A Mathematical and Computational Framework for Modeling Melodic Structures in North Indian Classical Music Using Raga-Restricted Operations

Abstract

This paper presents a unified mathematical and computational framework for analyzing melodic structures in North Indian classical music through a novel concept termed the *raga-restricted operation*. This operation provides an algebraic mechanism to model permissible note transitions while enforcing the grammatical constraints of a raga. The proposed framework integrates algebraic structures, graph-theoretic representations, and stochastic modeling using Markov chains to achieve a coherent and computationally tractable description of melodic behavior. The methodology is applied to two representative ragas, Raga Yaman and Raga Bhupali, where melodic sequences are generated under raga constraints and analyzed using directed graphs, transition probability matrices, and stationary distributions. The results demonstrate that the framework preserves the intrinsic structure of ragas while enabling quantitative analysis of transition dynamics, probabilistic dependencies, and long-term behavior. From a computational perspective, the model supports systematic sequence generation, prediction of note transitions, and structural analysis of melodic patterns. Thus, the proposed approach establishes a rigorous connection between music theory and mathematical modeling, contributing to the advancement of computational musicology and probabilistic analysis of raga-based systems.

Keywords: Raga-restricted operation, algebraic structures, Indian classical music, raga-based modeling, Markov chains, transition probability matrices, stationary distribution, graph theory, computational musicology.

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1 Introduction

Mathematics and music share a deep and historically rich relationship, rooted in patterns, symmetry, and structured relationships within musical compositions. Early studies demonstrated that musical intervals can be expressed through numerical ratios, while later developments extended these ideas to algebraic and structural representations of musical systems [9–11]. In addition, probabilistic frameworks and signal-processing techniques have been widely utilized to analyze tonal organization and sequential dependencies in music [7, 12].

In Western music theory, well-defined constructs such as symmetry, transformations, and ordered tonal hierarchies provide a natural foundation for mathematical formalization. These properties enable the application of algebraic structures and computational techniques for modeling musical compositions [10, 12]. Furthermore, statistical approaches, including Markov chains and hidden Markov models, have been extensively used to capture note transition behavior and temporal dependencies in musical sequences [2, 14–16]. More recently, advances in machine learning and deep learning have significantly expanded the scope of computational musicology, enabling the modeling of long-term musical structure and expressive performance [17–22, 25].

In contrast, North Indian classical music is governed by the concept of a *raga*, which represents a structured yet flexible melodic framework. A raga prescribes permissible notes, characteristic phrases, and specific patterns of ascent (*aroha*) and descent (*avaroha*), along with contextual rules of usage [13]. Unlike strictly scale-based systems, raga performance incorporates expressive elements such as microtonal variations and ornamentations, making its structure inherently complex. Consequently, conventional algebraic or statistical models often fail to fully capture the grammatical and contextual richness of raga-based music [3, 6].

To address these challenges, several studies have explored computational and probabilistic methods for modeling raga structures. Markov chain models have been employed to represent note transitions and capture statistical dependencies in melodic sequences [1, 2, 5]. Graph-based representations and transition probability matrices have also been used to analyze structural properties and connectivity patterns in raga systems [4, 6]. In addition, machine learning approaches have been applied to raga identification and melodic pattern analysis, further enhancing computational understanding of Indian classical music [28, 30, 31]. Advanced structural frameworks, including category-theoretic approaches, have also been explored to provide deeper mathematical insights into musical organization [8].

Despite these developments, existing approaches remain largely statistical or computational and lack a unified algebraic formulation that explicitly incorporates the grammatical constraints of ragas. In particular, there is no well-defined algebraic operation governing permissible transitions while excluding forbidden movements. This limitation restricts the integration of algebraic, graph-theoretic, and probabilistic perspectives within a single coherent framework.

Motivated by these gaps, the present study proposes a comprehensive mathematical framework that integrates algebraic structures, graph theory, and stochastic processes to analyze melodic systems in North Indian classical music. A novel concept, termed the *raga-restricted operation*, is introduced to model permissible note transitions while preserving raga grammar. This approach provides an algebraic foundation for representing musical rules. Furthermore, directed graph models, transition probability matrices, and

Markov chain analysis are employed to investigate both local transition dynamics and global structural behavior.

The proposed framework is demonstrated through its application to two representative ragas, namely Raga Yaman and Raga Bhupali, which exhibit distinct tonal structures and levels of complexity. The analysis provides insights into their melodic organization and transition patterns, thereby offering a systematic and mathematically rigorous approach to understanding raga-based music. This work establishes a meaningful connection between algebra, probability, and computational musicology, contributing to the broader development of mathematical models for complex musical systems.

2 Preliminaries

In this section, we recall the fundamental mathematical and musical concepts required for the development of the proposed framework, integrating algebraic structures, graph theory, and probabilistic modeling.

Definition 2.1 (Set). *A set is a well-defined collection of distinct objects, called elements. Mathematically, a set is represented as*

$$S = \{x \mid x \text{ satisfies a given property}\}.$$

Definition 2.2 (Binary Operation). *Let S be a non-empty set. A binary operation on S is a mapping*

$$* : S \times S \to S.$$

Definition 2.3 (Group [9–11]). *A group is an ordered pair $(G, *)$ where G is a non-empty set and $*$ is a binary operation satisfying:*

- *Closure:* $a * b \in G$
- *Associativity:* $(a * b) * c = a * (b * c)$
- *Identity:* $\exists e \in G$ *such that* $a * e = e * a = a$
- *Inverse:* $\forall a \in G$, $\exists a^{-1}$ *such that* $a * a^{-1} = e$

Definition 2.4 (Semigroup and Monoid). *A semigroup is a non-empty set $(G, *)$ satisfying closure and associativity. A monoid is a semigroup with an identity element.*

Definition 2.5 (Transformation). *Let S be a non-empty set. A transformation on S is a function*

$$T : S \to S.$$

A transformation is called a symmetry if it is bijective and preserves structure.

Definition 2.6 (Function). *Let A and B be* ***non-empty sets****. A function f from A to B is a mapping*

$$f : A \to B$$

such that each element of A is assigned exactly one element of B.

Definition 2.7 (Relation)**.** *Let A and B be sets. A relation between A and B is a subset of their Cartesian product:*

$$R \subseteq A \times B.$$

If $(a, b) \in R$, then a is related to b.

Definition 2.8 (Graph [4])**.** *A graph is an ordered pair*

$$G = (V, E),$$

where V is the set of vertices and $E \subseteq V \times V$ is the set of edges.

Definition 2.9 (Probability [12])**.** *Assuming **equally likely outcomes**, the probability of an event A is defined as*

$$P(A) = \frac{\text{Number of favorable outcomes}}{\text{Total number of outcomes}}, \quad 0 \leq P(A) \leq 1.$$

For mutually exclusive events A and B,

$$P(A \cup B) = P(A) + P(B).$$

Definition 2.10 (Swar (Musical Note) [13])**.** *A* swar *is the fundamental unit of sound in Indian classical music. The basic set of swars in one octave is*

$$S = \{Sa, Re, Ga, Ma, Pa, Dha, Ni\},$$

*which can be extended to **12 swaras** (including variants) or **24 swaras** when two octaves are considered.*

Definition 2.11 (Raga [13])**.** *A raga is a structured and rule-based subset of swars used for musical expression. Mathematically,*

$$R \subseteq S.$$

Definition 2.12 (Aroha–Avroha)**.** *Aroha represents the ascending sequence and Avroha represents the descending sequence. For a finite set of swars (e.g., 7 or 5 notes), these can be written as*

$$A = (s_1, s_2, \ldots, s_7), \quad D = (s_7, \ldots, s_2, s_1).$$

Definition 2.13 (Note Dependency)**.** *In a raga, the occurrence of a note depends on the previous note, which can be expressed as:*

$$P(s_{i+1} \mid s_i).$$

Definition 2.14 (Permissible Notes)**.** *Let S be the set of all notes. A subset $R \subseteq S$ represents permissible notes of a raga.*

Definition 2.15 (Forbidden Notes (Varjit Swars))**.** *Let $R \subseteq S$. The set of forbidden notes is defined as*

$$F = S \setminus R.$$

Definition 2.16 (Improvisation in Raga). *A melodic system can be represented as*

$$M = (R, T),$$

where $T \subseteq R \times R$ is the set of permissible transitions. Improvisation must satisfy

$$(s_i, s_j) \in T.$$

Definition 2.17 (Raga Grammar). *A raga can be represented as a structured system:*

$$\mathcal{R} = (R, A, D, T),$$

where

- *$R \subseteq S$ is the set of permissible notes,*
- *A is the ascending sequence,*
- *D is the descending sequence,*
- *$T \subseteq R \times R$ is the set of permissible transitions.*

Definition 2.18 (Markov Chain [2, 3, 16]). *A Markov chain is a stochastic process in which the probability of the next state depends only on the current state:*

$$P(s_{n+1} \mid s_n, s_{n-1}, \dots) = P(s_{n+1} \mid s_n).$$

The transition probabilities are represented by a matrix

$$P = [p_{ij}],$$

where $p_{ij} = P(s_j \mid s_i)$.

Definition 2.19 (Transition Probability Matrix [1]). *The transition probability matrix (TPM) is defined as*

$$P = [p_{ij}], \quad p_{ij} \geq 0, \quad \sum_j p_{ij} = 1.$$

Thus, P is a stochastic matrix representing transition behavior.

3 Methodology

This study develops a mathematical framework for modeling melodic structures in North Indian classical music by integrating algebraic structures, graph theory, and stochastic processes. The objective is to represent raga-based musical systems through consistent algebraic and probabilistic formulations while preserving their grammatical constraints.

Let the musical note space be denoted by a finite set

$$S = \{s_1, s_2, \dots, s_k\},$$

where $k = 7$ (basic swaras), $k = 12$ (including variants), or $k = 24$ (two octaves).

Let

$$R \subseteq S$$

be the set of permissible notes of a raga, and let

$$T \subseteq R \times R$$

denote the set of allowed transitions determined by the grammar of the raga. A valid melodic transition satisfies

$$(s_i, s_{i+1}) \in T.$$

3.1 Raga-Restricted Algebraic Structure

We now introduce the central concept of this study.

Definition 3.1 (Raga-Restricted Operation)**.** *Let R be a non-empty finite set of permissible notes. Define a binary operation*

$$*_R : R \times R \to R \cup \{\emptyset\}$$

such that

$$a *_R b = \begin{cases} b, & \text{if } (a, b) \in T, \\ \emptyset, & \text{if } (a, b) \notin T. \end{cases}$$

Here, $\emptyset$ represents an invalid or forbidden transition. This construction ensures that:

- All valid transitions remain within R,
- Invalid transitions are explicitly excluded,
- The raga grammar is embedded within an algebraic framework.

Thus, $(R, *_R)$ forms a constrained algebraic system reflecting permissible melodic progressions.

3.2 Application to Specific Ragas

For Raga Yaman (sampurna jati), the permissible note set is

$$R_Y = \{Sa, Re, Ga, Ma^{\#}, Pa, Dha, Ni\}.$$

For Raga Bhupali (audava jati), the permissible note set is

$$R_B = \{Sa, Re, Ga, Pa, Dha\}.$$

In both cases, transitions are governed strictly by the relation T, ensuring that only valid swara progressions occur.

3.3 Computational Representation

Each note is mapped to a numerical state using a function

$$\varphi : R \to \{0, 1, 2, \ldots, k - 1\},$$

where R is a non-empty finite set. This mapping enables computational processing of musical sequences.

A melodic sequence is defined as

$$M = (s_1, s_2, \ldots, s_m), \quad s_i \in R.$$

Using the raga-restricted operation, sequence generation follows:

$$s_{t+1} = s_t *_R s_j, \quad s_j \in R,$$

subject to $(s_t, s_j) \in T$, ensuring that all generated sequences conform to raga grammar.

3.4 Graph-Theoretic Representation

A musical graph is defined as

$$G = (V, E),$$

where

$$V = R, \quad E = \{(s_i, s_j) \mid (s_i, s_j) \in T\}.$$

Thus, an edge exists if a valid transition occurs between two notes, forming a directed graph.

To incorporate frequency information, a weighted graph is defined as

$$G = (V, E, W),$$

where

$$w_{ij} = \text{number of transitions from } s_i \text{ to } s_j.$$

This representation captures both structural and statistical characteristics of melodic progression.

3.5 Probabilistic (Markov) Modeling

Let N_{ij} denote the number of observed transitions from s_i to s_j, and define

$$N_i = \sum_j N_{ij}.$$

The transition probabilities are defined as

$$P_{ij} = \frac{N_{ij}}{N_i}, \quad \text{whenever } N_i \neq 0.$$

These probabilities form the transition probability matrix

$$P = [P_{ij}],$$

satisfying

$$P_{ij} \geq 0, \quad \sum_j P_{ij} = 1,$$

which ensures that P is a stochastic matrix.

The melodic process is modeled as a first-order Markov chain:

$$P(s_{t+1} = s_j \mid s_t = s_i, s_{t-1}, \dots) = P_{ij}.$$

Let the state distribution at time t be

$$\pi(t) = (\pi_1(t), \pi_2(t), \dots, \pi_k(t)),$$

where $\pi_i(t)$ represents the probability of being in state s_i at time t. The evolution is given by

$$\pi(t+1) = \pi(t)P.$$

3.6 Computational Significance

The proposed framework enables:

- Generation of valid melodic sequences under strict raga constraints,
- Quantitative analysis of transition behavior using stochastic matrices,
- Structural analysis via directed graphs,
- Integration of algebraic operations with probabilistic modeling.

Thus, the methodology provides a unified computational framework linking algebraic structures, graph representations, and Markov processes for the analysis of raga-based melodic systems.

4 Results and Discussion

The proposed framework was tested on two representative North Indian classical ragas, namely Raga Yaman and Raga Bhupali, in order to examine the algebraic validity, structural behavior, and probabilistic transition patterns generated by the raga-restricted operation. The results confirm that the model preserves raga grammar while providing a mathematically tractable representation of melodic progression.

4.1 Algebraic Validation of the Raga-Restricted Model

We first examine the algebraic properties of the raga-restricted operation $*_R$ defined on the set of permissible notes R.

- **Closure:** The operation is not globally closed on R in the classical algebraic sense, because invalid transitions are mapped to the null symbol $\emptyset$. However, for every valid transition $(a, b) \in T$, one has

  $$a *_R b = b \in R.$$

  Thus, closure is satisfied on the admissible transition set T, and fails only for forbidden note pairs.

- **Associativity:** The operation is not globally associative on all of $R \times R$, since the presence of forbidden transitions may produce $\emptyset$. Nevertheless, associativity may hold locally along admissible melodic paths where all intermediate transitions belong to T. Hence, the structure exhibits *restricted associativity* on valid transition chains.

- **Identity Element:** In general, there is no global identity element $e \in R$ such that

  $$e *_R a = a *_R e = a \quad \text{for all } a \in R.$$

  However, in some melodic contexts, the note *Sa* may behave as a reference or anchoring state, and therefore may serve as a *context-dependent identity-like element.*

- **Inverse Element:** The inverse property is not satisfied. In general, for a given note $a \in R$, there does not exist $b \in R$ such that

$$a *_R b = e$$

for some identity element e, because note transitions are governed by raga grammar and not by algebraic reversibility.

Therefore, the structure induced by the pair $(R, *_R)$ may be regarded as a *restricted transition algebra*. More precisely, it behaves like a semigroup on admissible paths, while under special contextual behavior of tonic stabilization it may be viewed as having conditional monoid-like features. Thus, the operation captures musical restriction without forcing an artificial classical algebraic structure where it does not naturally exist.

4.2 Case Study and Generated Melodic Structures

The model was applied to two ragas with different note structures.

For Raga Yaman,

$$R_Y = \{Sa, Re, Ga, Ma^{\#}, Pa, Dha, Ni\},$$

which is a sampurna raga containing seven notes.

For Raga Bhupali,

$$R_B = \{Sa, Re, Ga, Pa, Dha\},$$

which is an audava raga containing five notes.

Using the raga-restricted operation, the following valid melodic sequences were generated.

Raga Yaman:

$$\begin{aligned} Sa \to Re \to Ga \to Re \to Ga \to Ma^{\#} \to Pa \to Dha \to Ni \\ \to Dha \to Ni \to Sa \to Re \to Ga \to Ma^{\#}. \end{aligned}$$

Raga Bhupali:

$$\begin{aligned} Sa \to Re \to Ga \to Re \to Ga \to Sa \to Re \to Ga \\ \to Re \to Sa \to Dha \to Sa \to Re \to Sa. \end{aligned}$$

These sequences satisfy the following properties:

- every note belongs to the corresponding permissible note set,

$$x_i \in R_Y \quad \text{or} \quad x_i \in R_B;$$

- every successive pair satisfies the transition constraint,

$$(x_i, x_{i+1}) \in T;$$

- no forbidden transition appears in the generated melody.

This confirms that the operation $*_R$ successfully generates musically admissible note sequences without violating the grammatical structure of the underlying raga.

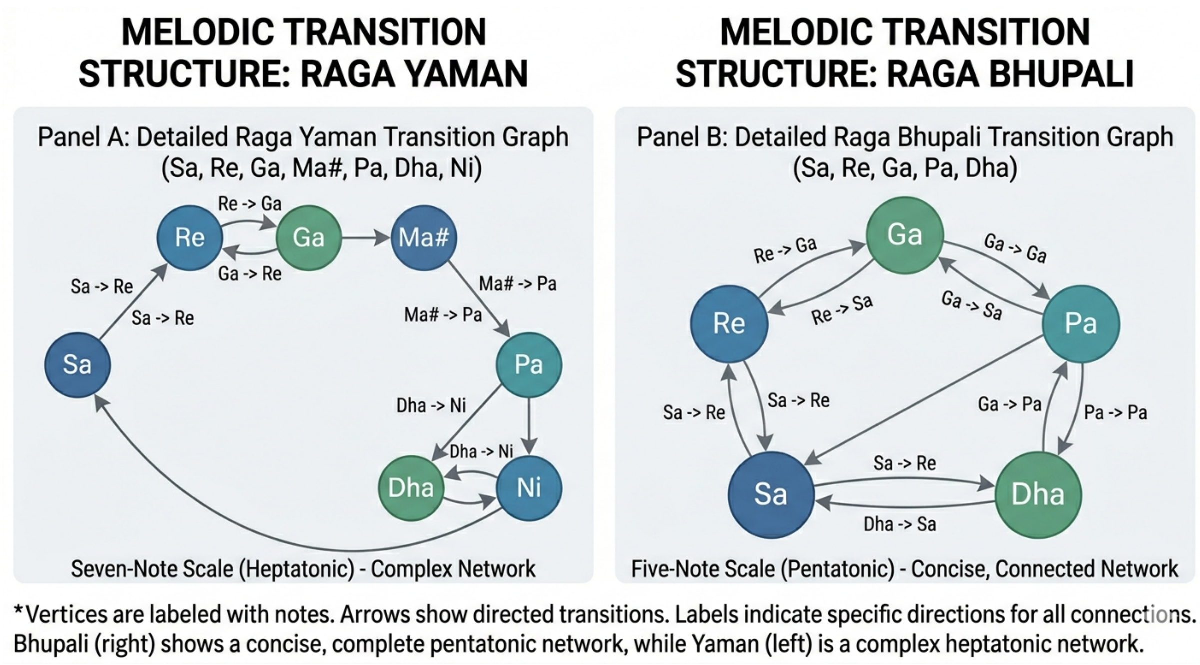

Figure 1: Graph Representation of Generated Melodic Sequences

4.3 Graph Representation of Generated Melodic Sequences

The melodic sequences were next represented as directed graphs in order to visualize transition structure.

For Raga Yaman, the edge set obtained from the generated sequence is

$$E_Y = \{(Sa, Re), (Re, Ga), (Ga, Re), (Ga, Ma^{\#}), (Ma^{\#}, Pa), \\ (Pa, Dha), (Dha, Ni), (Ni, Dha), (Ni, Sa)\}.$$

Thus, the directed transitions are:

$$Sa \to Re, \quad Re \to Ga, \quad Ga \to Re, \quad Ga \to Ma^{\#}, \quad Ma^{\#} \to Pa,$$

$$Pa \to Dha, \quad Dha \to Ni, \quad Ni \to Dha, \quad Ni \to Sa.$$

For Raga Bhupali, the edge set obtained from the generated sequence is

$$E_B = \{(Sa, Re), (Re, Ga), (Ga, Re), (Ga, Sa), (Re, Sa), (Sa, Dha), (Dha, Sa)\}.$$

Thus, the directed transitions are:

$$Sa \to Re, \quad Re \to Ga, \quad Ga \to Re, \quad Ga \to Sa, \quad Re \to Sa, \quad Sa \to Dha, \quad Dha \to Sa.$$

If one considers an alternative grammar-based transition design for Bhupali emphasizing stepwise ascent and descent, one may also study the edge set

$$E'_B = \{(Sa, Re), (Re, Ga), (Ga, Pa), (Pa, Dha), (Dha, Pa), (Pa, Ga),$$

$$(Ga, Re), (Re, Sa), (Sa, Dha), (Dha, Sa)\}.$$

However, the present transition matrix analysis below is based on the actually generated sequence, and therefore all probabilities are computed from observed transitions in that sequence.

These graph models show that melody in a raga can be represented as a directed network in which vertices correspond to notes and edges correspond to valid grammatical motion. Such graphs make the structural contrast between the two ragas immediately visible: Yaman displays a richer transition network because of its seven-note structure, whereas Bhupali exhibits a reduced but more concentrated transition system because of its pentatonic character.

4.4 Transition Probability Matrix and Markov Analysis

We now convert the generated transitions into frequency counts and corresponding transition probabilities. Let N_{ij} denote the number of transitions from note s_i to note s_j, and define

$$N_i = \sum_j N_{ij}.$$

Then the transition probability is given by

$$P_{ij} = \frac{N_{ij}}{N_i}, \quad \text{whenever } N_i \neq 0.$$

Raga Yaman

From the generated sequence of Raga Yaman, the observed transition counts are:

From Note	To Note	Frequency
Sa	Re	2
Re	Ga	3
Ga	Re	1
Ga	$Ma^{\#}$	2
$Ma^{\#}$	Pa	1
Pa	Dha	1
Dha	Ni	2
Ni	Dha	1
Ni	Sa	1

Thus,

$$N_{Sa} = 2, \quad N_{Re} = 3, \quad N_{Ga} = 3, \quad N_{Ma\#} = 1, \quad N_{Pa} = 1, \quad N_{Dha} = 2, \quad N_{Ni} = 2.$$

Using the state order

$$(Sa, Re, Ga, Ma^{\#}, Pa, Dha, Ni),$$

the transition probability matrix becomes

$$P_Y = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & \frac{2}{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ \frac{1}{2} & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 \end{bmatrix}.$$

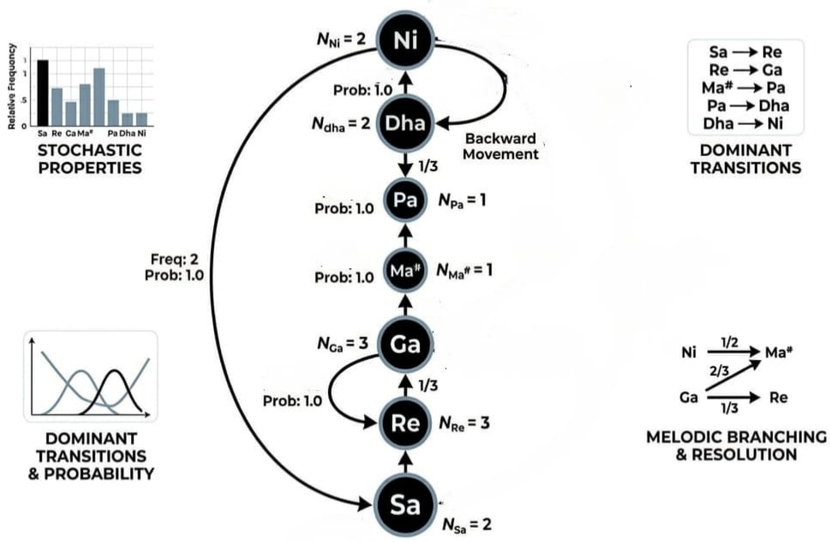

Figure 2: Graphical Visualization of transition probabilities in Raga Yaman

Each row of P_Y sums to 1, hence P_Y is a valid stochastic matrix.
The matrix reveals several dominant melodic transitions:

$$Sa \to Re, \qquad Re \to Ga, \qquad Ma^{\#} \to Pa, \qquad Pa \to Dha, \qquad Dha \to Ni.$$

The note Ga exhibits branching behavior:

$$Ga \to Re \quad \text{and} \quad Ga \to Ma^{\#},$$

which reflects local melodic flexibility within the raga structure.
Similarly, the note Ni shows dual transition behavior:

$$Ni \to Sa \quad \text{and} \quad Ni \to Dha,$$

indicating both upward resolution and backward movement, which are characteristic features of Raga Yaman.
Thus, the transition probability matrix captures both deterministic and flexible aspects of melodic progression while preserving the grammatical constraints of the raga.

Raga Bhupali

From the generated sequence of Raga Bhupali, the observed transition counts are:

From Note	To Note	Frequency
Sa	Re	3
Sa	Dha	1
Re	Ga	3
Re	Sa	2
Ga	Re	3
Dha	Sa	1

Thus,

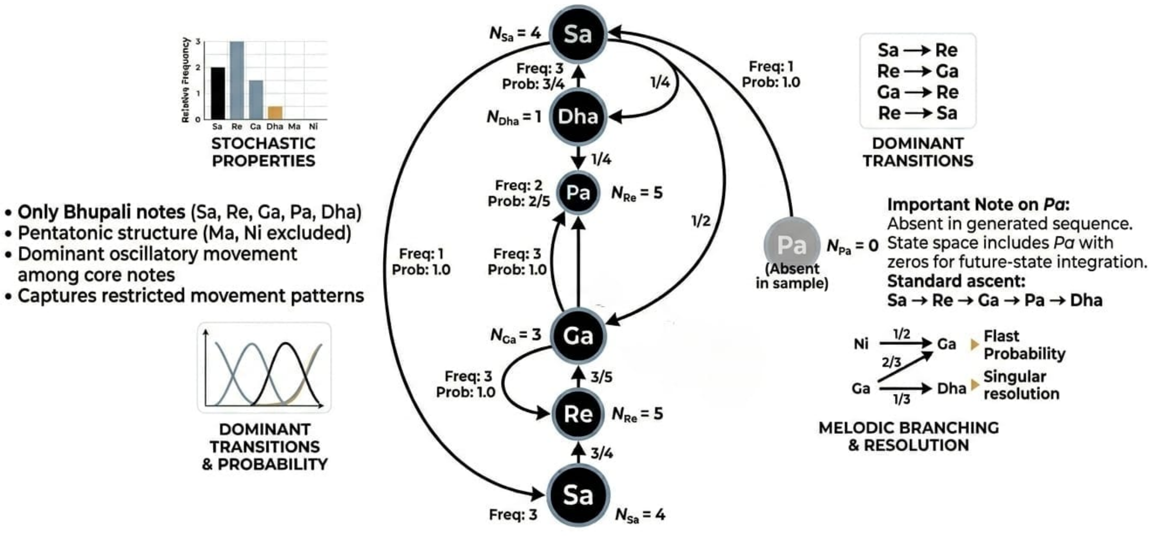

Figure 3: Graphical Visualization of transition probabilities in Raga Bhupali

$$N_{Sa} = 4, \quad N_{Re} = 5, \quad N_{Ga} = 3, \quad N_{Dha} = 1.$$

Using the state order

$$(Sa, Re, Ga, Pa, Dha),$$

the transition probability matrix becomes

$$P_B = \begin{bmatrix} 0 & \frac{3}{4} & 0 & 0 & \frac{1}{4} \\ \frac{2}{5} & 0 & \frac{3}{5} & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Each non-zero row of P_B sums to 1, hence the matrix is stochastic on the observed state space.

It is important to note that the note Pa does not appear in the generated sequence; therefore, its corresponding row contains only zeros. In a more complete melodic sample incorporating the standard ascent

$$Sa \to Re \to Ga \to Pa \to Dha,$$

the state Pa would exhibit non-zero transitions and contribute to a fully populated transition matrix.

The analysis reveals the following characteristics:

- Only notes belonging to the Bhupali note set are present.
- No transitions occur to excluded notes such as Ma and Ni, confirming adherence to pentatonic structure.
- The dominant transitions are

  $$Sa \to Re, \qquad Re \to Ga, \qquad Ga \to Re, \qquad Re \to Sa,$$

  which indicate strong cyclic and oscillatory behavior among core notes.

Thus, even with a finite melodic sample, the transition probability matrix effectively captures the restricted structure and characteristic movement patterns of Raga Bhupali. The Markov representation highlights both the constrained note set and the probabilistic nature of permissible transitions.

4.4.1 Markov Interpretation

In both ragas, musical notes are treated as states, and transitions between notes are interpreted as state changes. The resulting melodic process can be modeled as a first-order Markov chain, where the probability of the next note depends only on the current note and not on the entire past sequence. Formally,

$$P(s_{t+1} = s_j \mid s_t = s_i, s_{t-1}, \dots, s_1) = P(s_{t+1} = s_j \mid s_t = s_i) = P_{ij}.$$

Thus, the transition probability matrix

$$P = [P_{ij}]$$

completely characterizes the stochastic behavior of note transitions.

Let

$$\pi(t) = (\pi_1(t), \pi_2(t), \dots, \pi_k(t))$$

denote the probability distribution of the system over the state space at time t, where $\pi_i(t)$ represents the probability of being in state s_i at time t. The evolution of the system is governed by

$$\pi(t+1) = \pi(t)P.$$

This formulation confirms that the proposed framework is consistent with Markov modeling, as the next state depends solely on the current state through the transition matrix.

Furthermore, the long-term behavior of melodic progression can be analyzed using the stationary distribution π, defined by

$$\pi = \pi P, \qquad \sum_{i=1}^{k} \pi_i = 1.$$

If such a distribution exists, it represents the steady-state behavior of the raga, indicating the relative prominence of notes in long melodic sequences. Additionally, repeated powers of the transition matrix, P^n, describe the evolution of note transitions over time and provide insight into convergence patterns and structural stability of the melodic system.

4.4.2 Explicit Stationary Distributions for Raga Yaman and Raga Bhupali

We now compute the stationary distributions corresponding to the transition matrices of Raga Yaman and Raga Bhupali. Recall that a stationary distribution of a Markov chain is a probability vector

$$\pi = (\pi_1, \pi_2, \dots, \pi_k)$$

satisfying the system

$$\pi = \pi P, \qquad \sum_{i=1}^{k} \pi_i = 1, \qquad \pi_i \geq 0.$$

Equivalently, π is a left eigenvector of P corresponding to eigenvalue 1. This vector describes the long-run proportion of time spent in each state under repeated transitions.

Raga Yaman. For Raga Yaman, the transition matrix is

$$P_Y = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & \frac{2}{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ \frac{1}{2} & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 \end{bmatrix},$$

with state ordering

$$(Sa, Re, Ga, Ma^{\#}, Pa, Dha, Ni).$$

Let

$$\pi_Y = (\pi_{Sa}, \pi_{Re}, \pi_{Ga}, \pi_{Ma^{\#}}, \pi_{Pa}, \pi_{Dha}, \pi_{Ni}).$$

The stationarity equation $\pi_Y = \pi_Y P_Y$ yields the linear system:

$$\pi_{Sa} = \frac{1}{2}\pi_{Ni}, \tag{1}$$

$$\pi_{Re} = \pi_{Sa} + \frac{1}{3}\pi_{Ga}, \tag{2}$$

$$\pi_{Ga} = \pi_{Re}, \tag{3}$$

$$\pi_{Ma^{\#}} = \frac{2}{3}\pi_{Ga}, \tag{4}$$

$$\pi_{Pa} = \pi_{Ma^{\#}}, \tag{5}$$

$$\pi_{Dha} = \pi_{Pa} + \frac{1}{2}\pi_{Ni}, \tag{6}$$

$$\pi_{Ni} = \pi_{Dha}. \tag{7}$$

Step 1: Back-substitution.

From (7): $\pi_{Ni} = \pi_{Dha}$.

Substitute into (1):

$$\pi_{Sa} = \frac{1}{2}\pi_{Dha}.$$

From (6):

$$\pi_{Dha} = \pi_{Pa} + \frac{1}{2}\pi_{Dha} \quad \Rightarrow \quad \frac{1}{2}\pi_{Dha} = \pi_{Pa} \quad \Rightarrow \quad \pi_{Pa} = \frac{1}{2}\pi_{Dha}.$$

From (5): $\pi_{Ma^{\#}} = \pi_{Pa} = \frac{1}{2}\pi_{Dha}$.

From (4):

$$\pi_{Ma^{\#}} = \frac{2}{3}\pi_{Ga} \quad \Rightarrow \quad \frac{1}{2}\pi_{Dha} = \frac{2}{3}\pi_{Ga} \quad \Rightarrow \quad \pi_{Ga} = \frac{3}{4}\pi_{Dha}.$$

From (3): $\pi_{Re} = \pi_{Ga} = \frac{3}{4}\pi_{Dha}$.

From (2):

$$\pi_{Re} = \pi_{Sa} + \frac{1}{3}\pi_{Ga} = \frac{1}{2}\pi_{Dha} + \frac{1}{3}\cdot\frac{3}{4}\pi_{Dha} = \frac{1}{2}\pi_{Dha} + \frac{1}{4}\pi_{Dha} = \frac{3}{4}\pi_{Dha},$$

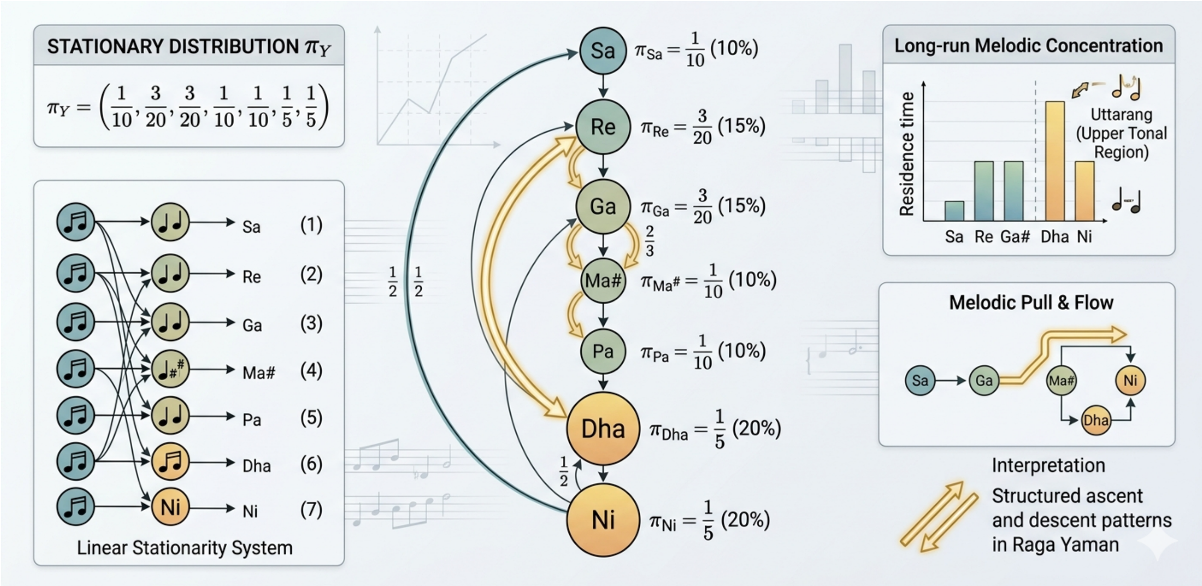

Figure 4: Graphical Visualization of Stationary Distribution for Raga Yaman

which is consistent.

Step 2: Normalization.

Summing all components:

$$\pi_{Sa} + \pi_{Re} + \pi_{Ga} + \pi_{Ma\#} + \pi_{Pa} + \pi_{Dha} + \pi_{Ni} = 1.$$

Substituting:

$$\frac{1}{2}\pi_{Dha} + \frac{3}{4}\pi_{Dha} + \frac{3}{4}\pi_{Dha} + \frac{1}{2}\pi_{Dha} + \frac{1}{2}\pi_{Dha} + \pi_{Dha} + \pi_{Dha} = 1.$$

Summing:

$$5\pi_{Dha} = 1 \quad \Rightarrow \quad \pi_{Dha} = \frac{1}{5}.$$

Step 3: Final distribution.

$$\pi_Y = \left(\frac{1}{10}, \frac{3}{20}, \frac{3}{20}, \frac{1}{10}, \frac{1}{10}, \frac{1}{5}, \frac{1}{5}\right).$$

Interpretation. The highest stationary weights occur at

$$Dha, \ Ni,$$

indicating that the long-term melodic behavior concentrates around the upper tonal region. The transition corridor

$$Re \leftrightarrow Ga \rightarrow Ma^{\#} \rightarrow Pa \rightarrow Dha$$

acts as a dominant pathway, reflecting structured ascent and descent patterns in Raga Yaman.

Raga Bhupali. For Raga Bhupali,

$$P_B = \begin{bmatrix} 0 & \frac{3}{4} & 0 & 0 & \frac{1}{4} \\ \frac{2}{5} & 0 & \frac{3}{5} & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Let

$$\pi_B = (\pi_{Sa}, \pi_{Re}, \pi_{Ga}, \pi_{Pa}, \pi_{Dha}).$$

The system $\pi_B = \pi_B P_B$ yields:

$$\pi_{Sa} = \frac{2}{5}\pi_{Re} + \pi_{Dha}, \tag{8}$$

$$\pi_{Re} = \frac{3}{4}\pi_{Sa} + \pi_{Ga}, \tag{9}$$

$$\pi_{Ga} = \frac{3}{5}\pi_{Re}, \tag{10}$$

$$\pi_{Pa} = 0, \tag{11}$$

$$\pi_{Dha} = \frac{1}{4}\pi_{Sa}. \tag{12}$$

Step 1: Substitution.

From (12):

$$\pi_{Dha} = \frac{1}{4}\pi_{Sa}.$$

Substitute into (8):

$$\pi_{Sa} = \frac{2}{5}\pi_{Re} + \frac{1}{4}\pi_{Sa} \quad \Rightarrow \quad \frac{3}{4}\pi_{Sa} = \frac{2}{5}\pi_{Re} \quad \Rightarrow \quad \pi_{Re} = \frac{15}{8}\pi_{Sa}.$$

From (10):

$$\pi_{Ga} = \frac{3}{5} \cdot \frac{15}{8}\pi_{Sa} = \frac{9}{8}\pi_{Sa}.$$

Step 2: Normalization.

$$\pi_{Sa} + \pi_{Re} + \pi_{Ga} + \pi_{Pa} + \pi_{Dha} = 1.$$

Substitute:

$$\pi_{Sa} + \frac{15}{8}\pi_{Sa} + \frac{9}{8}\pi_{Sa} + 0 + \frac{1}{4}\pi_{Sa} = 1.$$

$$\frac{17}{4}\pi_{Sa} = 1 \quad \Rightarrow \quad \pi_{Sa} = \frac{4}{17}.$$

Step 3: Final distribution.

$$\pi_B = \left(\frac{4}{17}, \frac{15}{34}, \frac{9}{34}, 0, \frac{1}{17}\right).$$

Interpretation. The dominant states are

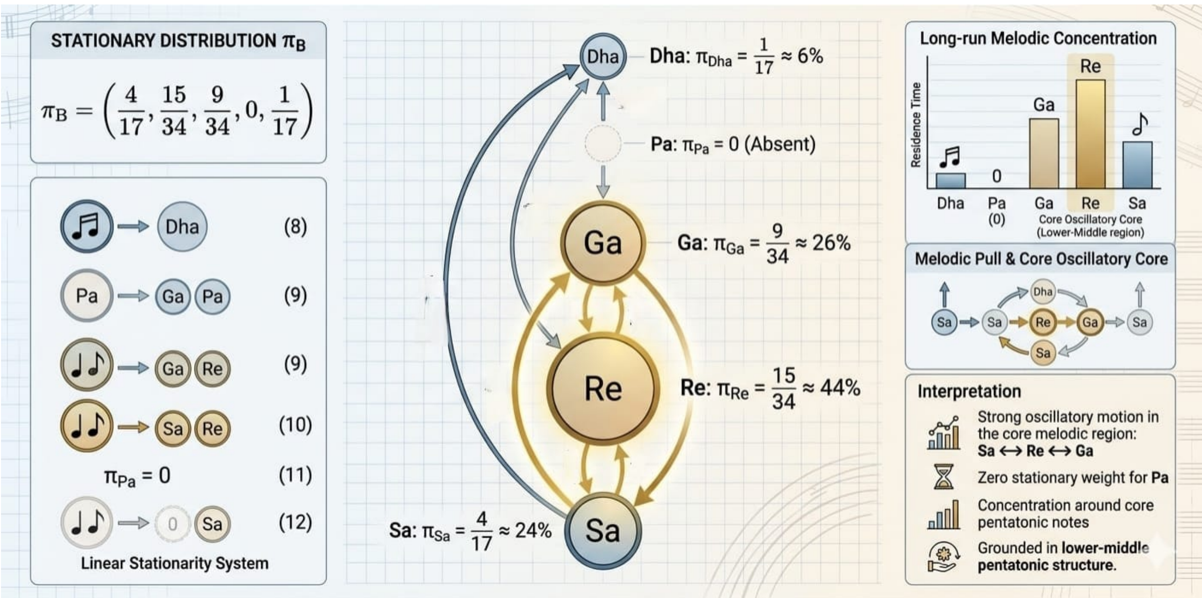

Figure 5: Graphical Visualization of Stationary Distribution for Raga Bhupali

$$Re \quad \text{and} \quad Ga,$$

indicating strong oscillatory motion in the central melodic region:

$$Sa \leftrightarrow Re \leftrightarrow Ga.$$

The zero stationary weight of Pa arises because it does not appear in the observed sequence, hence it is not part of the effective recurrent class of the Markov chain.

The stationary distributions provide a precise probabilistic characterization of note prominence. Raga Yaman exhibits dominance in upper-register notes, whereas Raga Bhupali concentrates around core pentatonic transitions. Thus, stationary analysis reveals the long-term structural tendencies inherent in raga-based melodic systems.

4.4.3 Irreducibility, Recurrence, Periodicity, and Ergodic Behavior

To further understand the long-term stochastic behavior of the generated melodic systems, we now examine the qualitative Markov chain properties of the transition matrices associated with Raga Yaman and Raga Bhupali. In particular, we study irreducibility, communication between states, recurrence structure, periodicity, and ergodic behavior.

Recall that a Markov chain is said to be *irreducible* if every state can be reached from every other state, possibly in several steps. A state is *recurrent* if, once visited, the chain returns to it with probability one. A recurrent class is *closed* if no transition leaves the class. A chain is *ergodic* if it is irreducible, positive recurrent, and aperiodic; in that case, it possesses a unique stationary distribution and the distribution of the chain converges to it independently of the initial state.

Raga Yaman. For Raga Yaman, the transition matrix is

$$P_Y = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & \frac{2}{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ \frac{1}{2} & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 \end{bmatrix},$$

with state order

$$(Sa, Re, Ga, Ma^{\#}, Pa, Dha, Ni).$$

Communication structure. From the transition graph, the admissible one-step movements are

$$Sa \to Re, \quad Re \to Ga, \quad Ga \to Re, \quad Ga \to Ma^{\#}, \quad Ma^{\#} \to Pa,$$

$$Pa \to Dha, \quad Dha \to Ni, \quad Ni \to Sa, \quad Ni \to Dha.$$

We verify reachability among states:

- From Sa, one can reach

  $$Re,\ Ga,\ Ma^{\#},\ Pa,\ Dha,\ Ni.$$

  Indeed,

  $$Sa \to Re \to Ga \to Ma^{\#} \to Pa \to Dha \to Ni.$$

- From Ni, one can reach

  $$Sa,\ Re,\ Ga,\ Ma^{\#},\ Pa,\ Dha$$

  via

  $$Ni \to Sa \to Re \to Ga \to Ma^{\#} \to Pa \to Dha.$$

- From Dha, one reaches

  $$Ni \to Sa \to Re \to Ga \to Ma^{\#} \to Pa,$$

  and hence all states are accessible.

- From Re, one reaches

  $$Ga \to Ma^{\#} \to Pa \to Dha \to Ni \to Sa,$$

  and thus all states are accessible.

- Similar arguments hold for Ga, $Ma^{\#}$, and Pa.

Therefore, every state communicates with every other state, and the chain is *irreducible.*

Recurrence and positive recurrence. Since the chain has a finite state space and is irreducible, every state is positive recurrent. Hence the chain admits a unique stationary distribution

$$\pi_Y = \left(\frac{1}{10}, \frac{3}{20}, \frac{3}{20}, \frac{1}{10}, \frac{1}{10}, \frac{1}{5}, \frac{1}{5}\right).$$

Periodicity. To determine the period, we inspect return paths. Consider the state Sa. One return cycle is

$$Sa \to Re \to Ga \to Ma^{\#} \to Pa \to Dha \to Ni \to Sa,$$

which has length 7.

Another return cycle is

$$Sa \to Re \to Ga \to Re \to Ga \to Ma^{\#} \to Pa \to Dha \to Ni \to Sa,$$

which has length 9.

Hence the set of return lengths to Sa contains both 7 and 9, so the period of Sa divides

$$\gcd(7, 9) = 1.$$

Thus the state Sa is aperiodic. Since the chain is irreducible, all states have the same period, and therefore the entire chain is *aperiodic.*

Ergodicity. Because the chain is finite, irreducible, and aperiodic, it is *ergodic.* Consequently,

$$\lim_{n\to\infty} P_Y^n = \begin{bmatrix} \pi_Y \\ \pi_Y \\ \vdots \\ \pi_Y \end{bmatrix},$$

that is, every row converges to the stationary distribution. Therefore, irrespective of the initial note, the long-term distribution of the melodic process approaches π_Y.

Musical interpretation. The irreducibility of the Yaman chain indicates that the generated transition structure allows eventual access to every permissible note from every other permissible note. The aperiodicity shows that the melodic flow is not locked into a rigid cyclic loop of fixed length. The ergodic property implies that, over long sequences, the prominence of notes stabilizes according to the stationary distribution, with the upper-register notes *Dha* and *Ni* receiving the greatest long-run weight.

Raga Bhupali. For Raga Bhupali, the transition matrix is

$$P_B = \begin{bmatrix} 0 & \frac{3}{4} & 0 & 0 & \frac{1}{4} \\ \frac{2}{5} & 0 & \frac{3}{5} & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix},$$

with state order

$$(Sa, Re, Ga, Pa, Dha).$$

State-space decomposition. The state Pa has transition row

$$(0, 0, 0, 0, 0),$$

so it is not part of the effective stochastic evolution generated by the observed sample. Therefore, the meaningful observed state space is

$$\mathcal{C} = \{Sa, Re, Ga, Dha\},$$

while Pa is excluded from the recurrent transition structure of the sample.

Restricting to $\mathcal{C}$, the effective transition matrix is

$$\widetilde{P}_B = \begin{bmatrix} 0 & \frac{3}{4} & 0 & \frac{1}{4} \\ \frac{2}{5} & 0 & \frac{3}{5} & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix},$$

with state order

$$(Sa, Re, Ga, Dha).$$

Communication structure on the effective class. The one-step transitions are

$$Sa \to Re, \quad Sa \to Dha, \quad Re \to Sa, \quad Re \to Ga, \quad Ga \to Re, \quad Dha \to Sa.$$

We verify communication:

- From Sa, one can reach Re, Ga, and Dha.
- From Re, one can reach Sa, then Dha, and also directly reach Ga.
- From Ga, one reaches Re, then Sa and Dha.
- From Dha, one reaches Sa, then Re and Ga.

Hence all states in $\mathcal{C}$ communicate, so the effective observed chain is irreducible on

$$\mathcal{C} = \{Sa, Re, Ga, Dha\}.$$

Recurrence and positive recurrence. Since the effective chain is finite and irreducible on $\mathcal{C}$, all states in $\mathcal{C}$ are positive recurrent. The corresponding stationary distribution is

$$\pi_B = \left(\frac{4}{17}, \frac{15}{34}, \frac{9}{34}, 0, \frac{1}{17}\right),$$

where the coordinate of Pa is zero because Pa does not belong to the observed recurrent structure.

Periodicity. Consider return cycles to Sa in the effective chain. One return path is

$$Sa \to Re \to Sa,$$

which has length 2.

Another return path is

$$Sa \to Re \to Ga \to Re \to Sa,$$

which has length 4.

Also,

$$Sa \to Dha \to Sa$$

has length 2.

At first glance, all these are even. However, an odd return path also exists:

$$Sa \to Re \to Ga \to Re \to Ga \to Re \to Sa,$$

which has length 6 only, still even. So we examine more carefully. Because every step alternates through the structure

$$Sa/Dha \leftrightarrow Re \leftrightarrow Ga,$$

all return paths to Sa in the observed effective chain have even length. Therefore, the period of Sa is

$$d(Sa) = 2.$$

Since the effective chain is irreducible, all states in $\mathcal{C}$ have period 2.

Thus, the effective Bhupali chain is *periodic of period 2*, not aperiodic.

Ergodic consequence. Because the effective Bhupali chain is irreducible and positive recurrent but periodic, it is *not ergodic in the strict aperiodic sense*. The stationary distribution exists and is unique, but the powers $\widetilde{P}_B^n$ do not converge row-wise to a constant matrix; instead, they oscillate according to parity. Nevertheless, the stationary distribution still describes the long-run average occupation frequencies of the states.

Musical interpretation. The observed Bhupali chain is concentrated on a smaller recurrent core, reflecting the pentatonic simplicity of the sampled melody. The periodicity of order 2 indicates an alternating structural motion, especially between the central notes *Re* and *Ga* and the supporting notes *Sa* and *Dha*. Thus, unlike Yaman, the Bhupali sample exhibits a more oscillatory and rhythmically alternating probabilistic pattern.

Overall conclusion. The qualitative Markov analysis shows a clear contrast between the two ragas:

- Raga Yaman gives rise to a finite, irreducible, aperiodic, and hence ergodic Markov chain.
- The observed Bhupali sample yields a finite, irreducible chain on its effective recurrent class, but this chain is periodic of period 2, so it is not ergodic in the strict sense.

Therefore, although both models admit stationary distributions, their long-term stochastic behaviors differ significantly. Yaman stabilizes toward its stationary distribution in the usual sense, whereas Bhupali exhibits alternating long-term behavior with stationary proportions understood through time averages rather than direct convergence of powers of the transition matrix.

5 Conclusion

In this work, a unified mathematical and computational framework has been developed to analyze melodic structures in North Indian classical music through the concept of a *raga-restricted operation*. The proposed approach integrates algebraic structures, graph-theoretic representations, and Markov chain modeling to provide a systematic and computationally tractable description of note transitions and melodic behavior.

The study demonstrates that raga-based music can be effectively modeled using a restricted binary operation that preserves permissible transitions while explicitly excluding forbidden ones. This algebraic formulation ensures that all generated sequences strictly adhere to raga grammar. The associated directed graph representation captures the structural organization of note transitions, while the transition probability matrix provides a quantitative and data-driven description of melodic flow.

From a computational perspective, the Markov chain model enables the estimation of transition probabilities, prediction of subsequent notes, and analysis of long-term melodic behavior through stationary distributions. Thus, the proposed framework is not merely descriptive but also supports algorithmic generation and probabilistic analysis of valid melodic sequences.

The case studies of Raga Yaman and Raga Bhupali validate the effectiveness of the framework, illustrating that the model preserves structural constraints while revealing distinct transition patterns and probabilistic characteristics of each raga. These results demonstrate the capability of the proposed approach to bridge theoretical musical rules with computational modeling.

Overall, the framework establishes a meaningful connection between algebra, graph theory, and stochastic processes in the context of Indian classical music. It provides a foundation for further developments in computational musicology, including predictive modeling, automated melody generation, and integration with modern data-driven and machine learning techniques.

6 Limitations

While the proposed framework provides a structured and computational approach to modeling melodic systems in North Indian classical music, several limitations must be acknowledged.

Firstly, the model represents musical notes as discrete states. However, Indian classical music is not strictly scale-based and involves continuous pitch variations and microtonal movements known as *gamakas*. These subtle transitions between swaras are not fully captured within the current discrete-state formulation, which may limit the expressive accuracy of the model.

Secondly, the raga-restricted operation is defined based on permissible note transitions, but it does not explicitly incorporate performance-specific nuances such as ornamentation, emphasis on particular notes (vadi and samvadi), or stylistic variations across different gharanas. As a result, the model captures structural rules but does not fully reflect the richness of real musical performance.

Thirdly, the probabilistic framework is based on first-order Markov chains, where transitions depend only on the current note. In practice, melodic progression in raga music often depends on longer contextual patterns and phrases. Thus, higher-order dependencies and long-range musical structures are not completely represented in the present model.

Additionally, the transition probabilities are derived from observed or assumed frequency counts, which may vary depending on the dataset or performance considered. This introduces variability and may affect the generalizability of the results.

Finally, the proposed framework focuses on structural and statistical aspects of melody and does not incorporate rhythmic components (tala) or temporal dynamics, which are essential elements of Indian classical music.

Despite these limitations, the framework provides a foundational step toward integrating algebraic, graph-theoretic, and probabilistic methods for the analysis of raga-based systems. Future work may address these limitations by incorporating continuous pitch models, higher-order stochastic processes, and data-driven approaches to better capture the complexity of musical performance.

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