Big\}, \end{aligned} \tag{2.1}$$

where

$$\mathcal{E}(0,T)\triangleq \sup_{0\le t\le T}\|\left(u_x,v_x,\theta_x\right)(t)\|_{H^1}^2+\int_0^T(\|\theta_t\|^2+\|u_x\|^2+\|v_x\|^2)\mathrm{d}t.. \tag{2.2}$$

To simplify the presentation, it is assumed without loss of generality that

$$\int_0^1 v_0(x)dx=1 \quad\text{and}\quad \int_0^1\left(\theta_0+\frac{u_0^2}{2}\right)(x)dx=1. \tag{2.3}$$

Then it is easy to obtain from (1.1) and (1.3) that for any $t\in[0,T]$,

$$\int_0^1 v(x,t)dx=\int_0^1 v_0(x)dx=1 \tag{2.4}$$

and

$$\int_0^1\left(\theta+\frac{u^2}{2}\right)(x,t)dx=\int_0^1\left(\theta_0+\frac{u_0^2}{2}\right)(x)dx=1. \tag{2.5}$$

Therefore, similar to the Navier-Stokes equations, using (1.4), One can obtain an estimation pertaining to radiative heat flux q_x. The estimate processes are standard; These are not repeated here. (one can see Wei, Zhang and Zhu [19] for details).

Lemma 2.1. *It holds that for any $(x,t)\in[0,1]\times[0,T]$,*

$$q_x\le bv\theta^4, \tag{2.6}$$

Lemma 2.2. *It holds that*

$$\begin{aligned} &\sup_{0\le t\le T}\int_0^1\left(\frac{1}{2}u^2+v-\ln v-1+\theta-\ln\theta-1\right)(x,t)\mathrm{d}x\\ &\quad+\int_0^T\int_0^1\left(\frac{\theta^\alpha u_x^2}{v\theta}+\frac{\theta_x^2}{v\theta^2}+\frac{vq^2}{\theta^2}+\frac{\theta^{\beta-2}\theta_x^2}{v}+\frac{\theta_x^2}{v}+\frac{q_x^2}{v\theta^2}\right)(x,t)\mathrm{d}x\mathrm{d}t\le C. \end{aligned} \tag{2.7}$$

Proof. From (1.2) and (1.3), we obtain:

$$e_t+Pu_x+q_x=\left(\frac{\kappa\theta_x}{v}\right)_x+\frac{\mu u_x^2}{v}. \tag{2.8}$$

By multiplying (1.1), (1.2), and (2.3) by $1-v^{-1}$, u, and $1-\theta^{-1}$ respectively, and integrating over $[0,1]$, The following results are derived.

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1\left(\frac{1}{2}u^2+(v-\ln v-1)+(\theta-\ln\theta-1)\right)\mathrm{d}x\\ &\quad+\int_0^1\left(\frac{\mu u_x^2}{v\theta}+\frac{\kappa\theta_x^2}{v\theta^2}\right)\mathrm{d}x=-\int_0^1 q_x\frac{\theta-1}{\theta}\mathrm{d}x. \end{aligned} \tag{2.9}$$

Applying the Cauchy-Schwarz inequality yields

$$
\begin{aligned}
-\int_0^1 q_x\left(\frac{\theta-1}{\theta}\right)\mathrm{d}x &= \int_0^1 q\frac{\theta_x}{\theta^2}\mathrm{d}x \\
&\leq \frac{a}{4b}\int_0^1 \frac{vq^2}{\theta^2}\mathrm{d}x + \frac{b}{a}\int_0^1 \frac{\theta_x^2}{v\theta^2}\mathrm{d}x.
\end{aligned} \tag{2.10}
$$

Substituting (2.10) into (2.9), The following is obtained.

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1\left(\frac{1}{2}u^2+(v-\ln v-1)+(\theta-\ln\theta-1)\right)\mathrm{d}x \\
&+\int_0^1\left(\frac{\mu u_x^2}{v\theta}+\frac{\widetilde{\kappa}\theta_x^2}{v\theta^2}+\frac{\widetilde{\kappa}\theta^{\beta-2}\theta_x^2}{v}\right)\mathrm{d}x \\
&\leq \frac{a}{4b}\int_0^1\frac{vq^2}{\theta^2}\mathrm{d}x+\frac{b}{a}\int_0^1\frac{\theta_x^2}{v\theta^2}\mathrm{d}x.
\end{aligned} \tag{2.11}
$$

Multiplying (1.4) by $\frac{q}{\theta^2}$ and integrating over $[0,1]$, while applying (2.6), The following is obtained.

$$
\begin{aligned}
\int_0^1\left(\frac{q_x^2}{v\theta^2}+\frac{avq^2}{\theta^2}\right)\mathrm{d}x &= -4b\int_0^1\theta\theta_x q\mathrm{d}x+\int_0^1\frac{2qq_x\theta_x}{v\theta^3}\mathrm{d}x \\
&\leq -4b\int_0^1\theta\theta_x q\mathrm{d}x+2b\int_0^1\theta|q\theta_x|\mathrm{d}x \\
&\leq \frac{a}{2}\int_0^1\frac{vq^2}{\theta^2}\mathrm{d}x+\frac{18b^2}{a}\int_0^1\frac{1}{v}\theta^4\theta_x^2\mathrm{d}x \\
&\leq \frac{a}{2}\int_0^1\frac{vq^2}{\theta^2}\mathrm{d}x+\frac{18b^2}{a}\int_0^1\frac{1}{v}\left(\theta^{\beta-2}+\frac{1}{\theta^2}\right)\theta_x^2\mathrm{d}x.
\end{aligned} \tag{2.12}
$$

where

$$
\theta^4 \leq \begin{cases} \theta^{\beta-2}, & \text{if } \theta\geq 1, \\ \theta^{-2}, & \text{if } 0<\theta<1. \end{cases} \tag{2.13}
$$

Multiplying (2.11) by $2b$ and adding (2.12), when $\tilde{\kappa} > \frac{10b}{a}$ holds, The following is obtained.

$$
\begin{aligned}
&\int_0^1\left(\frac{1}{2}u^2+(v-\ln v-1)+(\theta-\ln\theta-1)\right)(x,t)\mathrm{d}x \\
&+\int_0^t\int_0^1\left(\frac{\theta^\alpha u_x^2}{v\theta}+\frac{\theta_x^2}{v\theta^2}+\frac{\theta^{\beta-2}\theta_x^2}{v}+\frac{q_x^2}{v\theta^2}\right)(x,\tau)\mathrm{d}x\mathrm{d}\tau \leq C
\end{aligned} \tag{2.14}
$$

Combining (2.12) with (2.14) yields

$$
\int_0^t\int_0^1\left(\frac{\theta_x^2}{v}+\frac{vq^2}{\theta^2}\right)\mathrm{d}x\mathrm{d}\tau \leq \int_0^t\int_0^1\left(\frac{\theta_x^2}{v\theta^2}+\frac{\theta^{\beta-2}\theta_x^2}{v}\right)\mathrm{d}x\mathrm{d}\tau + C \leq C.
$$

Then the proof of Lemma 2.2 is completed.

Therefore, similar to the Navier-Stokes equations, using (1.1), (1.2) and Lemma 2.2, One can obtain an expression for the specific volume $v(x,t)$ along with positive lower and upper bounds for $v(x,t)$.The estimate processes are standard; These are not repeated here. (one can see Ying Sun [18] for details).

Lemma 2.3. *Let $\mu_0 \triangleq \mu(\theta_0)$. Then for any $t \geq 0$, there is a $\eta_0(t) \in (0,1)$ such that*

$$
\begin{aligned}
v(x,t) = B(t)D(x,t) + \int_0^t \frac{B(t)D(x,t)}{B(\tau)D(x,\tau)} v(x,\tau) \\
\times \left(\frac{P}{\mu} + \int_0^x g dy - \int_0^1 v \left(\int_0^x g dy \right) dx \right)(x,\tau) d\tau,
\end{aligned}
\tag{2.15}
$$

where

$$
g(x,t) \triangleq - \left[u \left(\frac{1}{\mu} \right)_t + P \left(\frac{1}{\mu} \right)_x + \frac{\mu_x u_x}{\mu v} \right],
$$

$$
B(t) \triangleq \exp \left\{ - \int_0^t \int_0^1 \frac{\theta + u^2}{\mu} dx d\tau \right\},
$$

and

$$
D(x,t) \triangleq v_0(x) \exp \left\{ \int_{\eta_0(t)}^x \frac{u}{\mu} dy - \int_0^x \frac{u_0}{\mu_0} dy + \int_0^1 v_0 \left(\int_0^x \frac{u_0}{\mu_0} dy \right) dx \right\}.
$$

Lemma 2.4. *There exist two positive constants C_0 and ε_1, depending only on β, V_0 and M_0, such that if $(v,u,\theta) \in X(0,T;m_1,m_2,N)$ is a solution of the problem* (1.1)–(1.7) *on* $(0,T)$*, satisfying*

$$
m_2^{-\alpha} \leq 2, \quad N^\alpha \leq 2, \quad \alpha H(m_1,m_2,N) \leq \varepsilon_1, \tag{2.16}
$$

with $H(m_1,m_2,N) \triangleq \left(1 + m_1^{-1} + m_2^{-1} + N\right)^6$, then

$$
v(x,t) \geq C_0 \triangleq \frac{e^{-4T}}{2C_1}, \quad v(x,t) \leq C_2, \quad \forall\ (x,t) \in \overline{\Omega}_T \triangleq [0,1] \times [0,T]. \tag{2.17}
$$

Proof. In view of(2.7) and (2.5), it is inferred from Jensen's inequality that there exists a positive number $\gamma_1 \in (0,1)$ such that

$$
\bar{\theta}(t) \triangleq \int_0^1 \theta(x,t) dx \in [\gamma_1, 1], \quad \forall\ t \in [0,T] \tag{2.18}
$$

Next, the upper and lower bounds of specific volume are derived based on the representation formula (2.15), the a priori assumption (2.16), as well as the basic estimates (2.4), (2.5), (2.7) and (2.18). First, by virtue of (2.16) and (2.5), it is readily obtained that for $\eta_0(t) \in (0,1)$ as in Lemma 2.3,

$$
\left| \int_{\eta_0(t)}^x \frac{u}{\mu} dy \right| \leq \int_0^1 \frac{|u|}{\theta^\alpha} dy \leq m_2^{-\alpha} \|u(t)\|_{L^2} \leq C
$$

and consequently,

$$
C^{-1} \leq D(x,t) \leq C, \quad \forall\ (x,t) \in \overline{\Omega}_T \tag{2.19}
$$

It follows from the Sobolev's inequality that

$$
\|(\theta - \bar{\theta})(t)\|_{L^\infty} \leq \|\theta_x(t)\|_{L^2} \leq N, \quad \forall\ t \in [0,T] \tag{2.20}
$$

which, combined with (2.18), shows that $\|\theta\|_{L^\infty(\Omega_T)} \le 2N$. Thus,

$$\frac{1}{4} \le (2N)^{-\alpha} \int_0^1 \left(\theta + \frac{u^2}{2}\right) dx \le \int_0^1 \frac{\theta + u^2}{\mu} dx \le 2m_2^{-\alpha} \int_0^1 \left(\theta + \frac{u^2}{2}\right) dx \le 4,$$

so that, it holds for $0 \le \tau \le t \le T$ that

$$e^{-4t} \le B(t) \le e^{-\frac{t}{4}} \quad \text{and} \quad e^{-4(t-\tau)} \le \frac{B(t)}{B(\tau)} \le e^{-\frac{t-\tau}{4}} \tag{2.21}$$

In terms of the definition of g, from (2.16) it is seen that

$$\begin{aligned} \left| v \int_0^x g dy \right| &\le \alpha \|v(t)\|_{L^\infty} \int_0^1 \left(\left|\theta^{-\alpha-1}\theta_t u\right| + \left|\theta^{-\alpha-1}\theta_x P\right| + \left|\frac{u_x\theta_x}{v\theta}\right| \right) dx \\ &\le 2\alpha N m_2^{-\alpha} \left(\frac{1}{m_2} \|\theta_t\|_{L^2} \|u\|_{L^2} + \frac{1}{m_1} \|\theta_x\|_{L^2} \right) + \frac{2\alpha N}{m_1 m_2} \|u_x\|_{L^2} \|\theta_x\|_{L^2} \\ &\le C\alpha H(m_1, m_2, N) + \frac{\alpha N}{m_2} \|\theta_t\|_{L^2}^2 \end{aligned} \tag{2.22}$$

where the Cauchy-Schwarz's inequality and the following elementary facts have been used:

$$\|v\|_{L^\infty} \le \|v\|_{L^1} + \|v_x\|_{L^2} \le 1 + N \le 2N, \quad \|u\|_{L^2} \le \|u_x\|_{L^2} \le N.$$

In a similar manner,

$$\left| v \int_0^1 v \left(\int_0^x g dy \right) dx \right| \le C\alpha H(m_1, m_2, N) + \frac{\alpha N}{m_2} \|\theta_t\|_{L^2}^2. \tag{2.23}$$

Assume that for any fixed $T > 0$, and for $(x,t) \in (0,1) \times (0,T)$, from (2.15) it is derived by (2.16) and (2.19)–(2.23) that

$$\begin{aligned} v(x,t) &\ge B(t)D(x,t) - C\alpha H(m_1, m_2, N) \int_0^T e^{-\frac{T-\tau}{4}} d\tau - C\frac{\alpha N}{m_2} \int_0^T \|\theta_t\|_{L^2}^2 dt \\ &\ge C_1^{-1} e^{-4T} - C_1 \alpha H(m_1, m_2, N) \ge C_1^{-1} e^{-4T} - C_1 \varepsilon_1 \ge \frac{e^{-4T}}{2C_1} \end{aligned} \tag{2.24}$$

provided ε_1 is chosen to be such that $\varepsilon_1 \le e^{-4T}/(2C_1^2)$. So, it readily follows from(2.24) that

$$v(x,t) \ge C_0 \triangleq \frac{e^{-4T}}{2C_1} \tag{2.25}$$

Let $W(t) \triangleq \int_0^1 \left(\frac{\theta^\alpha u_x^2}{v\theta} + \frac{\theta_x^2}{v\theta^2} + \frac{\theta^{\beta-2}\theta_x^2}{v} \right)(x,t)\mathrm{d}x$. In view of (2.4) and (2.18), it holds that

$$\begin{aligned} \theta^{\frac{\beta+1}{2}}(x,t) - \bar{\theta}^{\frac{\beta+1}{2}}(t) &\le \frac{\beta+1}{2} \left(\int_0^1 \frac{\theta^\beta \theta_x^2}{v\theta^2} dx \right)^{\frac{1}{2}} \left(\int_0^1 \theta v dx \right)^{\frac{1}{2}} \\ &\le C W^{\frac{1}{2}}(t) \max_{x\in[0,1]} v^{\frac{1}{2}}(x,t) \end{aligned}$$

so that, using (2.18) and the Cauchy-Schwarz's inequality, it is found that

$$\max_{x\in[0,1]} \theta(x,t) \le C + CW(t) \max_{x\in[0,1]} v(x,t) \tag{2.26}$$

In terms of (2.19)–(2.23) and (2.26), from (2.16) it is seen that for any $(x,t)\in\Omega_T$,

$$\begin{aligned} v(x,t) &\le C + C\alpha H(m_1,m_2,N) + Cm_2^{-\alpha}\int_0^t e^{-\frac{t-\tau}{4}}\max_{x\in[0,1]}\theta d\tau \\ &\le C + C\int_0^t e^{-\frac{t-\tau}{4}}\left(1+W(\tau)\max_{x\in[0,1]}v(x,\tau)\right)d\tau \\ &\le C + C\int_0^t W(\tau)\max_{x\in[0,1]}v(x,\tau)d\tau \end{aligned}$$

which, combined with (2.7) and Grönwall's inequality, yields

$$v \le Ce^{\int_0^t W(\tau)d\tau} \le C_2 \tag{2.27}$$

This, together with(2.25), finishes the proof of Lemma 2.4. □

Corollary 2.1. *Assume that if the inequality*

$$\int_0^\infty\int_0^1\left(\frac{\theta^\alpha u_x^2}{v\theta}+\frac{\theta_x^2}{v\theta^2}+\frac{\theta^{\beta-2}\theta_x^2}{v}\right)(x,t)\mathrm{d}x\mathrm{d}t \le C$$

is satisfied,then on the basis of Lemma 2.4, it follows that

$$\bar{C}_0 \le v(x,t) \le \bar{C}_0^{-1}, \quad \forall\ (x,t)\in[0,1]\times[0,+\infty). \tag{2.28}$$

Proof. Let $f_+ \triangleq \max\{f,0\}$. In view of (2.4) and (2.18), it holds that

$$\begin{aligned} \left(\bar{\theta}^{\frac{\beta+1}{2}}(t)-\theta^{\frac{\beta+1}{2}}(x,t)\right)_+ &\le C\left(\int_0^1\frac{\theta^\beta\theta_x^2}{v\theta^2}dx\right)^{\frac12}\left(\int_0^1 v\theta\chi_{(\theta\le\bar{\theta})}dx\right)^{\frac12}, \\ &\le CW^{\frac12}(t) \end{aligned}$$

where $\chi_{(\theta\le\bar{\theta})}=1$ if $\theta\le\bar{\theta}$, and $\chi_{(\theta\le\bar{\theta})}=0$ if $\theta>\bar{\theta}$.As a result,

$$\min_{x\in[0,1]}\theta(x,t)\ge C_1 - C_2W(t) \tag{2.29}$$

Using (2.7),(2.19)–(2.23) and (2.29), it is inferred from (2.15) and (2.16) that

$$\begin{aligned} v(x,t) &\ge C^{-1}N^{-\alpha}\int_0^t e^{-4(t-\tau)}\min_{x\in[0,1]}\theta d\tau - C\frac{\alpha N}{m_2}\int_0^t\|\theta_\tau\|_{L^2}^2 d\tau \\ &\quad - C\alpha H(m_1,m_2,N)\int_0^t e^{-\frac{t-\tau}{4}}d\tau \\ &\ge C^{-1}\int_0^t e^{-4(t-\tau)}\left(C_2-C_3W(\tau)\right)d\tau - C\alpha H(m_1,m_2,N) \\ &\ge \frac{C_2}{4C}\left(1-e^{-4t}\right)-\frac{C_3}{C}\int_0^t e^{-4(t-\tau)}W(\tau)d\tau - C\varepsilon_1. \end{aligned} \tag{2.30}$$

It is easy to deduce from the assumptions in the corollary 2.1 that

$$\begin{aligned} \int_0^t e^{-4(t-\tau)}W(\tau)d\tau &= \int_0^{\frac{t}{2}} e^{-4(t-\tau)}W(\tau)d\tau + \int_{\frac{t}{2}}^t e^{-4(t-\tau)}W(\tau)d\tau \\ &\le e^{-2t}\int_0^{\frac{t}{2}}W(\tau)d\tau + \int_{\frac{t}{2}}^t W(\tau)d\tau \\ &\to 0, \quad \text{as} \quad t\to\infty \end{aligned}$$

Thus, if $\tilde{T} > 0$ is chosen to be sufficiently large such that

$$\frac{C_2 e^{-4t}}{4C} + \frac{C_3}{C}\int_0^t e^{-4(t-\tau)} W(\tau) d\tau \le \frac{C_2}{16C}, \quad \forall t \ge \tilde{T}$$

then one has

$$v(x,t) \ge \frac{C_2}{8C}, \quad \forall\, x \in [0,1],\ t \ge \tilde{T} \tag{2.31}$$

provided $\varepsilon_1 > 0$ is chosen to be small enough such that $\varepsilon_1 \le \min\{1, C_2/(16C^2)\}$. So, it readily follows from(2.24) and(2.31) that

$$v(x,t) \ge \bar{C}_0 \triangleq \min\left\{\frac{C_2}{8C}, \frac{e^{-4\tilde{T}}}{2C_1}\right\} \tag{2.32}$$

provided $\alpha H(m_1, m_2, N) \le \varepsilon_1$ with $\varepsilon_1 \le \min\{1, C_2/(16C^2), e^{-4\tilde{T}}/(2C_1^2)\}$.

This, together with the upper bound of v in Lemma 2.4, finishes the proof of corollary 2.1.

Lemma 2.5. *Let the conditions of Lemma 2.4 be in force. Then for any $p > 2$, there exists a positive constant C, which may depend on p, such that*

$$\int_0^T \int_0^1 \frac{\kappa \theta_x^2}{\theta^{p+1}} dxdt + \int_0^T \int_0^1 \frac{{q_x}^2}{v\theta^{p+1}} dxdt \le C(p) \quad \textit{and} \quad \int_0^T \|u_x\|_{L^2}^2 dt \le C. \tag{2.33}$$

Proof. It suffices to consider the case that $p > 2$. Indeed, multiplying (2.8) by θ^{-p} with $p > 2$ and integrating by parts, from (1.1), it follows that

$$\begin{aligned}
\frac{1}{p-1}\left(\int_0^1 \theta^{1-p} \mathrm{d}x\right)_t &+ p\int_0^1 \frac{\kappa\theta_x^2}{v\theta^{p+1}}\mathrm{d}x + \int_0^1 \frac{\mu u_x^2}{v\theta^p}\mathrm{d}x \\
&= \int_0^1 \frac{\left(\theta^{1-p} - 1\right) u_x}{v}\mathrm{d}x + \left(\int_0^1 \ln v \mathrm{d}x\right)_t + \int_0^1 q_x \theta^{-p}\mathrm{d}x \\
&\le C(p)\int_0^1 |\theta^{\frac{1}{2}} - 1| \left(\theta^{\frac{1}{2}-p} + 1\right)|u_x|\mathrm{d}x \\
&\quad + \left(\int_0^1 \ln v\mathrm{d}x\right)_t + p\int_0^1 \frac{q\cdot\theta_x}{\theta^{p+1}}\mathrm{d}x \\
&\le C(p)\left\|\theta^{\frac{1}{2}} - 1\right\|_{L^\infty}\left(\int_0^1 \frac{v\theta^{1-p}}{\mu}\mathrm{d}x\right)^{\frac{1}{2}}\left(\int_0^1 \frac{\mu u_x^2}{v\theta^p}\mathrm{d}x\right)^{\frac{1}{2}} \\
&\quad + C(p)\left\|\theta^{\frac{1}{2}} - 1\right\|_{L^\infty}\int_0^1 |u_x|\mathrm{d}x + \left(\int_0^1 \ln v\mathrm{d}x\right)_t \\
&\quad + C(p)\int_0^1 \frac{q^2}{\theta^{p+1}}\mathrm{d}x + \frac{1}{2}\int_0^1 \frac{\theta_x^2}{\theta^{p+1}}\mathrm{d}x \\
&\le \frac{1}{2}\int_0^1 \frac{\theta_x^2}{\theta^{p+1}}\mathrm{d}x + \frac{1}{2}\int_0^1 \frac{\mu u_x^2}{v\theta^p}\mathrm{d}x \\
&\quad + C(p)\left\|\theta^{\frac{1}{2}} - 1\right\|_{L^\infty}^2\left(1 + \int_0^1 \theta^{1-p}\mathrm{d}x\right) + C(p)\left(\int_0^1 |u_x|\mathrm{d}x\right)^2 \\
&\quad + C(p)\int_0^1 \frac{q^2}{\theta^{p+1}}\mathrm{d}x + \left(\int_0^1 \ln v\mathrm{d}x\right)_t
\end{aligned} \tag{2.34}$$

where (2.16) and (2.17) have also been used to obtain that

$$\int_0^1 \frac{v\theta^{1-p}}{\mu}dx \le C_0^{-1}m_2^{-\alpha}\int_0^1 \theta^{1-p}dx \le C\int_0^1 \theta^{1-p}dx. \tag{2.35}$$

In view of (2.5), (2.16) and (2.17), it is found that

$$\int_0^1 |u_x|dx \le C_0^{-\frac{1}{2}}m_2^{-\frac{\alpha}{2}}\left(\int_0^1 \frac{\mu u_x^2}{v\theta}dx\right)^{\frac{1}{2}}\left(\int_0^1 \theta dx\right)^{\frac{1}{2}} \le CW^{\frac{1}{2}}(t). \tag{2.36}$$

By virtue of (2.5) and (2.18), we obtain that for any $(x,t)\in\overline{\Omega}_T$,

$$\begin{aligned}\left|1-\bar{\theta}^{\frac{1}{2}}\right| &\le C|1-\bar{\theta}| = C\left|1-\int_0^1 \theta dx\right| \le C\|u\|_{L^2}^2 \le C\|u(t)\|_{L^2}\|u(t)\|_{L^\infty}\\ &\le C\int_0^1 |u_x|dx \le CW^{\frac{1}{2}}(t).\end{aligned} \tag{2.37}$$

Analogously, using (2.4), (2.5) and (2.17), we see that for any $\beta\ge 6$,

$$\begin{aligned}\left|\theta^{\frac{1}{2}}-\bar{\theta}^{\frac{1}{2}}\right| &\le \left|\theta^{\frac{\beta+1}{2}}(x,t)-\bar{\theta}^{\frac{\beta+1}{2}}(t)\right|\\ &\le \frac{\beta+1}{2}\left(\int_0^1 \frac{\theta^\beta\theta_x^2}{v\theta^2}dx\right)^{\frac{1}{2}}\left(\int_0^1 \theta v dx\right)^{\frac{1}{2}}\\ &\le CW^{\frac{1}{2}}(t).\end{aligned} \tag{2.38}$$

Combining (2.35)–(2.38) with (2.7) gives

$$\int_0^t\left(\int_0^1 |u_x|dx\right)^2 d\tau + \int_0^t \left\|\theta^{\frac{1}{2}}-1\right\|_{L^\infty}^2 d\tau \le C\int_0^t W(\tau)d\tau \le C. \tag{2.39}$$

Multiplying (1.4) by $\frac{q}{\theta^{p+1}}$ where $q>2$ yields

$$-(\frac{q_x}{v})_x\frac{q}{\theta^{p+1}} + av\frac{q^2}{\theta^{p+1}} + b(\theta^4)_x\frac{q}{\theta^{p+1}} = 0$$

and

$$\begin{aligned}&\int_0^1 av\frac{q^2}{\theta^{p+1}}\,\mathrm{d}x + \int_0^1 \frac{{q_x}^2}{v\theta^{p+1}}\mathrm{d}x\\ &\le \int_0^1 \left(\frac{q_x}{v}\right)_x\frac{q}{\theta^{p+1}}\,\mathrm{d}x + \left|\int_0^1 \frac{4b\theta^3\theta_x q}{\theta^{2p+2}}\,\mathrm{d}x\right| + \int_0^1 \frac{{q_x}^2}{v\theta^{p+1}}\mathrm{d}x\\ &\le -\int_0^1 \frac{q_x}{v}\left(\frac{q_x}{\theta^{p+1}}-(p+1)\frac{\theta_x q}{\theta^{p+2}}\right)\mathrm{d}x + \left|4b\int_0^1 \frac{\theta^2\theta_x q}{\theta^{p+1}}\,\mathrm{d}x\right| + \int_0^1 \frac{{q_x}^2}{v\theta^{p+1}}\mathrm{d}x\\ &\le (p+1)\int_0^1 \frac{q_x\theta_x q}{v\theta^{p+2}} + C\int_0^1 \frac{\theta^5\theta_x^2}{\theta^p}dx + \varepsilon\int_0^1 \frac{q^2}{\theta^{p+1}}dx\\ &\le C\int_0^1 \frac{{q_x}^2\theta_x^2}{\theta^{p+3}}dx + \varepsilon\int_0^1 \frac{q^2}{\theta^{p+1}}dx + C\int_0^1 \frac{\theta^5\theta_x^2}{\theta^p}dx + \varepsilon\int_0^1 \frac{q^2}{\theta^{p+1}}dx\\ &\le C\int_0^1 \frac{\theta^5\theta_x^2}{\theta^p}dx + 2\varepsilon\int_0^1 \frac{q^2}{\theta^{p+1}}dx\\ &\le \frac{\tilde{\kappa}}{4}\int_0^1 \frac{\theta^5\theta_x^2}{v\theta^{p+1}}\mathrm{d}x + C\int_0^1 \frac{\theta^5\theta_x^2}{\theta^2}dx + 2\varepsilon\int_0^1 \frac{q^2}{\theta^{p+1}}dx\end{aligned} \tag{2.40}$$

where

$$\int_0^1 \theta^3\theta_x^2 \, \mathrm{d}x \le C\left(\int_0^1 \frac{\theta^{\beta-2}\theta_x^2}{v} \, \mathrm{d}x + \int_0^1 \frac{\theta_x^2}{v} \, \mathrm{d}x\right) \tag{2.41}$$

Thus, by virtue of the Grönwall's inequality and the fact that $\|\ln v(t)\|_{L^1} \le C$ due to (2.17), we easily infer from (2.40), (2.41), (2.7) and (2.34) that$(2.33)_1$ with $p > 2$ holds.

To prove $(2.33)_2$, we multiply (1.2) by u in L^2 and integrate by parts to get that

$$\begin{aligned}
&\left(\int_0^1 \frac{u^2}{2} dx\right)_t + \int_0^1 \frac{\mu u_x^2}{v} dx = \int_0^1 \frac{\theta - 1}{v} u_x dx + \left(\int_0^1 \ln v dx\right)_t \\
&\le \frac{1}{2}\int_0^1 \frac{\mu u_x^2}{v} dx + \frac{1}{2}\int_0^1 \frac{(\theta^{\frac{1}{2}} - 1)^2(\theta^{\frac{1}{2}} + 1)^2}{\mu v} dx + \left(\int_0^1 \ln v dx\right)_t \\
&\le \frac{1}{2}\int_0^1 \frac{\mu u_x^2}{v} dx + C_0^{-1} m_2^{-\alpha} \left\|\theta^{\frac{1}{2}} - 1\right\|_{L^\infty}^2 \int_0^1 (1+\theta) \, dx + \left(\int_0^1 \ln v dx\right)_t \\
&\le \frac{1}{2}\int_0^1 \frac{\mu u_x^2}{v} dx + C\left\|\theta^{\frac{1}{2}} - 1\right\|_{L^\infty}^2 + \left(\int_0^1 \ln v dx\right)_t,
\end{aligned}$$

where we have used (2.5) and (2.16). This, combined with (2.17) and (2.39), shows that for any $t \in [0, T]$,

$$\|u(t)\|_{L^2}^2 + \int_0^t \|u_x\|_{L^2}^2 d\tau \le \|u(t)\|_{L^2}^2 + 2C_0^{-1}\int_0^t \int_0^1 \frac{\mu u_x^2}{v} dx d\tau \le C,$$

since $\mu \ge m_2^\alpha \ge 1/2$ due to (2.16). This finishes the proof of $(2.33)_2$. □

With the help of Lemma 2.5, we can now estimate the L^2-norm of the first-order derivative of specific volume.

Thus, similar to the Navier-Stokes equations, using Lemmas 2.2-2.5, based on the case where $\beta \ge 6$, we can obtain the following two lemmas. These are not repeated here. (one can see Ying Sun [18]for details).

Lemma 2.6. *Let the conditions of Lemma 2.4 be in force. Then,*

$$\sup_{0\le t\le T} \|v_x(t)\|_{L^2}^2 + \int_0^T \int_0^1 \left(v_x^2 + \theta v_x^2\right) dx dt \le C. \tag{2.42}$$

and

$$\sup_{0\le t\le T} \|u_x(t)\|_{L^2}^2 + \int_0^T \left(\|u_t\|_{L^2}^2 + \|u_{xx}\|_{L^2}^2 + \|\theta_x\|_{L^2}^2\right) dt \le C. \tag{2.43}$$

Lemma 2.7. *Let the conditions of Lemma 2.4 be in force. Then,*

$$\theta(x,t) \ge C_4 \triangleq [C(T+1)]^{-1}, \quad \theta(x,t) \le C_3, \quad \forall\ (x,t) \in \overline{\Omega}_T \triangleq [0,1]\times[0,T]. \tag{2.44}$$

and

$$\sup_{0\le t\le T} \|\theta_x(t)\|_{L^2}^2 + \int_0^T \left(\|\theta_t\|_{L^2}^2 + \|\theta_{xx}\|_{L^2}^2\right) dt \le C. \tag{2.45}$$

Proof. we multiply (2.8) by θ in L^2 and integrated by parts to deduce

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\theta\|_{L^2}^2+\int_0^1\frac{\kappa\theta_x^2}{v}\,\mathrm{d}x&=\int_0^1\frac{\mu\theta u_x^2}{v}\,\mathrm{d}x-\int_0^1\frac{\theta^2u_x}{v}\,\mathrm{d}x-\int_0^1 q_x\theta dx\\&\le\left|\int_0^1\frac{\mu\theta u_x^2}{v}\,\mathrm{d}x\right|+\left|\int_0^1\frac{\theta^2u_x}{v}\,\mathrm{d}x\right|+\left|\int_0^1 q_x\theta\,\mathrm{d}x\right|,\end{aligned}\tag{2.46}$$

By the Cauchy-Schwarz's inequality, we have

$$\begin{aligned}\left|\int_0^1 q_x\theta\,\mathrm{d}x\right|&\le\int_0^1|q\theta_x|\,\mathrm{d}x\\&\le C\int_0^1\frac{q^2}{\theta^2}dx+C\int_0^{\prime}\theta^2\theta_x^2dx\\&\le C\int_0^1\frac{q^2}{\theta^2}dx+C\int_0^1\theta^4\theta_x^2dx+C\int_0^1\theta_x^2dx.\end{aligned}\tag{2.47}$$

It follows from (2.5), (2.16), (2.17) and (2.43) that

$$\begin{aligned}\int_0^1\frac{\mu\theta u_x^2}{v}dx&\le CN^\alpha\|u_x\|_{L^\infty}^2\|\theta\|_{L^1}\le C\|u_x\|_{L^2}\|u_{xx}\|_{L^2}\\&\le C\|u_{xx}\|_{L^2}.\end{aligned}\tag{2.48}$$

Thanks to(2.18), (2.28)and (2.34) ,we easily obtain

$$\begin{aligned}\left|\int_0^1\frac{\theta^2u_x}{v}\,\mathrm{d}x\right|&=\left|-\int_0^1\frac{(\theta^2-\bar\theta^2)u_x}{v}dx+(1-\bar\theta)\int_0^1\frac{u_x}{v}dx-\int_0^1\frac{u_x}{v}dx\right|\\&\le\|(\theta^2-\bar\theta^2)(t)\|_{L^\infty}\|u_x\|_{L^1}+CW(t)+\left(\int_0^1\ln v\mathrm{d}x\right)_t,\end{aligned}\tag{2.49}$$

where

$$\begin{aligned}\|(\theta^2-\bar\theta^2)(t)\|_{L^\infty}&\le C\|\theta\|_{L^2}\|\theta_x\|_{L^2}\\&\le C\|\theta\|_{L^2}W^{\frac{1}{2}}(t).\end{aligned}$$

Putting (2.47)–(2.49) into(2.46), by (2.7) we deduce that,

$$\sup_{0\le t\le T}\|\theta(t)\|_{L^2}^2+\int_0^T\int_0^1\kappa\theta_x^2\,\mathrm{d}x\mathrm{d}t\le C.\tag{2.50}$$

Next, multiplying (2.8) by $\kappa\theta_t$ in L^2 and integrating by parts, we have

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1\frac{(\kappa\theta_x)^2}{v}\,\mathrm{d}x+\int_0^1\kappa\theta_t^2\,\mathrm{d}x\\&=\int_0^1\left(\frac{\mu u_x^2-\theta u_x}{v}\right)\kappa\theta_t\,\mathrm{d}x-\frac{1}{2}\int_0^1\frac{u_x}{v^2}(\kappa\theta_x)^2\,\mathrm{d}x-\int_0^1 q_x\kappa\theta_t\,\mathrm{d}x\\&\le\frac{1}{4}\int_0^1\kappa\theta_t^2\,\mathrm{d}x+\frac{1}{4}\int_0^1\kappa\theta_t^2dx+C\int_0^1 q_x^2\kappa dx\\&+C\int_0^1(\kappa u_x^4\mu^2+\kappa\theta^2u_x^2)dx+\int_0^1|u_x|\,(\kappa\theta_x)^2\mathrm{d}x\\&\triangleq\frac{1}{2}\int_0^1\kappa\theta_t^2\,\mathrm{d}x+I_1+I_2+I_3.\end{aligned}\tag{2.51}$$

in which the second term on the right-hand side can be estimated as follows:

$$
\begin{aligned}
I_1 \le C\int_0^1 \frac{q_x^2}{\theta^2}(\theta^2+\theta^{\beta+2})\,\mathrm{d}x &\le C\left(1+\|\theta\|_{L^\infty}^{\beta+2}\right)\int_0^1 \frac{q_x^2}{\theta^2}dx\\
&\le C\left(1+\|\theta^{\beta+1}\theta_x\|_{L^2}\right)\int_0^1 \frac{q_x^2}{\theta^2}dx \qquad (2.52)\\
&\le C\int_0^1 \frac{q_x^2}{\theta^2}dx + C\int_0^1 \frac{q_x^2}{\theta^2}dx\|\kappa\theta_x\|_{L^2},
\end{aligned}
$$

and

$$
\begin{aligned}
I_2 &\le C\left(1+\|\theta\|_{L^\infty}^{\beta+1}\right)\|u_x\|_{L^\infty}\|\theta\|_{L^2}\|u_x\|_{L^2}\\
&\quad + CN^{2\alpha}\left(1+\|\theta\|_{L^\infty}^{\beta}\right)\|u_x\|_{L^\infty}^2\|u_x\|_{L^2}^2\\
&\le C\left(1+\|\theta\|_{L^\infty}^{\beta+1}\right)\|u_{xx}\|_{L^2}\|u_x\|_{L^2} \qquad (2.53)\\
&\le C\left(1+\|\theta^{\beta}\theta_x\|_{L^2}\right)\|u_{xx}\|_{L^2}\|u_x\|_{L^2}\\
&\le C\left(\|u_x\|_{L^2}^2+\|u_{xx}\|_{L^2}^2+\|\kappa\theta_x\|_{L^2}^2\right),
\end{aligned}
$$

where we have used (2.16), (2.43), (2.50) and the following Sobolev inequalities:

$$
\|u_x\|_{L^\infty} \le \|u_{xx}\|_{L^2}, \quad \|u_x\|_{L^\infty}^2 \le 2\|u_x\|_{L^2}\|u_{xx}\|_{L^2}.
$$

In a similar manner, we have

$$
I_3 \le C\|u_x\|_{L^\infty}\|\kappa\theta_x\|_{L^2}^2 \le C\left(\|u_{xx}\|_{L^2}^2+1\right)\|\kappa\theta_x\|_{L^2}^2. \qquad (2.54)
$$

Thus, substituting (2.52), (2.53), and (2.54) into (2.51), we conclude from (2.7),$(2.33)_2$, (2.43), (2.50) and the Grönwall's inequality that

$$
\sup_{0\le t\le T}\|\kappa\theta_x(t)\|_{L^2}^2 + \int_0^T\int_0^1 \kappa\theta_t^2 dxdt \le C, \qquad (2.55)
$$

which particularly implies

$$
\theta(x,t) \le C_3, \quad \forall\ (x,t)\in\overline{\Omega}_T. \qquad (2.56)
$$

Assume that for any fixed $T>0$, and for $(x,t)\in(0,1)\times(0,T)$, Multiplying (2.8) by θ^{-p} with $p>2$, and integrating by parts, by (2.16), (2.40), (2.41) we have

$$
\begin{aligned}
&\frac{1}{p-1}\frac{d}{dt}\|\theta^{-1}\|_{L^{p-1}}^{p-1} + p\int_0^1 \frac{\kappa\theta_x^2}{v\theta^{p+1}}dx + \int_0^1 \frac{\mu u_x^2}{v\theta^p}dx\\
&= \int_0^1 \frac{u_x}{v\theta^{p-1}}\,dx + p\int_0^1 \frac{q\theta_x}{\theta^{p+1}}\,dx\\
&\le C(p)\int_0^1 \frac{q^2}{\theta^{p+1}}dx + \frac{1}{2}\int_0^1 \frac{\theta_x^2}{\theta^{p+1}}dx + \frac{1}{2}\int_0^1 \frac{\mu u_x^2}{v\theta^p}dx + \frac{1}{2}m_2^{-\alpha}C_0^{-1}\int_0^1 \frac{1}{\theta^{p-2}}dx\\
&\le \frac{1}{2}\int_0^1 \frac{\theta_x^2}{\theta^{p+1}}dx + \frac{1}{2}\int_0^1 \frac{\mu u_x^2}{v\theta^p}dx\\
&\quad + \varepsilon\int_0^1 \frac{\theta_x^2\theta^5}{\theta^{p+1}}dx + C(\varepsilon)\left(\int_0^1 \frac{\theta^{\beta-2}\theta_x^2}{v}\,\mathrm{d}x + \int_0^1 \frac{\theta_x^2}{v}\,\mathrm{d}x\right) + C\left\|\theta^{-1}\right\|_{L^{p-1}}^{p-2},
\end{aligned}
$$

and consequently,

$$\frac{d}{dt}\left\|\theta^{-1}\right\|_{L^{p-1}} \le C,$$

where C is a generic positive constant independent p. Thus, integrating the above inequality over $(0,t)$ and letting $p \to \infty$, we arrive at

$$\theta^{-1}(x,t) \le C(T+1) \Leftrightarrow \theta(x,t) \ge C_4 \triangleq [C(T+1)]^{-1},\ \forall\ (x,t) \in \overline{\Omega}_T.$$

This, together with (2.56) and (2.59), proves (2.44). Finally, using (2.17), (2.42), (2.43), (2.44) and (2.50), we deduce from (2.8) that

$$\begin{aligned}\int_0^T \|\theta_{xx}\|_{L^2}^2 dt &\le C\int_0^T\int_0^1 \left(\theta_t^2 + u_x^2 + \theta_x^4 + \theta_x^2 v_x^2 + u_x^4\right) dxdt \\ &\le C + C\int_0^T \|\theta_x\|_{L^\infty}^2 dt \le C + C\int_0^T \|\theta_x\|_{L^2}\|\theta_{xx}\|_{L^2} dt \\ &\le C + \frac{1}{2}\int_0^T \|\theta_{xx}\|_{L^2}^2 dt,\end{aligned}$$

which, combined with (2.44) and (2.55), leads to (2.45). this completes the proof of the Lemma 2.7.

□

Corollary 2.2. *Assume that if the inequality*

$$\int_0^\infty \int_0^1 \left(\frac{\theta^\alpha u_x^2}{v\theta} + \frac{\theta_x^2}{v\theta^2} + \frac{\theta^{\beta-2}\theta_x^2}{v}\right)(x,t)\mathrm{d}x\mathrm{d}t \le C$$

is satisfied,then on the basis of Lemma 2.4, it follows that

$$C_0 \le \theta(x,t) \le C_0^{-1}, \quad \forall\ (x,t) \in [0,1]\times[0,+\infty). \tag{2.57}$$

Proof. In view of (2.50) and (2.56), one has

$$\begin{aligned}\int_0^T\int_0^1 \left(\theta^{\beta+1} - \bar\theta^{\beta+1}\right)^2 dxdt &\le C\int_0^T\int_0^1 \theta^{2\beta}\theta_x^2 dxdt \\ &\le C\sup_{0\le t\le T}\|\theta(t)\|_{L^\infty}^\beta \int_0^T\int_0^1 \theta^\beta\theta_x^2 dxdt \le C,\end{aligned} \tag{2.58}$$

which, together with (2.55) and (2.56), yields

$$\begin{aligned}&\int_0^T \left|\frac{d}{dt}\int_0^1 \left(\theta^{\beta+1} - \bar\theta^{\beta+1}\right)^2 dx\right| dt \\ &\le C\int_0^T\int_0^1 \left(\theta^{\beta+1} - \bar\theta^{\beta+1}\right)^2 dxdt + C\int_0^T \left(\|\theta^\beta\theta_t\|_{L^2}^2 + \bar\theta^{2\beta}\bar\theta_t^2\right) dt \\ &\le C + C\int_0^T \|u\|_{L^2}^2\|u_t\|_{L^2}^2 dt \le C,\end{aligned} \tag{2.59}$$

since it follows from (2.5), $(2.33)_2$ and (2.43) that

$$\int_0^T \bar\theta_t^2 dt \le C\int_0^T \|u\|_{L^2}^2\|u_t\|_{L^2}^2 dt \le C.$$

As an immediate result of (2.55), (2.58) and (2.59), one has

$$\lim_{t\to+\infty}\int_0^1 \left(\theta^{\beta+1}-\bar\theta^{\beta+1}\right)^2 dx=0,$$

which, combined with (2.55) again, shows that as $t\to+\infty$,

$$\left\|\left(\theta^{\beta+1}-\bar\theta^{\beta+1}\right)(t)\right\|_{L^\infty}^2 \le C\left\|\left(\theta^{\beta+1}-\bar\theta^{\beta+1}\right)(t)\right\|_{L^2}\|\theta^\beta\theta_x(t)\|_{L^2}\to 0 \quad \text{when} \quad t\to\infty.$$

Thus, by (2.18) we conclude that there exists a time $T_0\gg 1$ such that

$$\theta(x,t)\ge\frac{\gamma_1}{2},\quad \forall\ (x,t)\in[0,1]\times[T_0,+\infty). \tag{2.60}$$

Lemma 2.8. *Let the conditions of Lemma 2.4 be in force. Then,*

$$\sup_{0\le t\le T}\|(q,q_x)(t)\|^2+\int_0^T\|(q,q_x)\|^2\mathrm{d}t\le C, \tag{2.61}$$

and

$$\sup_{0\le t\le T}\sigma(t)\|(u_t,u_{xx},\theta_t,\theta_{xx})(t)\|_{L^2}^2+\int_0^T\sigma(t)\|(u_{xt},q_t,q_{xt},\theta_{xt})\|_{L^2}^2dt\le C, \tag{2.62}$$

where

$$\sigma(t)\triangleq\min\{1,t\}.$$

Proof. It follows from (2.7) and (2.12) that

$$\sup_{0\le t\le T}\|(q,q_x)(t)\|^2+\int_0^T\|(q,q_x)\|^2\mathrm{d}t\le C. \tag{2.63}$$

First, differentiating (1.2) with respect to t, by (1.1) we have

$$u_{tt}+\left(\frac{v\theta_t-\theta u_x}{v^2}\right)_x=\left(\frac{\mu u_x}{v}\right)_{xt}=\left(\left(\frac{\mu}{v}\right)_t u_x+\frac{\mu}{v}u_{xt}\right)_x,$$

which, multiplied by u_t in L^2 and integrated by parts, gives

$$\begin{aligned}
&\frac12\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 u_t^2\mathrm{d}x+\int_0^1\frac{\mu}{v}u_{xt}^2\mathrm{d}x\\
&=\int_0^1\left(\frac{\theta_t}{v}-\frac{\theta u_x}{v^2}-\left(\frac{\mu}{v}\right)_t u_x\right)u_{xt}\mathrm{d}x\\
&\le\eta\int_0^1\frac{\mu u_{xt}^2}{v}\mathrm{d}x+C\int_0^1(\theta_t^2+u_x^2+u_x^4+\theta_t^2u_x^2)\mathrm{d}x\\
&\le\eta\int_0^1\frac{\mu u_{xt}^2}{v}\mathrm{d}x+C\left(1+\|u_x\|_{L^\infty}^2\right)\left(\|\theta_t\|_{L^2}^2+\|u_x\|_{L^2}^2\right),
\end{aligned} \tag{2.64}$$

where we have also used (2.17), (2.44) and the Cauchy-Schwarz's inequality.

Similarly, differentiating (2.8) with respect to t gives

$$\theta_{tt}-\left(\frac{\kappa\theta_x}{v}\right)_{xt}=-\frac{\theta_t u_x}{v}+\frac{\theta u_x^2}{v^2}-\frac{\theta}{v}u_{xt}+\left(\frac{\mu}{v}\right)_t u_x^2+\frac{2\mu}{v}u_xu_{xt}-q_{xt},$$

which, multiplied by θ_t in L^2 and integrated by parts, yields

$$\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 \theta_t^2 \mathrm{d}x + \int_0^1 \frac{\kappa}{v}\theta_{xt}^2 \mathrm{d}x \leq & -\int_0^1 \left(\frac{\kappa}{v}\right)_t \theta_x \theta_{xt} \mathrm{d}x + \eta \int_0^1 \frac{\mu u_{xt}^2}{v} \mathrm{d}x \\
& + C\left(1 + \|u_x\|_{L^\infty}^2 + \|\theta_x\|_{L^\infty}^2\right)\left(\|u_x\|_{L^2}^2 + \|\theta_x\|_{L^2}^2\right) \\
& + C\left(1 + \|u_x\|_{L^\infty}^2 + \|\theta_x\|_{L^\infty}^2\right)\|\theta_t\|_{L^2}^2 + \int_0^1 q_{xt}\theta_t \mathrm{d}x \\
\leq & \, \eta \int_0^1 \frac{\kappa\theta_{xt}^2}{v} \mathrm{d}x + \eta \int_0^1 \frac{\mu u_{xt}^2}{v} \mathrm{d}x + \eta \int_0^1 \frac{q_{xt}^2}{v} \mathrm{d}x \\
& + C\left(1 + \|u_x\|_{L^\infty}^2 + \|\theta_x\|_{L^\infty}^2\right)\left(\|u_x\|_{L^2}^2 + \|\theta_x\|_{L^2}^2\right) \\
& + C\left(1 + \|u_x\|_{L^\infty}^2 + \|\theta_x\|_{L^\infty}^2\right)\|\theta_t\|_{L^2}^2,
\end{aligned} \tag{2.65}$$

Similarly, differentiating (1.4) with respect to t gives

$$-(\frac{q_x}{v})_{xt} + a(vq)_t + b(\theta^4)_{xt} = 0,$$

which, multiplied by q_t in L^2 and integrated by parts, yields

$$\begin{aligned}
& \int_0^1 \frac{q_{xt}^2}{v} \mathrm{d}x + \int_0^1 avq_t^2 \mathrm{d}x \\
= & \int_0^1 \frac{1}{v^2} q_x u_x q_{xt} \mathrm{d}x - \int_0^1 aqv_t q_t \mathrm{d}x + \int_0^1 4b\theta^3\theta_t q_{xt} \mathrm{d}x \\
\leq & \, \eta \int_0^1 \frac{q_{xt}^2}{v} \mathrm{d}x + \eta \int_0^1 avq_t^2 \mathrm{d}x + C\|u_x\|_{L^2}^2 + C\int_0^1 \frac{u_x^2 q^2}{v} dx + C\int_0^1 \theta_t^2 dx \\
\leq & \, \eta \int_0^1 \frac{q_{xt}^2}{v} \mathrm{d}x + \eta \int_0^1 avq_t^2 \mathrm{d}x + C(\|u_x\|_{L^2}^2 + \|\theta_t\|_{L^2}^2) + C\|u_x\|_{L^\infty}^2 \|q\|_{L^2}^2 \\
\leq & \, \eta \int_0^1 \frac{q_{xt}^2}{v} \mathrm{d}x + \eta \int_0^1 avq_t^2 \mathrm{d}x + C(\|u_x\|_{L^2}^2 + \|u_{xx}\|_{L^2}^2 + \|\theta_t\|_{L^2}^2),
\end{aligned} \tag{2.66}$$

which, combined with (2.64) and (2.65), leads to

$$\begin{aligned}
& \frac{d}{dt}\left(\|u_t\|_{L^2}^2 + \|\theta_t\|_{L^2}^2\right) + \left(\|\theta_{xt}\|_{L^2}^2 + \|u_{xt}\|_{L^2}^2 + \|q_{xt}\|_{L^2}^2 + \|q_t\|_{L^2}^2\right) \\
& \leq C\left(1 + \|u_x\|_{L^\infty}^2 + \|\theta_x\|_{L^\infty}\right)\left(\|u_x\|_{L^2}^2 + \|\theta_x\|_{L^2}^2 + \|\theta_t\|_{L^2}^2\right),
\end{aligned} \tag{2.67}$$

where we have also used (2.17) and (2.44). Due to $(2.33)_2$, (2.43) and (2.45), one has

$$\|u_x\|_{L^\infty}^2 + \|\theta_x\|_{L^\infty}^2 \leq C\left(\|u_x\|_{L^2}^2 + \|\theta_x\|_{L^2}^2 + \|u_{xx}\|_{L^2}^2 + \|\theta_{xx}\|_{L^2}^2\right) \in L^1(0,T),$$

Multiplying (2.67) by $\sigma(t)$ and Grönwall's inequality, we arrive at

$$\sup_{0\leq t\leq T} \sigma(t)\|(u_t,\theta_t)(t)\|_{L^2}^2 + \int_0^T \sigma(t)\|(u_{xt}, q_t, q_{xt}, \theta_{xt})\|_{L^2}^2 dt \leq C, \tag{2.68}$$

Furthermore, it follows from (1.2) that

$$\begin{aligned}
\|u_{xx}\|_{L^2}^2 & \leq C\left(\|u_t\|_{L^2}^2 + \|\theta_x\|_{L^2}^2 + \|v_x\|_{L^2}^2\right) + C\|u_x\|_{L^\infty}\left(\|\theta_x\|_{L^2}^2 + \|v_x\|_{L^2}^2\right) \\
& \leq \frac{1}{2}\|u_{xx}\|_{L^2}^2 + C\|(u_x, u_t, \theta_x, v_x)\|_{L^2}^2.
\end{aligned} \tag{2.69}$$

Similarly, it follows from (1.3) that

$$\begin{aligned}\|\theta_{xx}\|_{L^2}^2 &= \left\|\frac{v}{\kappa}\left(e_t + Pu_x + q_x - \frac{\mu}{v}u_x^2 - \left(\frac{\kappa}{v}\right)_x\theta_x\right)\right\|_{L^2}^2\\ &\le C\|(u_x, q_x, \theta_t, v_x, u_{xx})\|_{L^2}^2 + C\|\theta_x\|_{L^\infty}^2\|(\theta_x, v_x)\|_{L^2}^2\\ &\le \frac{1}{2}\|\theta_{xx}\|_{L^2}^2 + C\|(u_x, q_x, \theta_t, v_x, u_{xx}, \theta_x)\|_{L^2}^2.\end{aligned} \tag{2.70}$$

Combining (2.69) and (2.70),we can derive that for $0 < t \le T$

$$\sigma(t)\|u_{xx}(t)\|_{L^2}^2 + \sigma(t)\|\theta_{xx}(t)\|_{L^2}^2 \le C. \tag{2.71}$$

The proof of Lemma 2.8 can be completed by (2.63), (2.68) and (2.71).

Lemma 2.9. *Let the conditions of Lemma 2.4 be in force. Then,*

$$\sup_{0\le t\le T}\|v_{xx}(t)\|_{L^2}^2 + \int_0^T \left(\|(v_{xx}, q_{xx})\|_{L^2}^2 + \sigma(t)\|(v_{xxt}, \partial_x^3 u, \partial_x^3\theta)\|_{L^2}^2\right) dt \le C. \tag{2.72}$$

Proof. First, differentiating (1.2) with respect to x, we have

$$u_{xt} + P_{xx} = (\frac{\mu}{v}v_{xx})_t - (\frac{\mu}{v})_t v_{xx} + 2(\frac{\mu}{v})_x u_{xx} + (\frac{\mu}{v})_{xx}u_x.$$

which, and multiplying it by $\frac{\mu}{v}v_{xx}$ in L^2, we obtain

$$\begin{aligned}&\sup_{0\le t\le T}\|v_{xx}(t)\|_{L^2}^2 + \int_0^T \|v_{xx}\|_{L^2}^2 \mathrm{d}t\\ &\le C + C\int_0^T\int_0^1 |u_x|(|v_t| + \alpha|\theta_t|)|v_{xx}|dxdt + C\int_0^T\int_0^1 (|v_t| + |\theta_t|)v_{xx}^2 dxdt\\ &+ C\int_0^T\int_0^1 \left(u_{xx}^2 + u_x^2(v_x^2 + \theta_x^2) + \theta_{xx}^2 + v_x^2\theta_x^2 + v_x^4\right) dxdt\\ &+ C\int_0^T\int_0^1 \left((|v_x| + |\theta_x|)|u_{xx}| + (|v_{xx}| + |\theta_{xx}| + v_x^2 + \theta_x^2)|u_x|\right)|v_{xx}|dxdt\\ &+ C\int_0^T |(u_x\frac{\mu}{v}u_{xx})|_{x=0}^{x=1}|dt\\ &\triangleq C + \sum_{i=1}^{5} W_i.\end{aligned} \tag{2.73}$$

In light of Lemmas 2.2–2.8 that

$$W_1 + W_3 \le \frac{1}{8}\int_0^T \|v_{xx}\|_{L^2}^2 \mathrm{d}t + C. \tag{2.74}$$

Set

$$\mathbf{1}_{[0,1]}(t) = \begin{cases} 1, & t \in [0,1],\\ 0, & t \in (1,T].\end{cases}$$

we then calculate from the Cauchy-Schwarz's inequality that

$$\begin{aligned}
W_2 &\leq C\int_0^T \left(\|u_x\|_{L^2}+\|u_x\|_{L^2}^{\frac{1}{2}}\|u_{xx}\|_{L^2}^{\frac{1}{2}}+\|\theta_t\|_{L^2}+\|\theta_t\|_{L^2}^{\frac{1}{2}}\|\theta_{xt}\|_{L^2}^{\frac{1}{2}}\right)\|v_{xx}\|_{L^2}^2 dt \\
&\leq C\int_0^T \left(\|(u_x,u_{xx},\theta_t)\|_{L^2}^2+\sigma(t)\|\theta_{xt}\|_{L^2}^2+t^{-\frac{1}{2}}\mathbf{1}_{[0,1]}(t)\right)\|v_{xx}\|_{L^2}^2 dt \qquad (2.75) \\
&+\frac{1}{8}\int_0^T \|v_{xx}\|_{L^2}^2 dt.
\end{aligned}$$

Next we can obtain from Sobolev's inequality that

$$\begin{aligned}
W_4 &\leq C\int_0^T \left(\|(v_x,\theta_x)\|_\infty\|u_{xx}\|+\|(v_{xx},\theta_{xx})\|\|u_x\|_\infty+\|(v_x,\theta_x)\|_\infty^2\|u_x\|\right)\|v_{xx}\|\mathrm{d}t \\
&\leq C\int_0^T \left(\|(v_x,\theta_x)\|^{\frac{1}{2}}\|(v_x,\theta_x)\|_1^{\frac{1}{2}}\|u_{xx}\|+\|(v_x,\theta_x)\|\|(v_x,\theta_x)\|_1\|u_x\|\right)\|v_{xx}\|\mathrm{d}t \\
&+\frac{1}{8}\int_0^T \|(v_{xx},\theta_{xx})\|^2\mathrm{d}t+C\int_0^T \|u_x\|_\infty^2\|v_{xx}\|^2\mathrm{d}t \\
&\leq \frac{1}{4}\int_0^T \|v_{xx}\|^2\mathrm{d}t+C\int_0^T \|(u_x,u_{xx})\|^2\|v_{xx}\|^2\mathrm{d}t+C.
\end{aligned} \qquad (2.76)$$

We have, according to (1.1) and (1.2),

$$(u_x\frac{\mu}{v}u_{xx})\Big|_{x=0}^{x=1}=\left(u_x(u_t+p_x-(\frac{\mu}{v})_x u_x)\right)\Big|_{x=0}^{x=1}=\left(u_x(-\frac{\theta}{v^2}v_x+\frac{\mu}{v^2}v_x u_x)\right)\Big|_{x=0}^{x=1}.$$

Thus

$$\begin{aligned}
W_5 &\leq C\int_0^t \|(u_x v_x, u_x^2 v_x)\|_{L^\infty} ds \\
&\leq C\int_0^t \left(\|u_x\|_{L^2}^{\frac{1}{2}}\|u_x\|_{H^1}^{\frac{1}{2}}+\|u_x\|_{L^2}\|u_x\|_{H^1}\right)\|v_x\|_{L^2}^{\frac{1}{2}}\|v_x\|_{H^1}^{\frac{1}{2}} ds \\
&\leq \frac{1}{8}\int_0^t \|v_{xx}\|_{L^2}^2 ds+C\int_0^t \|(v_x,u_x,u_{xx})\|_{L^2}^2 ds \\
&\leq \frac{1}{8}\int_0^t \|v_{xx}\|_{L^2}^2 ds+C.
\end{aligned} \qquad (2.77)$$

Therefore, substituting (2.74)-(2.77) into (2.73), we can obtain from(2.62), and Grönwall's inequality that

$$\begin{aligned}
&\sup_{0\leq t\leq T}\|v_{xx}(t)\|_{L^2}^2+\int_0^T \|v_{xx}\|_{L^2}^2\mathrm{d}t \\
&\leq C\exp\left(C\int_0^T \left(\|(u_x,u_{xx},\theta_t)\|_{L^2}^2+\sigma(t)\|\theta_{xt}\|_{L^2}^2+t^{-\frac{1}{2}}\mathbf{1}_{[0,1]}(t)\right)\mathrm{d}t\right)\leq C.
\end{aligned} \qquad (2.78)$$

We can also infer from (1.4) and (2.69) that

$$\int_0^T \|q_{xx}\|_{L^2}^2\mathrm{d}t\leq \int_0^T \|q_x\|_{L^2}^2\|v_x\|_{L^2}\|v_x\|_{H^1}\mathrm{d}t+\int_0^T \|(q,\theta_x)\|_{L^2}^2\mathrm{d}t\leq C. \qquad (2.79)$$

And, due to (2.17), (2.78) and Lemmas 2.2–2.8, we deduce from (1.2) that

$$
\begin{aligned}
&\int_0^T \sigma(t)\|u_{xxx}\|^2\mathrm{d}t \\
&\le \int_0^T (\sigma(t)\|u_{xt}\|^2 + C\|(\theta_{xx}, v_{xx})\|^2)\mathrm{d}t + C\int_0^T \|v_x\|\|v_x\|_{H^1}\|(\theta_x, u_{xx})\|^2\mathrm{d}t \\
&+ C\int_0^T \|\theta_x\|\sigma(t)^{\frac{1}{2}}\|\theta_x\|_{H^1}\|u_{xx}\|^2\mathrm{d}t + C\int_0^T \|u_x\|\sigma(t)^{\frac{1}{2}}\|u_x\|_{H^1}\|(\theta_{xx}, v_{xx})\|^2\mathrm{d}t \\
&+ C\int_0^T \|v_x\|\|v_x\|_{H^1}\|u_x\|\sigma(t)^{\frac{1}{2}}\|u_x\|_{H^1}\mathrm{d}t + \int_0^T \|\theta_x\|^2\sigma(t)\|\theta_x\|_{H^1}^2\|u_x\|^2\mathrm{d}t \\
&\le C.
\end{aligned} \tag{2.80}
$$

By means of above and (1.1), we can infe

$$\int_0^T \sigma(t)\|v_{xxt}\|_{L^2}^2\mathrm{d}t \le C.$$

Similarly,it is also easily derived from (2.8) that

$$\int_0^T \sigma(t)\|\theta_{xxx}\|_{L^2}^2\mathrm{d}t \le C$$

The proof of Lemma 2.9 is complete.

Corollary 2.3. *Let the conditions of Lemma 2.4 be in force. Then*

$$\sup_{0\le t\le T}\|q_{xx}(t)\|_{L^2}^2 + \sup_{0\le t\le T}\sigma(t)\|(q_t, q_{xt}, q_{xxx})(t)\|_{L^2}^2 + \int_0^T (\|q_{xxx}\|_{L^2}^2 + \sigma(t)\|q_{xxt}\|_{L^2}^2)\mathrm{d}t \le C.$$

Proof. we can infer from (1.4), (2.45), (2.63) and (2.72) that

$$
\begin{aligned}
\|q_{xx}(t)\|_{L^2}^2 &\le C\|q\|_{L^2}^2 + C\|v_x q_x\|_{L^2}^2 + C\int_0^1 (4b\theta^3\theta_x)^2 dx \\
&\le C\|q\|_{L^2}^2 + C\|\theta_x\|_{L^2}^2 + C\|q_x\|_{L^\infty}^2\|v_x\|_{L^2}^2 \\
&\le C + C\|q_x\|_{L^2}\|q_{xx}\|_{L^2} \le C + \frac{1}{2}\|q_{xx}\|_{L^2}^2,
\end{aligned}
$$

and consequently,

$$\sup_{0\le t\le T}\|q_{xx}(t)\|_{L^2}^2 \le C.$$

Differentiating (1.4) with respect to x and combining the results from (2.17), (2.43), (2.60), (2.61) and (2.72), we obtain

$$
\begin{aligned}
\sigma(t)\|q_{xxx}(t)\|_{L^2}^2 &\le \sigma(t)\int_0^1 (q_{xx}^2 v_x^2 + q_x^2 v_{xx}^2 + v_x^2 + \theta_x^4 + q_x^2 + q_x^2 v_x^4 + \theta_{xx}^2)\mathrm{d}x \\
&\le C(\|v_x\|_{L^\infty}^2\|q_{xx}\|_{L^2}^2 + \|q_x\|_{L^\infty}^2\|v_{xx}\|_{L^2}^2 + \|q_x\|_{L^2}^2 + \|v_x\|_{L^2}^2) \\
&+ C(\|q_x\|_{L^\infty}^2\|v_x\|_{L^\infty}^2\|v_x\|_{L^2}^2 + \|\theta_x\|_{L^2}^3\cdot\sigma(t)^{\frac{1}{2}}\|\theta_x\|_{H^1} + \sigma(t)\|\theta_{xx}\|_{L^2}^2) \\
&\le C,
\end{aligned}
$$

and

$$\int_0^T \|q_{xxx}(t)\|_{L^2}^2 dt \leq C\int_0^T (\|v_x\|_{L^\infty}^2\|q_{xx}\|_{L^2}^2 + \|q_x\|_{L^\infty}^2\|v_{xx}\|_{L^2}^2 + \|q_x\|_{L^2}^2 + \|v_x\|_{L^2}^2)dt$$
$$+ C\int_0^T (\|q_x\|_{L^\infty}^2\|v_x\|_{L^\infty}^2\|v_x\|_{L^2}^2 + \|\theta_x\|_{L^\infty}^2\|\theta_x\|_{L^2}^2 + \|\theta_{xx}\|_{L^2}^2)dt$$
$$\leq C.$$

On the other hand, differentiating (1.2) with respect to x, we have

$$\frac{q_{xxt}}{v} = \frac{v_t q_{xx}}{v^2} + av_t q + avq_t + 4b(\theta^3\theta_x)_t + \frac{v_{xt}q_t + v_x q_{xt}}{v^2} - \frac{2v_t v_x q_x}{v^3}$$

Then we integrate it over $[0,1]\times[0,t]$ to compute

$$\int_0^T \sigma(t)\|q_{xxt}\|_{L^2}^2 \mathrm{d}t$$
$$\leq C\int_0^T \|u_x\|_{L^2}\sigma(t)\|u_x\|_{H^1}\cdot\|q_{xx}\|_{L^2}^2 + \|v_x\|_{L_2}\|v_x\|_{H^1}\cdot\sigma(t)\|q_{xt}\|_{L^2}^2 \mathrm{d}t$$
$$+ C\int_0^T \|q_x\|_{L^2}\|q_x\|_{H^1}\|u_{xx}\|_{L^2}^2 + \|q_x\|_{L^2}\|q_x\|_{H^1}\|u_x\|_{L^2}\|u_x\|_{H^1}\mathrm{d}t$$
$$+ C\int_0^T (\sigma(t)\|(q_t,\theta_{xt})\|_{L^2}^2 + C\|u_x\|_{L^2}^2) + \|\theta_x\|_{L^2}\sigma(t)\|\theta_x\|_{H^1}\cdot\|\theta_t\|_{L^2}^2\mathrm{d}t$$
$$\leq C$$

In addition, by means of (2.62) and(2.66), we can also obtain that

$$\sup_{0\leq t\leq T}\sigma(t)\|q_{xt}(t)\|_{L^2}^2 + \sup_{0\leq t\leq T}\sigma(t)\|q_t(t)\|_{L^2}^2 \leq C.$$

The proof of corollary 2.1 is complete.

Corollary 2.4. *Let the conditions of Lemma 2.4 be in force. Then there exists* $C(m_2) > 0$*, which depends only on* m_2*,* V_0*,* M_0*, such that for all* $t\in[0,T]$*,*

$$\sup_{0\leq t\leq T}\left(\|(v-1,u,\theta-1)(t)\|_{H^2}^2 + \|q\|_{H^3}^2\right) + \int_0^T\left(\|v_x\|_{H^1}^2 + \|(u_x,\theta_x)\|_{H^2}^2 + \|q\|_{H^3}^2\right)\mathrm{d}t \leq C(m_2).$$

3 Proof of Theorem 1.1

3.1 Global Well-Posedness

Relying on the a priori estimates established in Section 2, we now prove our main result, Theorem 1.1. To this end, we start with the local well-posedness of the Cauchy problem (1.1)-(1.8). which may be established by means of the standard iteration argument (see [21]).

Lemma 3.1. *Assume that* (1.9) *holds. Then there exists* $T_0 = T_0(V_0, V_0, M_0) > 0$*, depending only on* β, V_0 *and* M_0*, such the initial and boundary problem* (1.1)–(1.8) *has a unique solution* $(v,u,\theta,q)\in X\left(0,T_0;\frac{1}{2}V_0,\frac{1}{2}V_0,2M_0\right)$.

Building on the local existence and uniqueness of strong solutions, we will prove Theorem 1.1 step by step as follows.

Step 1. Through Lemmas 2.4–2.8, we can find a constant C_5 satisfying

$$\|(v-1,u,\theta-1,q)(t)\|_{H^1}^2+\int_0^t\left[\|\sqrt{\theta}v_x\|_{L^2}^2+\|(u_x,\theta_x,q)\|_{H^1}^2\right]\mathrm{d}s\le C_5^2. \tag{3.1}$$

Set $T_1=128C_5^4$. With (1.9) recalled and Lemma 3.1 applied, we can find a positive constant $t_1=\min\{T_1,T_0(V_0,V_0,M_0)\}$ such that the Cauchy problem (1.1)–(1.4), and (1.9) has a unique solution $(v,u,\theta,z)\in X(0,t_1;\frac{1}{2}V_0,\frac{1}{2}V_0,2\Pi_0)$.

Take $\alpha\le\alpha_1$ with α_1 is some positive constant such that

$$\left(\frac{1}{2}V_0\right)^{-\alpha_1}\le 2,\quad (2M_0)^{\alpha_1}\le 2,\quad \alpha_1 H(\frac{1}{2}V_0,\frac{1}{2}V_0,2M_0)\le\varepsilon_1, \tag{3.2}$$

where $\varepsilon_1>0$ is chosen in Lemma 2.4. Next, by applying Lemma 2.4 and 2.7 with $T=t_1$, we can conclude that for each $t\in[0,t_1]$, the above-constructed local solution (v,u,θ,z) satisfies:

$$v(x,t)\ge\frac{e^{-4T_1}}{2C_1}=:C_0\quad\text{for all } x\in[0,1], \tag{3.3}$$

$$\theta(x,t)\ge[C_6(T_1+1)]^{-1}=:C_4\quad\text{for all } x\in[0,1], \tag{3.4}$$

$$v(x,t)\le C_2,\quad \theta(x,t)\le C_3,\quad\text{for all } x\in[0,1], \tag{3.5}$$

$$\|(v-1,u,\theta-1,q)(t)\|_{H^1}^2+\int_0^t\left[\|\sqrt{\theta}v_x\|_{L^2}^2+\|(u_x,\theta_x,q)\|_{H^1}^2\right]\mathrm{d}s\le C_5^2. \tag{3.6}$$

Combining Corollary 2.2, (3.3) and (3.4), we can find a positive constant C_7, which depends on C_1,C_4,V_0 and M_0, such that for each $t\in[0,t_1]$,

$$\|(v-1,u,\theta-1)(t)\|_{H^2}^2+\|q(t)\|_{H^3}^2+\int_0^t\left(\|v_x\|_{H^1}^2+\|(u_x,\theta_x)\|_{H^2}^2+\|q\|_{H^3}^2\right)\mathrm{d}s\le C_7^2. \tag{3.7}$$

Step 2. Taking $(v(\cdot,t_1),u(\cdot,t_1),\theta(\cdot,t_1),z(\cdot,t_1))$ as the initial data and reapplying Lemma 3.1 allows us to extend the local solution (v,u,θ,z) to the time interval $[0,t_1+t_2]$ where

$$t_2=\min\{T_1-t_1,T_0(C_0,C_4,C_7)\}.$$

Moreover,

$$v(x,t)\ge\frac{1}{2}C_0,\quad \theta(x,t)\ge\frac{1}{2}C_4,\quad (x,t)\in[0,1]\times[t_1,t_1+t_2],$$

and

$$\|(v-1,u,\theta-1)(t)\|_{H^2}^2+\|q(t)\|_{H^3}^2+\int_0^t\left(\|v_x\|_{H^1}^2+\|(u_x,\theta_x)\|_{H^2}^2+\|q\|_{H^3}^2\right)\mathrm{d}s\le 4C_7^2.$$

which combined with (3.7) implies that for all $t\in[0,t_1+t_2]$,

$$\|(v-1,u,\theta-1)(t)\|_{H^2}^2+\|q(t)\|_{H^3}^2+\int_0^t\left(\|v_x\|_{H^1}^2+\|(u_x,\theta_x)\|_{H^2}^2+\|q\|_{H^3}^2\right)\mathrm{d}s\le 5C_7^2. \tag{3.8}$$

Choose $|\alpha| \le \min\{\alpha_1, \alpha_2\}$, with $\alpha_1 > 0$ being determined by (3.2) and α_2 being a positive constant that satisfies

$$\left(\frac{1}{2}C_0\right)^{-\alpha_2} \le 2, \quad \left(\frac{1}{2}C_4\right)^{-\alpha_2} \le 2, \quad \left(2\sqrt{5}C_7\right)^{\alpha_2} \le 2,, \quad \alpha_2 H(\frac{1}{2}C_0, \frac{1}{2}C_4, \sqrt{5}C_7) \le \varepsilon_1, \tag{3.9}$$

where $\varepsilon_1 > 0$ is chosen as in Lemma 2.4. Next, by applying Lemma 2.4, Lemma 2.7 and Corollary 2.2 with $T = t_1 + t_2$, we can conclude that for each $t \in [0, t_1 + t_2]$, the above-constructed local solution (v, u, θ, z) satisfies(3.3)-(3.7).

Step 3. Repeating the argument from Step 2 enables us to extend the solution (v, u, θ, z) to the time interval $[0, t_1 + t_2 + t_3]$, with

$$t_3 = \min\left\{T_1 - (t_1 + t_2), T_0\left(C_0, C_4, C_7\right)\right\}.$$

Suppose $|\alpha| \le \min\{\alpha_1, \alpha_2\}$, where the constants α_1 and α_2 satisfy (3.2) and (3.9) respectively. Moving forward, after a finite number of iterations, we construct the unique solution (v, u, θ, z) existing on $[0, t_1]$ and fulfilling (3.3)-(3.7) for each $t \in [0, T_1]$.

Step 4. Since $T_1 = 128C_5^4$ and

$$\sup_{0\le t\le T_1} \|(\theta-1)(t)\|_{H^1}^2 + \int_{T_1/2}^{T_1} \|\theta_x(t)\|_{H^1}^2 \mathrm{d}t \le C_5^2, \tag{3.10}$$

we can find a $t_0' \in [T_1/2, T_1]$ such that

$$\|\theta(t_0') - 1\|_{L^2} \le C_5, \quad \|\theta_x(t_0')\|_{L^2} \le \frac{1}{8}C_5^{-1}.$$

For if not, we have that $\|\theta_x(t)\|_{L^2} > \frac{1}{8}C_5^{-1}$ for each $t \in [T_1/2, T_1]$ and hence

$$\int_{T_1/2}^{T_1} \|\theta_x(t)\|_{H^1}^2 \mathrm{d}t > \frac{1}{2}T_1\left(\frac{1}{8}C_5^{-1}\right)^2 = C_5^2.$$

This stands in contradiction to (3.10), and from Sobolev's inequality, it thus follows that

$$\|(\theta-1)(t_0')\|_{L^\infty} \le \sqrt{2}\|(\theta-1)(t_0')\|_{L^2}^{\frac{1}{2}}\|\theta_x(t_0')\|_{L^2}^{\frac{1}{2}} \le \frac{1}{2},$$

from which we get

$$\theta(t_0', x) \ge 1 - \|(\theta-1)(t_0')\|_{L^\infty} \ge \frac{1}{2} \quad \text{for all } x \in [0,1]. \tag{3.11}$$

We observe that

$$\|(v-1, u, \theta-1)(t_0')\|_{H^2}^2 + \|q(t_0')\|_{H^3}^2 \le C_7, \quad v(t_0', x) \ge C_0 \quad \text{for all } x \in [0,1].$$

We proceed to apply Lemma 3.1 again, choosing $(v, u, \theta, z)(\cdot, t_0')$ as the initial data. We then deduce that the solution (v, u, θ, z) exists on the interval $[t_0', t_0' + t_1']$, where $t_1' = \min\{T_1, T_0(C_0, \frac{1}{2}, C_7)\}$, and for all $(x,t) \in [0,1] \times [t_0', t_0' + t_1']$,

$$\|(v-1, u, \theta-1)(t)\|_{H^2}^2 + \|q(t)\|_{H^3}^2 + \int_{t_0'}^{t} \left(\|v_x\|_{H^1}^2 + \|(u_x, \theta_x)\|_{H^2}^2 + \|q\|_{H^3}^2\right) \mathrm{d}s \le 4C_7^2.$$

and

$$v(x,t) \ge \frac{1}{2}C_0, \quad \theta(x,t) \ge \frac{1}{4}.$$

Therefore, the solution (v,u,θ,z) satisfies (3.8) for all $t\in[0,t_0'+t_1']$.

Let $|\alpha|\le\min\{\alpha_1,\alpha_2,\alpha_3\}$, where each $\alpha_i(i=1,2,3)$ being positive constants satisfying (3.2), (3.9) and

$$\left(\frac{1}{4}\right)^{-\alpha_3}\le 2,\quad \left(2\sqrt{5}C_7\right)^{\alpha_3}\le 2,,\quad \alpha_3 H(\frac{1}{2}C_0,\frac{1}{4},\sqrt{5}C_7)\le\varepsilon_1, \tag{3.12}$$

where $\varepsilon_1>0$ is chosen as in Lemma 2.4. Then we can deduce from Lemma 2.4, Lemma 2.7 and (3.1) with $T=t_0'+t_1'$, that for each $t\in[t_0',t_0'+t_1']$, the local solution $(v,u,\theta,z)(x,t)$ satisfies (3.3), (3.4), (3.5) and (3.6).

Here we have used the estimate (3.11).

We deduce from (3.4) and Corollary 2.2 that there exists some positive constant C_8, depending on C_0,C_4,V_0 and M_0, such that for each $t\in[0,t_0'+t_1']$,

$$\|(v-1,u,\theta-1)(t)\|_{H^2}^2+\|q(t)\|_{H^3}^2+\int_0^t\left(\|v_x\|_{H^1}^2+\|(u_x,\theta_x)\|_{H^2}^2+\|q\|_{H^3}^2\right)\mathrm{d}s\le C_8^2. \tag{3.13}$$

Step 5. Next, with $(v,u,\theta,z)(\cdot,t_0'+t_1')$ as the initial data, we apply Lemma 3.1 to construct the solution $(v,u,\theta,z)(x,t)$ existing on the time interval $[0,t_0'+t_1'+t_2']$ with

$$t_2'=\min\{T_1-t_1',T_0(C_0,C_4,C_8)\},$$

such that for all $(x,t)\in[0,1]\times[t_0'+t_1',t_0'+t_1'+t_2']$

$$v(x,t)\ge\frac{1}{2}C_0,\quad \theta(x,t)\ge\frac{1}{2}C_4,$$

and

$$\|(v-1,u,\theta-1)(t)\|_{H^2}^2+\|q(t)\|_{H^3}^2+\int_{t_0'+t_1'}^t\left(\|v_x\|_{H^1}^2+\|(u_x,\theta_x)\|_{H^2}^2+\|q\|_{H^3}^2\right)\mathrm{d}s\le 4C_8^2.$$

which combined with (3.13) implies that for each $t\in[0,t_0'+t_1'+t_2']$,

$$\|(v-1,u,\theta-1)(t)\|_{H^2}^2+\|q(t)\|_{H^3}^2+\int_0^t\left(\|v_x\|_{H^1}^2+\|(u_x,\theta_x)\|_{H^2}^2+\|q\|_{H^3}^2\right)\mathrm{d}s\le 5C_8^2.$$

Let $|\alpha|\le\min\{\alpha_1,\alpha_2,\alpha_3,\alpha_4\}$, where each $\alpha_i(i=1,2,3)$ being positive constants satisfying (3.2), (3.9), (3.12) and

$$\left(\frac{1}{2}C_0\right)^{-\alpha_4}\le 2,\quad \left(\frac{1}{2}C_4\right)^{-\alpha_4}\le 2,\quad \left(2\sqrt{5}C_8\right)^{\alpha_4}\le 2,,\quad \alpha_4 H(\frac{1}{2}C_0,\frac{1}{2}C_4,\sqrt{5}C_8)\le\varepsilon_1, \tag{3.14}$$

where $\varepsilon_1>0$ is chosen as in Lemma 2.4. Then we can deduce from Lemma 2.4,(3.1) and Corollary 2.2 with $T=t_0'+t_1'+t_2'$, we conclude that the local solution $(v,u,\theta,z)(x,t)$ satisfies (3.3), (3.4) and (3.13) for each $t\in[0,t_0'+t_1'+t_2']$. With $|\alpha|\le\min\{\alpha_1,\alpha_2,\alpha_3,\alpha_4\}$ assumed, we may repeatedly employ the argument above to extend the local solution to the time interval $[0,t_0'+T_1]$. Furthermore,we infer that (3.3), (3.4) and (3.13) hold for each $t\in[0,t_0'+T_1]$, In view of $t_0'+T_1\ge 3T_1/2$, we have shown that the Cauchy problem (1.1)-(1.8) admits a unique solution $(v,u,\theta,z)(x,t)\in X(0,\frac{3}{2}T_1;C_0,C_4,C_8)$ on the time interval $[0,\frac{3}{2}T_1]$.

Step 6. Let $|\alpha|\le\min\{\alpha_1,\alpha_2,\alpha_3,\alpha_4\}$, Following the same reasoning as in Steps 4 and 5, we are able to find $t_0''\in[t_0'+T_1/2,t_0'+T_1]$ where the Cauchy problem (1.1)-(1.8) has a unique solution (v,u,θ,z) on $[0,t_0''+T_1]$ that satisfies (3.3), (3.4) and (3.13) for every $t\in[0,t_0''+T_1]$. Since $t_0''+T_1\geq t_0'+3T_1/2\geq 2T_1$, we have extended the local solution (v,u,θ,z) to the time interval $[0,2T_1]$. Repeating the above process allows us to extend the solution (v,u,θ,z) incrementally to a global solution, on the condition that $|\alpha|\leq\min\{\alpha_1,\alpha_2,\alpha_3,\alpha_4\}$.

Choosing

$$\varepsilon_0=\min\{\alpha_1,\alpha_2,\alpha_3,\alpha_4\} \tag{3.15}$$

where $\alpha_i(i=1,2,3,4)$ are given by (3.2), (3.9), (3.12) and (3.14), we then derive that the Cauchy problem (1.1)-(1.8) has a unique solution (v,u,θ,z) satisfying (3.3), (3.4) and (3.13) for all $t\in[0,\infty)$. Thus we have

$$\|(v-1,u,\theta-1)(t)\|_{H^2}^2+\|q(t)\|_{H^3}^2+\int_0^\infty\left(\|v_x\|_{H^1}^2+\|(u_x,\theta_x)\|_{H^2}^2+\|q\|_{H^3}^2\right)\mathrm{d}s\leq C_8^2.$$

which implies that the solution $(v,u,\theta,z)\in X(0,\infty;C_0,C_4,C_8)$.

The constants $\varepsilon_1,C_i(i=0,1,2,3,4,5,7,8)$ depend only on V_0 and M_0. According to the definition (3.15) of ε_0, we can conclude the proof of Theorem 1.1.

3.2 Nonlinearly Exponential Stability

For completeness, we sketch the proof of large-time behavior of the solutions, based on the t-independent regularities (1.10) and (1.11).

Relying on (1.1) and (1.2), direct computations allow us to obtain that

$$\begin{aligned}\left(\frac{\mu(\theta)v_x}{v}\right)_t&=u_t+P_x+\frac{\alpha\theta^\alpha}{v\theta}(\theta_tv_x-u_x\theta_x)\\&=u_t+\frac{\theta_x}{v}-\frac{\theta v_x}{v^2}+\frac{\alpha\theta^\alpha}{v\theta}(\theta_tv_x-u_x\theta_x),\end{aligned}$$

Multiplying it by $u-\mu\frac{v_x}{v}$ in L^2, integrating by parts, and using the Cauchy-Schwarz inequality, we have

$$\begin{aligned}&\frac{d}{dt}\int_0^1\left(u-\mu\frac{v_x}{v}\right)^2dx+C_3\int_0^1v_x^2dx\\&\leq C\int_0^1|uv_x|+|\theta_xu|+|\theta_xv_x|+|\theta_tv_x|+|\theta_xu_x|dx\\&\leq C\|(u_x,\theta_x,\theta_t)\|_{L^2}^2+\frac{C_3}{2}\|v_x\|_{L^2}^2.\end{aligned} \tag{3.16}$$

Multiplying (1.1) by u_{xx}, we have from integration by parts that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1u_x^2\mathrm{d}x+\int_0^1\frac{\mu u_{xx}^2}{v}\mathrm{d}x=\int_0^1P_xu_{xx}\mathrm{d}x-\int_0^1\left(\frac{\mu}{v}\right)_xu_{xx}u_x\mathrm{d}x.$$

It is evident from this that

$$\frac{d}{dt}\int_0^1u_x^2dx+C_4\int_0^1u_{xx}^2dx\leq C\left\|(u_x,v_x,\theta_x)\right\|_{L^2}^2, \tag{3.17}$$

Multiplying (2.8) by $\kappa\theta_x$ in L^2, integrating by parts, we deduce that for $\delta \in (0,1)$

$$\frac{d}{dt}\int_0^1 \frac{(\kappa\theta_x)^2}{v}dx + C_5\int_0^1 \theta_t^2 dx \le C(\delta)\|(u_x,\theta_x,q_x)\|_{L^2}^2 + \delta\|u_{xx}\|_{L^2}^2 \tag{3.18}$$

Choosing $\delta > 0$ sufficiently small,Based on (3.6)-(3.8), we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\|(\mu v_x, u_x, \kappa\theta_x)(t)\|_{L^2}^2 \le C\|(u_x,u_t,v_x,\theta_x,\theta_t,q_x)\|_{L^2}^2. \tag{3.19}$$

Integrating (3.9) over $[s,t]$, we have

$$\left|\|(\mu v_x(t),u_x(t),\kappa\theta_x(t))\|_{L^2}^2 - \|(\mu v_x(s),u_x(s),\kappa\theta_x(s))\|_{L^2}^2\right| \le C(t-s). \tag{3.20}$$

Since $\|(\mu v_x(t),u_x(t),\kappa\theta_x(t))\|_{L^2}^2$ is a nonnegative and uniformly continuous function, we can infer from (3.10) that

$$\|(\mu v_x(t),u_x(t),\kappa\theta_x(t))\|_{L^2}^2 \to 0 \quad as \quad t\to\infty. \tag{3.21}$$

From (2.12) and (3.11),we can infer that

$$\int_0^1 (q_x^2+q^2)\mathrm{d}x \le C\int_0^1 \kappa\theta_x^2\mathrm{d}x \le C\int_0^1 \theta_x^2\mathrm{d}x. \tag{3.22}$$

By means of (2.7), (2.17), (2.30), (3.21), and (3.22), it follows

$$\|(v-1,u,\theta-1,q)(t)\|_{L^\infty} \to 0 \quad as \quad t\to\infty.$$

The proof of Theorem 1.1 is complete.

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