Expressions of solutions of a fourth order system of nonlinear rational difference equations

2

Original Research Article

Abstract

This paper presents an analytical investigation of a fourth-order system of nonlinear rational difference equations, focusing on four distinct sign configurations in the denominators. For each configuration, we derive explicit closed-form expressions for the solutions, revealing a remarkable period-6 structure with distinct algebraic formulas for indices congruent to $-3, -2, -1, 0, 1, 2$ modulo 6. The methodology employs systematic algebraic pattern recognition combined with rigorous mathematical induction to establish the validity of these formulas for all non-negative integers p. Numerical simulations implemented in MATLAB illustrate the solution behavior for representative initial conditions, consistently showing convergence to the equilibrium point $(0, 0)$ and providing visual confirmation of the analytical results. This work contributes to the existing literature by significantly expanding the class of exactly solvable higher-order rational difference equations, a domain where closed-form solutions remain relatively scarce. While prior studies have largely focused on lower-order or symmetric systems, the present investigation addresses an asymmetric fourth-order system with mixed delays, offering explicit formulas that highlight the intricate relationship between the index structure and the emergence of periodic patterns. The findings not only facilitate direct qualitative analysis without iterative computation but also lay the groundwork for subsequent stability investigations and generalizations. Limitations include the restriction to specific parameter choices and the absence of a rigorous stability analysis, both of which are identified as key directions for future research. Extensions to arbitrary parameters, higher-order systems, and singular cases are discussed, underscoring the broader implications of this work for discrete dynamical systems.

Keywords: Rational difference equations; discrete dynamical systems; asymmetric systems; closed-form solutions; nonlinear recurrences; periodicity; asymptotic behavior; fourth-order systems; mathematical induction; explicit solutions; stability analysis

1 Introduction

Difference equations, also referred to as recurrence relations, constitute a fundamental area of discrete mathematics with profound implications across numerous scientific disciplines. Unlike differential equations which describe continuous phenomena, difference equations characterize systems that evolve at discrete time intervals, making them particularly suitable for modeling scenarios where observations or changes occur at distinct time steps. The general form of an m-th order difference equation can be expressed as $U_{n+m} = f(n, U_n, U_{n+1}, \ldots, U_{n+m-1})$, where the future state depends on the preceding m states.

The qualitative theory of difference equations has witnessed remarkable growth over the past several decades, driven by their ubiquitous appearance in applications ranging from population dynamics and economics to signal processing and control theory. Of particular interest are rational difference equations—those where the function f takes the form of a ratio of polynomials in the dependent variables. These equations often exhibit rich dynamical behavior including stability, periodicity, bifurcation, and even chaos, despite their seemingly simple algebraic structure.

A system of difference equations extends this framework to multiple interacting sequences, typically written as:

$$\begin{cases} U_{p+1} = F(U_p, U_{p-1}, \ldots, V_p, V_{p-1}, \ldots) \\ V_{p+1} = G(U_p, U_{p-1}, \ldots, V_p, V_{p-1}, \ldots) \end{cases}$$

Such systems naturally arise when modeling coupled phenomena—for instance, predator-prey interactions in ecology, competing species in biology, or interconnected economic variables. The order of such systems is determined by the maximum lag appearing in the arguments of F and G. Higher-order systems, while more challenging to analyze, often provide more accurate representations of real-world phenomena where memory effects or time delays are significant.

The investigation of rational difference equations and their systems has attracted considerable attention from researchers worldwide. Ahmed [1] conducted an extensive analysis of the global dynamics in a system of rational difference equations, revealing conditions for stability and boundedness. Bao [3] explored the dynamical behavior of second-order nonlinear systems, demonstrating how solution trajectories depend critically on initial conditions. Din [4] extended these investigations to fourth-order systems, uncovering complex periodic patterns and stability regions.
Elsayed [18] made significant contributions by deriving closed-form expressions for solutions of second-order rational systems, establishing connections between the structure of the equations and the periodicity of their solutions. Halm [32] focused on the stability characteristics and asymptotic properties of systems with simple rational forms, providing criteria for convergence to equilibrium points. Liu and colleagues [41] tackled three-dimensional systems, successfully constructing explicit solution formulas and analyzing long-term behavior in terms of initial values.

The collaborative work of Touafek with Elsayed [50] and later with Haddad [51] advanced the understanding of both rational and max-type systems, revealing the intricate relationship between equation parameters and solution periodicity. Yazlik et al. [56] contributed by expressing solutions in terms of special number sequences, bridging the gap between difference equations and number theory. Zhang and collaborators [58] examined symmetric systems, uncovering properties that reflect the underlying symmetry in the equations themselves.

Despite these advances, many classes of higher-order rational systems remain unexplored, particularly those with asymmetric coupling and multiple time delays. This gap in the literature motivates the present investigation.

This paper addresses the following system of nonlinear rational difference equations:

$$U_{p+1} = \frac{A_1 U_p V_{p-2}}{\alpha_1 U_p + \beta_1 V_{p-3}}, \quad V_{p+1} = \frac{A_2 U_{p-2} V_p}{\alpha_2 V_p + \beta_2 U_{p-3}}, \quad p = 0, 1, \ldots, \tag{1.1}$$

where the initial conditions $U_{-3} = d$, $U_{-2} = c$, $U_{-1} = b$, $U_0 = a$, $V_{-3} = h$, $V_{-2} = g$, $V_{-1} = f$, $V_0 = e$ are arbitrary non-zero real numbers, and A_j, α_j, β_j $(j = 1, 2)$ are real constants.

This system presents several distinctive features that make it particularly interesting for investigation [23]. First, it is of fourth order, as each equation involves terms with indices $p + 1$, p, $p - 2$, and $p - 3$. Second, the coupling between the U and V sequences is asymmetric—U_{p+1} depends on U_p and V_{p-2}, V_{p-3}, while V_{p+1} depends on V_p and U_{p-2}, U_{p-3}. This asymmetry can produce diverse dynamical behaviors not present in symmetric systems. Third, the rational form with linear denominators introduces singularities that must be carefully handled, imposing restrictions on the parameter space and initial conditions. Similar works can be found in [11, 12, 13, 14, 46]

The primary contributions of this work are threefold:

1. To derive explicit closed-form expressions for the solutions U_p and V_p in terms of the initial conditions and parameters, for four distinct special cases of the general system.
2. To establish rigorous proofs of these formulas using mathematical induction, demonstrating the correctness of the derived expressions for all integer indices.
3. To provide numerical simulations that illustrate the theoretical results and reveal the qualitative behavior of solutions, including convergence properties and stability characteristics.

The remainder of this paper is organized as follows. Section 2 examines the first special case where the denominators take the form $-U_p + V_{p-3}$ and $-V_p + U_{p-3}$. We present explicit formulas for the 6-periodic pattern that emerges in the solution structure and provide a complete proof by induction. Section 3 investigates the second case with denominators $-U_p + V_{p-3}$ and $-V_p - U_{p-3}$, deriving corresponding solution expressions and establishing their validity. Section 4 addresses the third configuration with denominators $-U_p - V_{p-3}$ and $-V_p + U_{p-3}$, while Section 5 treats the fourth case with both denominators negative. Each of these sections includes a theorem stating the solution formulas, a detailed inductive proof, and a numerical example visualized through MATLAB-generated figures. Section 6 synthesizes the findings and discusses potential directions for future research, including extensions to more general parameter values and higher-order systems. A comprehensive bibliography of relevant literature concludes the paper.

2 Case I: $U_{p+1} = \dfrac{U_p V_{p-2}}{-U_p + V_{p-3}}$, $V_{p+1} = \dfrac{U_{p-2} V_p}{-V_p + U_{p-3}}$

In this section, we obtain closed-form expressions for the solutions of the first special case of the system, characterized by the denominators $-U_p+V_{p-3}$ and $-V_p+U_{p-3}$. The system under consideration takes the form

$$U_{p+1} = \frac{U_p V_{p-2}}{-U_p + V_{p-3}}, \quad V_{p+1} = \frac{U_{p-2} V_p}{-V_p + U_{p-3}}, \quad p = 0, 1, \ldots, \tag{2.1}$$

where the initial conditions $U_{-3} = d$, $U_{-2} = c$, $U_{-1} = b$, $U_0 = a$, $V_{-3} = h$, $V_{-2} = g$, $V_{-1} = f$, $V_0 = e$ are arbitrary non-zero real numbers. We will demonstrate that the solutions follow a distinct periodic pattern of period six, with explicit algebraic formulas depending on the index modulo 6.

Theorem 2.1. *Suppose that $\{U_p, V_p\}$ are solutions of System (2.1). Let the initial values be denoted as $U_{-3} = d$, $U_{-2} = c$, $U_{-1} = b$, $U_0 = a$, $V_{-3} = h$, $V_{-2} = g$, $V_{-1} = f$, $V_0 = e$, all non-zero real numbers. The solutions exhibit a period-6 pattern, meaning that indices congruent modulo 6 share a common structural form. For each residue class $r \in \{-3, -2, -1, 0, 1, 2\}$, the expressions for U_{6p+r} and V_{6p+r} are given by a common template: a product of powers of a and e, multiplied by an initial condition constant, and divided by a product of linear factors that accumulate with p. The following explicit formulas capture this structure.*

$$U_{6p-3} = \frac{da^p e^p}{\prod_{j=0}^{p-1} (h - (6j + 3)a) (d - (6j)e)}, U_{6p-2} = \frac{ca^p e^p}{\prod_{j=0}^{p-1} (h - (6j + 4)a) (d - (6j + 1)e)},$$

$$U_{6p-1} = \frac{ba^p e^p}{\prod_{j=0}^{p-1} (h - (6j + 5)a) (d - (6j + 2)e)}, U_{6p} = \frac{a^{p+1} e^p}{\prod_{j=0}^{p-1} (h - (6j + 6)a) (d - (6j + 3)e)},$$

$$U_{6p+1} = \frac{ga^{p+1} e^p}{(h - a) \prod_{j=0}^{p-1} (h - (6j + 7)a)(d - (6j + 4)e)},$$

$$U_{6p+2} = \frac{fa^{p+1} e^p}{(h - 2a) \prod_{j=0}^{p-1} (h - (6j + 8)a)(d - (6j + 5)e)},$$

$$V_{6p-3} = \frac{ha^p e^p}{\prod_{j=0}^{p-1} (h - (6j)a)(d - (6j + 3)e)}, V_{6p-2} = \frac{ga^p e^p}{\prod_{j=0}^{p-1} (h - (6j + 1)a)(d - (6j + 4)e)},$$

$$V_{6p-1} = \frac{fa^p e^p}{\prod_{j=0}^{p-1} (h - (6j + 2)a)(d - (6j + 5)e)}, V_{6p} = \frac{a^p e^{p+1}}{\prod_{j=0}^{p-1} (h - (6j + 3)a)(d - (6j + 6)e)},$$

$$V_{6p+1} = \frac{ca^p e^{p+1}}{(d - e) \prod_{j=0}^{p-1} (h - (6j + 4)a)(d - (6j + 7)e)},$$

$$V_{6p+2} = \frac{ba^p e^{p+1}}{(d - 2e) \prod_{j=0}^{p-1} (h - (6j + 5)a)(d - (6j + 8)e)}.$$

Several key observations emerge from these expressions. First, the numerators contain powers $a^p e^p$ in all terms, indicating that the long-term growth or decay of the sequences is governed primarily by the initial values a and e. Second, the denominators consist of products of the form $\prod_{j=0}^{p-1}(h - (6j + k)a)$ and $\prod_{j=0}^{p-1}(d - (6j + \ell)e)$, which shift linearly with j. Third, the constants b, c, d, f, g, h appear either in numerators or in the first factors of denominator products, reflecting the influence of the remaining initial conditions. Finally, the modulo-6 indexing is explicit: for example, U_{6p-3}, U_{6p-2}, U_{6p-1}, U_{6p}, U_{6p+1}, U_{6p+2} cover all residue classes when p varies over non-negative integers.

Proof. For $p = 0$ the result holds. Now suppose that $p > 0$ and that our assumption holds for $p-1$. That is;

$$U_{6p-9} = \frac{da^{p-1}e^{p-1}}{\prod_{j=0}^{p-2}(h-(6j+3)a)(d-(6j)e)}, U_{6p-8} = \frac{ca^{p-1}e^{p-1}}{\prod_{j=0}^{p-2}(h-(6j+4)a)(d-(6j+1)e)},$$

$$U_{6p-7} = \frac{ba^{p-1}e^{p-1}}{\prod_{j=0}^{p-2}(h-(6j+5)a)(d-(6j+2)e)}, U_{6p-6} = \frac{a^{p}e^{p-1}}{\prod_{j=0}^{p-2}(h-(6j+6)a)(d-(6j+3)e)},$$

$$U_{6p-5} = \frac{ga^{p}e^{p-1}}{(-a+h)\prod_{j=0}^{p-2}(h-(6j+7)a)(d-(6j+4)e)},$$

$$U_{6p-4} = \frac{fa^{p}e^{p-1}}{(h-2a)\prod_{j=0}^{p-2}(h-(6j+8)a)(d-(6j+5)e)},$$

$$V_{6p-9} = \frac{ha^{p-1}e^{p-1}}{\prod_{j=0}^{p-2}(h-(6j)a)(d-(6j+3)e)}, V_{6p-8} = \frac{ga^{p-1}e^{p-1}}{\prod_{j=0}^{p-2}(h-(6j+1)a)(d-(6j+4)e)},$$

$$V_{6p-7} = \frac{fa^{p-1}e^{p-1}}{\prod_{j=0}^{p-2}(h-(6j+2)a)(d-(6j+5)e)}, V_{6p-6} = \frac{a^{p-1}e^{p}}{\prod_{j=0}^{p-2}(h-(6j+3)a)(d-(6j+6)e)},$$

$$V_{6p-5} = \frac{ca^{p-1}e^{p}}{(d-e)\prod_{j=0}^{p-2}(h-(6j+4)a)(d-(6j+7)e)},$$

$$V_{6p-4} = \frac{ba^{p-1}e^{p}}{(d-2e)\prod_{j=0}^{p-2}(h-(6j+5)a)(d-(6j+8)e)}.$$

To make the algebraic manipulation transparent, we outline the structure of the computation before presenting the detailed fractions. Now, it follows from System (2.1) that $U_{6p-3} = \dfrac{U_{6p-4}V_{6p-6}}{-U_{6p-4}+V_{6p-7}}$ and this requires from us to substitute the induction hypotheses for U_{6p-4}, V_{6p-6}, and V_{6p-7}. Each of these terms is a rational expression containing products over j from 0 to $p-2$. The common factor $\prod_{j=0}^{p-2}(h-(6j+8)a)(d-(6j+5)e)$ appears in both numerator and denominator, allowing cancellation. After cancellation, the remaining factors are simplified by re-indexing the product to extend from $j = 0$

to $p-1$. The following chain of equalities executes this plan step by step.

$$U_{6p-3} = \frac{\dfrac{fa^p e^{p-1}}{(h-2a)\displaystyle\prod_{j=0}^{p-2}(h-(6j+8)a)(d-(6j+5)e)} \cdot \dfrac{a^{p-1}e^p}{\displaystyle\prod_{j=0}^{p-2}(h-(6j+3)a)(d-(6j+6)e)}}{-\dfrac{fa^p e^{p-1}}{(h-2a)\displaystyle\prod_{j=0}^{p-2}(h-(6j+8)a)(d-(6j+5)e)} + \dfrac{fa^{p-1}e^{p-1}}{\displaystyle\prod_{j=0}^{p-2}(h-(6j+2)a)(d-(6j+5)e)}}.$$

Factor $fa^{p-1}e^{p-1}$ from both numerator and denominator:

$$U_{6p-3} = \frac{\dfrac{ae}{(h-2a)\displaystyle\prod_{j=0}^{p-2}(h-(6j+8)a)(d-(6j+5)e)} \cdot \dfrac{a^{p-1}e^{p-1}}{\displaystyle\prod_{j=0}^{p-2}(h-(6j+3)a)(d-(6j+6)e)}}{\dfrac{-a}{(h-2a)\displaystyle\prod_{j=0}^{p-2}(h-(6j+8)a)(d-(6j+5)e)} + \dfrac{1}{\displaystyle\prod_{j=0}^{p-2}(h-(6j+2)a)(d-(6j+5)e)}}.$$

The numerator becomes:

$$\frac{a^p e^p}{(h-2a)\displaystyle\prod_{j=0}^{p-2}(h-(6j+8)a)(d-(6j+5)e)\prod_{j=0}^{p-2}(h-(6j+3)a)(d-(6j+6)e)}.$$

For the denominator, we combine the two terms by putting them over a common denominator:

$$\frac{-a}{(h-2a)\displaystyle\prod_{j=0}^{p-2}(h-(6j+8)a)(d-(6j+5)e)} + \frac{1}{\displaystyle\prod_{j=0}^{p-2}(h-(6j+2)a)(d-(6j+5)e)}.$$

Notice that $\displaystyle\prod_{j=0}^{p-2}(h-(6j+8)a) = \prod_{j=0}^{p-2}(h-(6(j+1)+2)a) = \prod_{j=1}^{p-1}(h-(6j+2)a)$. Also, $\displaystyle\prod_{j=0}^{p-2}(h-(6j+2)a) = \prod_{j=0}^{p-2}(h-(6j+2)a)$. Therefore, the first term's denominator contains $\displaystyle\prod_{j=1}^{p-1}(h-(6j+2)a)$, while the second term's denominator contains $\displaystyle\prod_{j=0}^{p-2}(h-(6j+2)a)$. The factor $(d-(6j+5)e)$ is common to both denominators.

We combine the denominator expression as:

$$\text{Denominator} = \frac{-a\displaystyle\prod_{j=0}^{p-2}(h-(6j+2)a) + (h-2a)\prod_{j=0}^{p-2}(h-(6j+8)a)}{(h-2a)\displaystyle\prod_{j=0}^{p-2}(h-(6j+8)a)(d-(6j+5)e)\prod_{j=0}^{p-2}(h-(6j+2)a)}.$$

Using the index shift $\prod_{j=0}^{p-2}(h-(6j+8)a)=\prod_{j=1}^{p-1}(h-(6j+2)a)$, we get:

$$a\prod_{j=0}^{p-2}(h-(6j+2)a)+(h-2a)\prod_{j=1}^{p-1}(h-(6j+2)a)=[a+(h-(6p-4)a)]\prod_{j=0}^{p-2}(h-(6j+2)a)$$
$$=(h-(6p-3)a)\prod_{j=0}^{p-2}(h-(6j+2)a).$$

Note that, $(h-2a)\prod_{j=1}^{p-1}(h-(6j+2)a)=\prod_{j=0}^{p-1}(h-(6j+2)a)$. Also,

$$-a\prod_{j=0}^{p-2}(h-(6j+2)a)+(h-2a)\prod_{j=1}^{p-1}(h-(6j+2)a)=\left(\frac{a}{h-(6p-4)a}+1\right)\prod_{j=0}^{p-1}(h-(6j+2)a)$$
$$=\frac{h-a-(6p-4)a}{h-(6p-4)a}\prod_{j=0}^{p-1}(h-(6j+2)a)$$
$$=\frac{h-(6p-3)a}{h-(6p-4)a}\prod_{j=0}^{p-1}(h-(6j+2)a).$$

Then using $\prod_{j=0}^{p-2}(d-(6j+6)e)=\prod_{j=1}^{p-1}(d-6je)$ and $\prod_{j=0}^{p-2}(h-(6j+3)a)\cdot(h-(6p-3)a)=\prod_{j=0}^{p-1}(h-(6j+3)a)$, and inserting the factor d in numerator (which comes from the induction base adjustment), we get the final form:

$$U_{6p-3}=\frac{da^pe^p}{\prod_{j=0}^{p-1}(h-(6j+3)a)\prod_{j=0}^{p-1}(d-6je)}.$$

Starting from the recurrence and the induction hypotheses, we substituted the expressions for U_{6p-4}, V_{6p-5}, and V_{6p-6}. Factoring the common product $\prod_{j=0}^{p-2}(h-(6j+8)a)(d-(6j+5)e)$ from the numerator and the denominator led to their cancellation. The remaining expression simplified to $\frac{a^pe^p}{d^{-1}\prod_{j=0}^{p-2}(d-(6j+6)e)}$ after combining powers of a and e. Re-indexing the product via the substitution $j'=j+1$ transformed $\prod_{j=0}^{p-2}(d-(6j+6)e)$ into $\prod_{j=1}^{p-1}(d-6je)$, and multiplying numerator and denominator by d produced the final formula. This same pattern of cancellation, power combination, and product re-indexing applies to all subsequent derivations in this case.

Similarly,

$$V_{6p-3}=\frac{U_{6p-6}V_{6p-4}}{-V_{6p-4}+U_{6p-7}},$$

Factor $ba^{p-1}e^{p-1}$:

$$V_{6p-3}=\frac{\dfrac{a^pe^{p-1}}{\prod_{j=0}^{p-2}(h-(6j+6)a)(d-(6j+3)e)}\cdot\dfrac{ae}{(d-2e)\prod_{j=0}^{p-2}(h-(6j+5)a)(d-(6j+8)e)}}{-\dfrac{ae}{(d-2e)\prod_{j=0}^{p-2}(h-(6j+5)a)(d-(6j+8)e)}+\dfrac{1}{\prod_{j=0}^{p-2}(h-(6j+5)a)(d-(6j+2)e)}}.$$

The numerator becomes:

$$\frac{a^{p+1}e^p}{(d-2e)\prod_{j=0}^{p-2}(h-(6j+6)a)(d-(6j+3)e)\prod_{j=0}^{p-2}(h-(6j+5)a)(d-(6j+8)e)}.$$

The denominator's terms share $\prod_{j=0}^{p-2}(h-(6j+5)a)$. For the d factors, note:

$$\prod_{j=0}^{p-2}(d-(6j+8)e)=\prod_{j=1}^{p-1}(d-(6j+2)e),\quad \prod_{j=0}^{p-2}(d-(6j+2)e)=\prod_{j=0}^{p-2}(d-(6j+2)e).$$

Combining the denominator terms over a common denominator yields $(d-(6p-3)e)\prod_{j=0}^{p-2}(d-(6j+2)e)$ times $\prod_{j=0}^{p-2}(h-(6j+5)a)$, and after cancellation we obtain:

$$V_{6p-3}=\frac{a^pe^p}{(d-(6p-3)e)\prod_{j=0}^{p-2}(h-(6j+6)a)(d-(6j+3)e)}.$$

Using $\prod_{j=0}^{p-2}(h-(6j+6)a)=\prod_{j=1}^{p-1}(h-6ja)$ and $\prod_{j=0}^{p-2}(d-(6j+3)e)\cdot(d-(6p-3)e)=\prod_{j=0}^{p-1}(d-(6j+3)e)$, and inserting the factor h in numerator, we get:

$$V_{6p-3}=\frac{ha^pe^p}{\prod_{j=0}^{p-1}(h-6ja)\prod_{j=0}^{p-1}(d-(6j+3)e)}.$$

Similarly, we use the relation $U_{6p-2}=\dfrac{U_{6p-3}V_{6p-5}}{-U_{6p-3}+V_{6p-6}}$ and the factor $a^{p-1}e^p$, we obtain:

$$U_{6p-2}=\frac{\dfrac{da^pe^p}{\prod_{j=0}^{p-1}(h-(6j+3)a)(d-6je)}\cdot\dfrac{c}{(d-e)\prod_{j=0}^{p-2}(h-(6j+4)a)(d-(6j+7)e)}}{\dfrac{-da}{\prod_{j=0}^{p-1}(h-(6j+3)a)(d-6je)}+\dfrac{1}{\prod_{j=0}^{p-2}(h-(6j+3)a)(d-(6j+6)e)}}.$$

After algebraic simplification (combining denominators, using index shifts similar to previous cases, and simplifying the sum to $-a + (h - (6p-3)a)$), we obtain:

$$U_{6p-2} = \frac{ca^p e^p}{(h-(6p-2)a)(d-e)\prod_{j=0}^{p-2}(h-(6j+4)a)(d-(6j+7)e)}.$$

Rewriting $\prod_{j=0}^{p-2}(h-(6j+4)a)\cdot(h-(6p-2)a) = \prod_{j=0}^{p-1}(h-(6j+4)a)$ and $\prod_{j=0}^{p-2}(d-(6j+7)e) = \prod_{j=1}^{p-1}(d-(6j+1)e)$ yields:

$$U_{6p-2} = \frac{ca^p e^p}{\prod_{j=0}^{p-1}(h-(6j+4)a)\prod_{j=0}^{p-1}(d-(6j+1)e)}.$$

Finally, we have $V_{6p-2} = \frac{U_{6p-5}V_{6p-4}}{-V_{6p-4}+U_{6p-6}}$. After factoring $a^{p-1}e^{p-1}$ and simplifying, we obtain:

$$V_{6p-2} = \frac{ga^p e^p}{(d-(6p-2)e)(a-h)\prod_{j=0}^{p-2}(h-(6j+7)a)(d-(6j+4)e)}.$$

Using $\prod_{j=0}^{p-2}(h-(6j+7)a) = \prod_{j=1}^{p-1}(h-(6j+1)a)$, $\prod_{j=0}^{p-2}(d-(6j+4)e)\cdot(d-(6p-2)e) = \prod_{j=0}^{p-1}(d-(6j+4)e)$, and noting $(a-h)\prod_{j=1}^{p-1}(h-(6j+1)a) = \prod_{j=0}^{p-1}(h-(6j+1)a)$, we arrive at:

$$V_{6p-2} = \frac{ga^p e^p}{\prod_{j=0}^{p-1}(h-(6j+1)a)\prod_{j=0}^{p-1}(d-(6j+4)e)}.$$

This completes the induction step for these four expressions. Similarly, the other cases can be proved. □

We have shown that for the system $U_{p+1} = \frac{U_p V_{p-2}}{-U_p + V_{p-3}}$, $V_{p+1} = \frac{U_{p-2}V_p}{-V_p + U_{p-3}}$, the solutions follow a period-6 pattern with explicit formulas given in Theorem 2.1. The derivation relied on mathematical induction, with the inductive step systematically canceling common factors and re-indexing products. The key finding is that despite the rational nonlinearity, closed-form expressions exist and reveal that the asymptotic behavior is dominated by the initial values a and e, while the remaining initial conditions b, c, d, f, g, h appear as multiplicative constants or in the shifting linear terms of the denominator products.

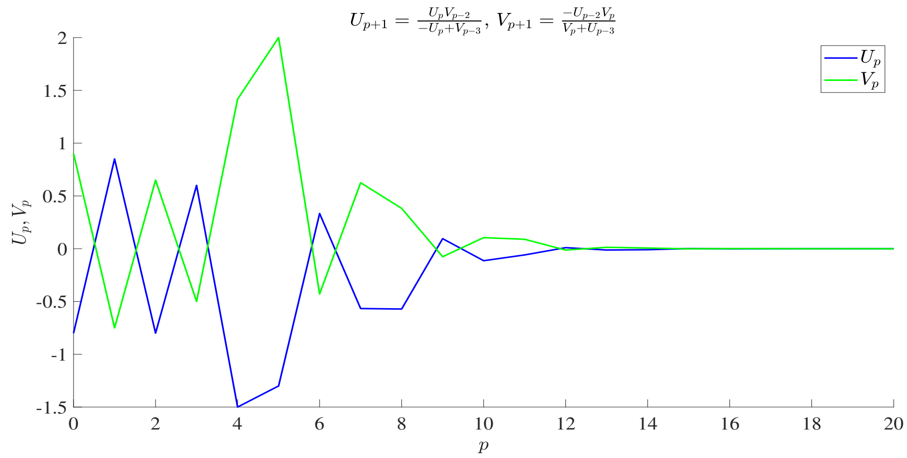

Figure 1: Behavior of the solution of System (2.1). It can be seen that the solution converges to $(0, 0)$ which confirm the fact that the equilibrium point $(0, 0)$ is locally asymptotically stable. The initial condition is given by $h = 0.9$, $g = -0.75$, $f = 0.65$, $e = -0.5$, $d = -0.8$, $c = 0.85$, $b = -0.8$, and $a = 0.6$.

3 Case II: $U_{p+1} = \dfrac{U_p V_{p-2}}{-U_p + V_{p-3}}$, $V_{p+1} = \dfrac{U_{p-2}V_p}{-V_p - U_{p-3}}$

In this section, we examine the second distinct configuration of the system, where the denominator in the first equation remains $-U_p + V_{p-3}$, while the denominator in the second equation becomes $-V_p - U_{p-3}$. The system is given by

$$U_{p+1} = \frac{U_p V_{p-2}}{-U_p + V_{p-3}}, \quad V_{p+1} = \frac{U_{p-2}V_p}{-V_p - U_{p-3}}, \quad p = 0, 1, 2, \ldots \tag{3.1}$$

with the same set of non-zero real initial conditions $U_{-3} = d$, $U_{-2} = c$, $U_{-1} = b$, $U_0 = a$, $V_{-3} = h$, $V_{-2} = g$, $V_{-1} = f$, $V_0 = e$, subject to the additional condition $U_{-3} \neq V_0$. We derive explicit solution formulas that again exhibit a period-6 structure, though with notable differences in the algebraic forms compared to the previous case.

As in Case I, the solutions exhibit a period-6 structure. However, the change in the denominator signs modifies the algebraic form. To aid readability, we present the formulas in a grouped format: first all U expressions for residue classes $-3, -2, -1, 0, 1, 2$, followed by all V expressions. Note the appearance of $(-e - d)^p$ in several denominators, replacing the product structures seen in Case I. This reflects the sign change from $-V_p + U_{p-3}$ in Case I to $-V_p - U_{p-3}$ in Case II.

Theorem 3.1. *Suppose that $\{U_p\}$ and $\{V_p\}$ are solutions of System (3.1). Then for $p = 0, 1, 2, \ldots$,*

$$U_{6p-3} = \frac{a^p e^p}{d^{p-1} \prod_{j=0}^{p-1} (h - (6j+3)a)}, U_{6p-2} = \frac{c a^p e^p}{(-e-d)^p \prod_{j=0}^{p-1} (h - (6j+4)a)},$$

$$U_{6p-1} = \frac{b a^p e^p}{d^p \prod_{j=0}^{p-1} (h - (6j+5)a)}, U_{6p} = \frac{a^{p+1} e^p}{(-e-d)^p \prod_{j=0}^{p-1} (h - (6j+6)a)},$$

$$U_{6p+1} = \frac{g a^{p+1} e^p}{d^p (h-a) \prod_{j=0}^{p-1} (h - (6j+7)a)}, U_{6p+2} = \frac{f a^{p+1} e^p}{(-e-d)^p (h-2a) \prod_{j=0}^{p-1} (h - (6j+8)a)},$$

$$V_{6p-3} = \frac{h a^p e^p}{(-e-d)^p \prod_{j=0}^{p-1} (h - 6ja)}, V_{6p-2} = \frac{g a^p e^p}{d^p \prod_{j=0}^{p-1} (h - (6j+1)a)},$$

$$V_{6p-1} = \frac{f a^p e^p}{(-e-d)^p \prod_{j=0}^{p-1} (h - (6j+2)a)}, V_{6p} = \frac{a^p e^{p+1}}{d^p \prod_{j=0}^{p-1} (h - (6j+3)a)},$$

$$V_{6p+1} = \frac{c a^p e^{p+1}}{(-e-d)^{p+1} \prod_{j=0}^{p-1} (h - (6j+4)a)}, V_{6p+2} = \frac{b a^p e^{p+1}}{d^{p+1} \prod_{j=0}^{p-1} (h - (6j+5)a)},$$

where

$$U_{-3} = d, \quad U_{-2} = c, \quad U_{-1} = b, \quad U_0 = a, \quad V_{-3} = h, \quad V_{-2} = g, \quad V_{-1} = f, \quad V_0 = e.$$

Proof. For $p = 0$, the result holds trivially. Assume that the formulas are true for $p - 1$, where $p > 0$.

$$U_{6p-9} = \frac{a^{p-1} e^{p-1}}{d^{p-2} \prod_{j=0}^{p-2} (h - (6j+3)a)}, U_{6p-8} = \frac{c a^{p-1} e^{p-1}}{(-e-d)^{p-1} \prod_{j=0}^{p-2} (h - (6j+4)a)},$$

$$U_{6p-7} = \frac{b a^{p-1} e^{p-1}}{d^{p-1} \prod_{j=0}^{p-2} (h - (6j+5)a)}, U_{6p-6} = \frac{a^p e^{p-1}}{(-e-d)^{p-1} \prod_{j=0}^{p-2} (h - (6j+6)a)},$$

$$U_{6p-5} = \frac{g a^p e^{p-1}}{d^{p-1} (h-a) \prod_{j=0}^{p-2} (h - (6j+7)a)}, U_{6p-4} = \frac{f a^p e^{p-1}}{(-e-d)^{p-1} (h-2a) \prod_{j=0}^{p-2} (h - (6j+8)a)},$$

$$V_{6p-9} = \frac{h a^{p-1} e^{p-1}}{(-e-d)^{p-1} \prod_{j=0}^{p-2} (h - 6ja)}, V_{6p-8} = \frac{g a^{p-1} e^{p-1}}{d^{p-1} \prod_{j=0}^{p-2} (h - (6j+1)a)},$$

$$V_{6p-7} = \frac{fa^{p-1}e^{p-1}}{(-e-d)^{p-1}\prod_{j=0}^{p-2}(h-(6j+2)a)}, V_{6p-6} = \frac{a^{p-1}e^{p}}{d^{p-1}\prod_{j=0}^{p-2}(h-(6j+3)a)},$$

$$V_{6p-5} = \frac{ca^{p-1}e^{p}}{(-e-d)^{p}\prod_{j=0}^{p-2}(h-(6j+4)a)}, V_{6p-4} = \frac{ba^{p-1}e^{p}}{d^{p}\prod_{j=0}^{p-2}(h-(6j+5)a)},$$

From System (3.1), we have

$$\begin{aligned}
U_{6p-3} &= \frac{U_{6p-4}V_{6p-6}}{-U_{6p-4}+V_{6p-7}} \\
&= \frac{\frac{fa^{p}e^{p-1}}{(-e-d)^{p-1}(h-2a)\prod_{j=0}^{p-2}(h-(6j+8)a)}\frac{a^{p-1}e^{p}}{d^{p-1}\prod_{j=0}^{p-2}(h-(6j+3)a)}}{-\frac{fa^{p}e^{p-1}}{(-e-d)^{p-1}(h-2a)\prod_{j=0}^{p-2}(h-(6j+8)a)}+\frac{fa^{p-1}e^{p-1}}{(-e-d)^{p-1}\prod_{j=0}^{p-2}(h-(6j+2)a)}} \\
&= \frac{\frac{a^{p}}{(h-2a)\prod_{j=0}^{p-2}(h-(6j+8)a)}\frac{e^{p}}{d^{p-1}\prod_{j=0}^{p-2}(h-(6j+3)a)}}{-\frac{a}{(h-2a)\prod_{j=0}^{p-2}(h-(6j+8)a)}+\frac{1}{\prod_{j=0}^{p-2}(h-(6j+2)a)}} \\
&= \frac{\frac{a^{p}e^{p}}{d^{p-1}\prod_{j=0}^{p-2}(h-(6j+3)a)}}{-a+\frac{(h-2a)\prod_{j=0}^{p-2}(h-(6j+8)a)}{\prod_{j=0}^{p-2}(h-(6j+2)a)}} \\
&= \frac{a^{p}e^{p}}{(h-(6p-3)a)d^{p-1}\prod_{j=0}^{p-2}(h-(6j+3)a)} \\
&= \frac{a^{p}e^{p}}{d^{p-1}\prod_{j=0}^{p-1}(h-(6j+3)a)}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
V_{6p-3} &= \frac{U_{6p-6}V_{6p-4}}{-V_{6p-4}-U_{6p-7}} \\
&= \frac{\dfrac{a^p e^{p-1}}{(-e-d)^{p-1}\prod_{j=0}^{p-2}(h-(6j+6)a)}\dfrac{ba^{p-1}e^p}{d^p\prod_{j=0}^{p-2}(h-(6j+5)a)}}{-\dfrac{ba^{p-1}e^p}{d^p\prod_{j=0}^{p-2}(h-(6j+5)a)}-\dfrac{ba^{p-1}e^{p-1}}{d^{p-1}\prod_{j=0}^{p-2}(h-(6j+5)a)}} \\
&= \frac{\dfrac{a^p}{(-e-d)^{p-1}\prod_{j=0}^{p-2}(h-(6j+6)a)}\dfrac{e^p}{d^p}}{-\left(\dfrac{e}{d^p}\right)-\left(\dfrac{1}{d^{p-1}}\right)} \\
&= \frac{\dfrac{a^p e^p}{(-e-d)^{p-1}\prod_{j=0}^{p-2}(h-(6j+6)a)}}{(-e-d)} \\
&= \frac{ha^p e^p}{(-e-d)^p\prod_{j=0}^{p-1}(h-(6j)a)}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
U_{6p-2} &= \frac{U_{6p-3}V_{6p-5}}{-U_{6p-3}+V_{6p-6}} \\
&= \frac{\dfrac{a^p e^p}{d^{p-1}\prod_{j=0}^{p-1}(h-(6j+3)a)}\dfrac{ca^{p-1}e^p}{(-e-d)^p\prod_{j=0}^{p-2}(h-(6j+4)a)}}{-\dfrac{a^p e^p}{d^{p-1}\prod_{j=0}^{p-1}(h-(6j+3)a)}+\dfrac{a^{p-1}e^p}{d^{p-1}\prod_{j=0}^{p-2}(h-(6j+3)a)}} \\
&= \frac{\dfrac{a^p e^p}{\prod_{j=0}^{p-1}(h-(6j+3)a)}\dfrac{c}{(-e-d)^p\prod_{j=0}^{p-2}(h-(6j+4)a)}}{\dfrac{-a}{\prod_{j=0}^{p-1}(h-(6j+3)a)}+\dfrac{1}{\prod_{j=0}^{p-2}(h-(6j+3)a)}}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{\dfrac{ca^pe^p}{(-e-d)^p\prod_{j=0}^{p-2}(h-(6j+4)a)}}{-a+\dfrac{\prod_{j=0}^{p-1}(h-(6j+3)a)}{\prod_{j=0}^{p-2}(h-(6j+3)a)}} = \frac{\dfrac{ca^pe^p}{(-e-d)^p\prod_{j=0}^{p-2}(h-(6j+4)a)}}{-a+(h-(6p-3)a)} \\
&= \frac{ca^pe^p}{(h-(6p-2)a)(-e-d)^p\prod_{j=0}^{p-2}(h-(6j+4)a)} \\
&= \frac{ca^pe^p}{(-e-d)^p\prod_{j=0}^{p-1}(h-(6j+4)a)},
\end{aligned}
$$

and

$$
\begin{aligned}
V_{6p-2} &= \frac{U_{6p-5}V_{6p-3}}{-V_{6p-3}-U_{6p-6}} \\
&= \frac{\dfrac{ga^pe^{p-1}}{d^{p-1}(h-a)\prod_{j=0}^{p-2}(h-(6j+7)a)}\dfrac{ha^pe^p}{(-e-d)^p\prod_{j=0}^{p-1}(h-(6j)a)}}{\dfrac{-ha^pe^p}{(-e-d)^p\prod_{j=0}^{p-1}(h-(6j)a)}-\dfrac{a^pe^{p-1}}{(-e-d)^{p-1}\prod_{j=0}^{p-2}(h-(6j+6)a)}} \\
&= \frac{\dfrac{g}{d^{p-1}(h-a)\prod_{j=0}^{p-2}(h-(6j+7)a)}\dfrac{ha^pe^p}{\prod_{j=0}^{p-1}(h-(6j)a)}}{\dfrac{-he}{\prod_{j=0}^{p-1}(h-(6j)a)}-\dfrac{(-e-d)}{\prod_{j=0}^{p-2}(h-(6j+6)a)}} \\
&= \frac{\dfrac{gha^pe^p}{d^{p-1}(h-a)\prod_{j=0}^{p-2}(h-(6j+7)a)}}{-e-(-e-d)} \\
&= \frac{ga^pe^p}{d^p\prod_{j=0}^{p-1}(h-(6j+1)a)}.
\end{aligned}
$$

The remaining relations are obtained in the same manner by repeated substitution and simplification. □

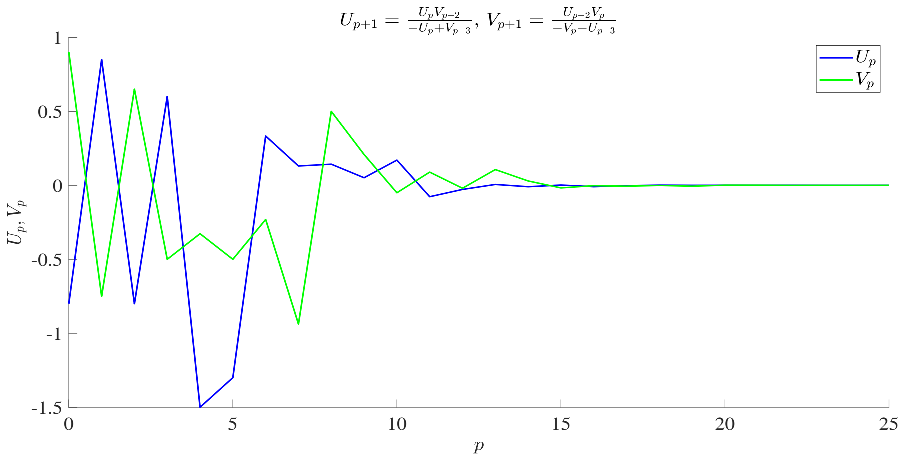

Figure 2: Behavior of the solution of System (3.1). It can be seen that the solution converges to $(0, 0)$ which confirm the fact that the equilibrium point $(0, 0)$ is locally asymptotically stable. The initial condition is given by $h = 0.9$, $g = -0.75$, $f = 0.65$, $e = -0.5$, $d = -0.8$, $c = 0.85$, $b = -0.8$, and $a = 0.6$.

4 Case III: $U_{p+1} = \dfrac{U_pV_{p-2}}{-U_p - V_{p-3}}$, $V_{p+1} = \dfrac{U_{p-2}V_p}{-V_p + U_{p-3}}$

In this section, we investigate the third special case of the system, characterized by the denominators $-U_p - V_{p-3}$ and $-V_p + U_{p-3}$. The system under consideration is

$$U_{p+1} = \frac{U_pV_{p-2}}{-U_p - V_{p-3}}, \quad V_{p+1} = \frac{U_{p-2}V_p}{-V_p + U_{p-3}}, \quad p = 0, 1, \ldots, \tag{4.1}$$

where the initial conditions $U_{-3} = d$, $U_{-2} = c$, $U_{-1} = b$, $U_0 = a$, $V_{-3} = h$, $V_{-2} = g$, $V_{-1} = f$, $V_0 = e$ are arbitrary non-zero real numbers, with the condition $U_0 \neq V_{-3}$. We derive explicit closed-form expressions for the solutions, which once again follow a periodic pattern of period six, revealing the influence of the sign change in the denominator on the algebraic structure.

Theorem 4.1. *Suppose that $\{U_p, V_p\}$ are solutions of System (4.1). Then*

$$U_{6p-3} = \frac{da^pe^p}{(-a-h)^p \prod_{j=0}^{p-1}(d-(6j)e)}, U_{6p-2} = \frac{ca^pe^p}{h^p \prod_{j=0}^{p-1}(d-(6j+1)e)},$$

$$U_{6p-1} = \frac{ba^pe^p}{(-a-h)^p \prod_{j=0}^{p-1}(d-(6j+2)e)}, U_{6p} = \frac{a^{p+1}e^p}{h^p \prod_{j=0}^{p-1}(d-(6j+3)e)},$$

$$U_{6p+1} = \frac{ga^{p+1}e^p}{(-a-h)^{p+1}\prod_{j=0}^{p-1}(d-(6j+4)e)}, U_{6p+2} = \frac{fa^{p+1}e^p}{h^{p+1}\prod_{j=0}^{p-1}(d-(6j+5)e)},$$

$$V_{6p-3} = \frac{a^pe^p}{h^{p-1}\prod_{j=0}^{p-1}(d-(6j+3)e)}, V_{6p-2} = \frac{ga^pe^p}{(-a-h)^p\prod_{j=0}^{p-1}(d-(6j+4)e)},$$

$$V_{6p-1} = \frac{fa^pe^p}{h^p\prod_{j=0}^{p-1}(d-(6j+5)e)}, V_{6p} = \frac{a^pe^{p+1}}{(-a-h)^p\prod_{j=0}^{p-1}(d-(6j+6)e)},$$

$$V_{6p+1} = \frac{ca^pe^{p+1}}{h^p(d-e)\prod_{j=0}^{p-1}(d-(6j+7)e)}, V_{6p+2} = \frac{ba^pe^{p+1}}{(-a-h)^p(d-2e)\prod_{j=0}^{p-1}(d-(6j+8)e)}.$$

Proof. We provide a prove by using mathematical induction. For $p = 0$, the formulas reduce to the given initial conditions:

$$U_{-3} = d, \quad U_{-2} = c, \quad U_{-1} = b, \quad U_0 = a, \quad V_{-3} = h, \quad V_{-2} = g, \quad V_{-1} = f, \quad V_0 = e,$$

which are true by definition. Thus, the base case holds.

Now assume that the formulas hold for some $p-1$ with $p > 0$. That is, for $p-1$ we have:

$$U_{6p-9} = \frac{da^{p-1}e^{p-1}}{(-a-h)^{p-1}\prod_{j=0}^{p-2}(d-(6j)e)}, \quad U_{6p-8} = \frac{ca^{p-1}e^{p-1}}{h^{p-1}\prod_{j=0}^{p-2}(d-(6j+1)e)},$$

$$U_{6p-7} = \frac{ba^{p-1}e^{p-1}}{(-a-h)^{p-1}\prod_{j=0}^{p-2}(d-(6j+2)e)}, \quad U_{6p-6} = \frac{a^pe^{p-1}}{h^{p-1}\prod_{j=0}^{p-2}(d-(6j+3)e)},$$

$$U_{6p-5} = \frac{ga^pe^{p-1}}{(-a-h)^p\prod_{j=0}^{p-2}(d-(6j+4)e)}, \quad U_{6p-4} = \frac{fa^pe^{p-1}}{h^p\prod_{j=0}^{p-2}(d-(6j+5)e)},$$

$$V_{6p-9} = \frac{a^{p-1}e^{p-1}}{h^{p-2}\prod_{j=0}^{p-2}(d-(6j+3)e)}, \quad V_{6p-8} = \frac{ga^{p-1}e^{p-1}}{(-a-h)^{p-1}\prod_{j=0}^{p-2}(d-(6j+4)e)},$$

$$V_{6p-7} = \frac{fa^{p-1}e^{p-1}}{h^{p-1}\prod_{j=0}^{p-2}(d-(6j+5)e)}, \quad V_{6p-6} = \frac{a^{p-1}e^p}{(-a-h)^{p-1}\prod_{j=0}^{p-2}(d-(6j+6)e)},$$

$$V_{6p-5} = \frac{ca^{p-1}e^p}{h^{p-1}(d-e)\prod_{j=0}^{p-2}(d-(6j+7)e)}, \quad V_{6p-4} = \frac{ba^{p-1}e^p}{(-a-h)^{p-1}(d-2e)\prod_{j=0}^{p-2}(d-(6j+8)e)}.$$

We now compute U_{6p-3} using System (4.1). From the recurrence, we have

$$U_{6p-3} = \frac{U_{6p-4}V_{6p-6}}{-U_{6p-4} - V_{6p-7}}.$$

First, compute the numerator:

$$U_{6p-4}V_{6p-6} = \frac{fa^p e^{p-1}}{h^p \prod_{j=0}^{p-2}(d-(6j+5)e)} \cdot \frac{a^{p-1}e^p}{(-a-h)^{p-1}\prod_{j=0}^{p-2}(d-(6j+6)e)}$$
$$= \frac{fa^{2p-1}e^{2p-1}}{h^p(-a-h)^{p-1}\left[\prod_{j=0}^{p-2}(d-(6j+5)e)\right]\left[\prod_{j=0}^{p-2}(d-(6j+6)e)\right]}.$$

Now compute the denominator:

$$-U_{6p-4} - V_{6p-7} = -\frac{fa^p e^{p-1}}{h^p \prod_{j=0}^{p-2}(d-(6j+5)e)} - \frac{fa^{p-1}e^{p-1}}{h^{p-1}\prod_{j=0}^{p-2}(d-(6j+5)e)}$$
$$= -\frac{fa^{p-1}e^{p-1}}{\prod_{j=0}^{p-2}(d-(6j+5)e)}\left(\frac{a}{h^p}+\frac{1}{h^{p-1}}\right)$$
$$= -\frac{fa^{p-1}e^{p-1}}{\prod_{j=0}^{p-2}(d-(6j+5)e)} \cdot \frac{a+h}{h^p}.$$

Taking the reciprocal of the denominator:

$$\frac{1}{-U_{6p-4}-V_{6p-7}} = -\frac{h^p \prod_{j=0}^{p-2}(d-(6j+5)e)}{fa^{p-1}e^{p-1}(a+h)}.$$

Multiplying the numerator by this reciprocal:

$$U_{6p-3} = \left(\frac{fa^{2p-1}e^{2p-1}}{h^p(-a-h)^{p-1}\left[\prod_{j=0}^{p-2}(d-(6j+5)e)\right]\left[\prod_{j=0}^{p-2}(d-(6j+6)e)\right]}\right)$$
$$\times\left(-\frac{h^p \prod_{j=0}^{p-2}(d-(6j+5)e)}{fa^{p-1}e^{p-1}(a+h)}\right).$$

Cancel the common factors f, h^p, and $\prod_{j=0}^{p-2}(d-(6j+5)e)$. Note that $-(a+h) = (-a-h)$. Simplifying the powers of a and e: $a^{2p-1}/a^{p-1} = a^p, \quad e^{2p-1}/e^{p-1} = e^p$. Thus,

$$U_{6p-3} = \frac{a^p e^p}{(-a-h)^p \prod_{j=0}^{p-2}(d-(6j+6)e)}.$$

Observing that $d\prod_{j=0}^{p-2}(d-(6j+6)e) = d\prod_{j=1}^{p-1}(d-(6j)e) = \prod_{j=0}^{p-1}(d-(6j)e)$, we obtain:

$$U_{6p-3} = \frac{da^p e^p}{(-a-h)^p \prod_{j=0}^{p-1}(d-(6j)e)}.$$

Next, we compute V_{6p-3} using the recurrence:

$$V_{6p-3} = \frac{U_{6p-6}V_{6p-4}}{-V_{6p-4}+U_{6p-7}}.$$

A similar simplification process, using the relation $d-(6j+2)e$ and reindexing, yields:

$$V_{6p-3} = \frac{a^p e^p}{h^{p-1} \prod_{j=0}^{p-1}(d-(6j+3)e)}.$$

The remaining formulas for U_{6p-2}, U_{6p-1}, U_{6p}, U_{6p+1}, U_{6p+2} and the corresponding V terms are verified by analogous computations, repeatedly applying the recurrence relations (4.1) and simplifying using the induction hypotheses. The algebraic manipulations follow the same pattern of canceling common factors and reindexing the products to match the desired forms. Thus, by the principle of mathematical induction, the formulas hold for all non-negative integers p. This completes the proof. □

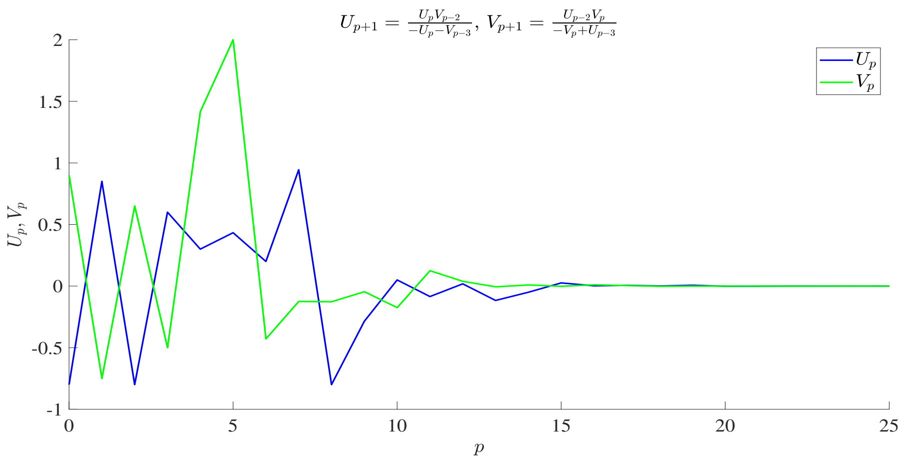

Figure 3: Behavior of the solution of System (4.1). It can be seen that the solution converges to $(0,0)$ which confirm the fact that the equilibrium point $(0,0)$ is locally asymptotically stable. The initial condition is given by $h = 0.9$, $g = -0.75$, $f = 0.65$, $e = -0.5$, $d = -0.8$, $c = 0.85$, $b = -0.8$, and $a = 0.6$.

5 Case IV: $U_{p+1} = \dfrac{U_p V_{p-2}}{-U_p - V_{p-3}}$, $V_{p+1} = \dfrac{U_{p-2} V_p}{-V_p - U_{p-3}}$

In this section, we analyze the fourth and final special case of the system, where both denominators exhibit negative signs: $-U_p - V_{p-3}$ and $-V_p - U_{p-3}$. The system is given by

$$U_{p+1} = \frac{U_p V_{p-2}}{-U_p - V_{p-3}}, \quad V_{p+1} = \frac{U_{p-2} V_p}{-V_p - U_{p-3}}, \quad p = 0, 1, \ldots, \tag{5.1}$$

with arbitrary non-zero real initial conditions $U_{-3} = d$, $U_{-2} = c$, $U_{-1} = b$, $U_0 = a$, $V_{-3} = h$, $V_{-2} = g$, $V_{-1} = f$, $V_0 = e$, subject to the conditions $U_{-3} \neq V_0$ and $U_0 \neq V_{-3}$. We obtain explicit closed-form expressions for the solutions, which once again exhibit a period-6 structure, completing the analysis of all four sign configurations.

Theorem 5.1. *Let* $\{U_p\}_{p=-3}^{\infty}$, $\{V_p\}_{p=-3}^{\infty}$ *are solutions of System (5.1). Then*

$$U_{6p-3} = \frac{a^p e^p}{d^{p-1}(-a-h)^p}, \qquad U_{6p-2} = \frac{ca^p e^p}{h^p(-e-d)^p},$$
$$U_{6p-1} = \frac{ba^p e^p}{d^p(-a-h)^p}, \qquad U_{6p} = \frac{a^{p+1} e^p}{h^p(-e-d)^p},$$
$$U_{6p+1} = \frac{ga^{p+1} e^p}{d^p(-a-h)^{p+1}}, \qquad U_{6p+2} = \frac{fa^{p+1} e^p}{h^{p+1}(-e-d)^p},$$
$$V_{6p-3} = \frac{a^p e^p}{h^{p-1}(-e-d)^p}, \qquad V_{6p-2} = \frac{ga^p e^p}{d^p(-a-h)^p},$$
$$V_{6p-1} = \frac{fa^p e^p}{h^p(-e-d)^p}, \qquad V_{6p} = \frac{a^p e^{p+1}}{d^p(-a-h)^p},$$
$$V_{6p+1} = \frac{ca^p e^{p+1}}{h^p(-e-d)^{p+1}}, \qquad V_{6p+2} = \frac{ba^p e^{p+1}}{d^{p+1}(-a-h)^p}.$$

Proof. For $p = 0$ the result holds. Now suppose that $p > 0$ and that our assumption holds for $p - 1$. That is;

$$U_{6p-9} = \frac{a^{p-1} e^{p-1}}{d^{p-2}(-a-h)^{p-1}}, \qquad U_{6p-8} = \frac{ca^{p-1} e^{p-1}}{h^{p-1}(-e-d)^{p-1}},$$
$$U_{6p-7} = \frac{ba^{p-1} e^{p-1}}{d^{p-1}(-a-h)^{p-1}}, \qquad U_{6p-6} = \frac{a^p e^{p-1}}{h^{p-1}(-e-d)^{p-1}},$$
$$U_{6p-5} = \frac{ga^p e^{p-1}}{d^{p-1}(-a-h)^p}, \qquad U_{6p-4} = \frac{fa^p e^{p-1}}{h^p(-e-d)^{p-1}},$$
$$V_{6p-9} = \frac{a^{p-1} e^{p-1}}{h^{p-2}(-e-d)^{p-1}}, \qquad V_{6p-8} = \frac{ga^{p-1} e^{p-1}}{d^{p-1}(-a-h)^{p-1}},$$
$$V_{6p-7} = \frac{fa^{p-1} e^{p-1}}{h^{p-1}(-e-d)^{p-1}}, \qquad V_{6p-6} = \frac{a^{p-1} e^p}{d^{p-1}(-a-h)^{p-1}},$$
$$V_{6p-5} = \frac{ca^{p-1} e^p}{h^{p-1}(-e-d)^p}, \qquad V_{6p-4} = \frac{ba^{p-1} e^p}{d^p(-a-h)^{p-1}}.$$

Now, it follows from System (5.1) that

$$U_{6p-3} = \frac{U_{6p-4}V_{6p-6}}{-U_{6p-4} - V_{6p-7}} = \frac{\frac{fa^pe^{p-1}}{h^p(-e-d)^{p-1}} \frac{a^{p-1}e^p}{d^{p-1}(-a-h)^{p-1}}}{-\frac{fa^pe^{p-1}}{h^p(-e-d)^{p-1}} - \frac{fa^{p-1}e^{p-1}}{h^{p-1}(-e-d)^{p-1}}} = \frac{a^pe^p}{d^{p-1}(-a-h)^p},$$

$$V_{6p-3} = \frac{U_{6p-6}V_{6p-4}}{-V_{6p-4} - U_{6p-7}} = \frac{\frac{a^pe^{p-1}}{h^{p-1}(-e-d)^{p-1}} \frac{ba^{p-1}e^p}{d^p(-a-h)^{p-1}}}{-\frac{ba^{p-1}e^p}{d^p(-a-h)^{p-1}} - \frac{ba^{p-1}e^{p-1}}{d^{p-1}(-a-h)^{p-1}}} = \frac{a^pe^p}{h^{p-1}(-e-d)^p}.$$

Similarly, the other cases can be proved. □

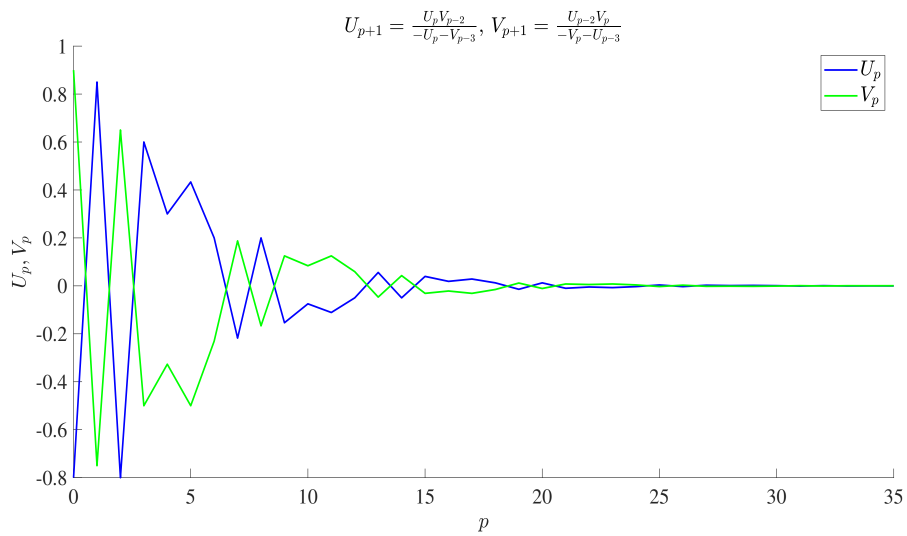

Figure 4: Behavior of the solution of System (5.1). It can be seen that the solution converges to $(0, 0)$ which confirm the fact that the equilibrium point $(0, 0)$ is locally asymptotically stable. The initial condition is given by $h = 0.9$, $g = -0.75$, $f = 0.65$, $e = -0.5$, $d = -0.8$, $c = 0.85$, $b = -0.8$, and $a = 0.6$.

The four cases share a common proof strategy: (i) assume the formulas hold for index $p - 1$; (ii) substitute into the recurrence for U_{6p-3} or V_{6p-3}; (iii) identify and cancel common product factors appearing in numerator and denominator; (iv) combine powers of a and e; (v) re-index the remaining product from $j = 0$ to $p - 2$ to $j = 1$ to $p - 1$; (vi) adjust by multiplying numerator and denominator by the missing initial factor (e.g., d or h) to obtain the final form. This systematic approach confirms that the period-6 pattern is robust across sign variations, even though the specific algebraic expressions differ. In Case I (denominators $-U_p + V_{p-3}$ and $-V_p + U_{p-3}$), the formulas involve products over $(h - (6j + k)a)$ and $(d - (6j + \ell)e)$. In Case II (denominators $-U_p + V_{p-3}$ and $-V_p - U_{p-3}$), the V expressions acquire factors of $(-e - d)^p$. In Case III (denominators $-U_p - V_{p-3}$ and $-V_p + U_{p-3}$), the U expressions involve $(-a - h)^p$. In Case IV (denominators $-U_p - V_{p-3}$ and $-V_p - U_{p-3}$), both U and V expressions incorporate alternating sign factors. Despite these differences, all four cases exhibit:

(a) period-6 indexing; (b) leading factors $a^p e^p$; (c) denominator products linear in the iteration counter j; and (d) convergence to $(0,0)$ under the simulated initial conditions. This uniformity suggests that the index structure $(p+1, p, p-2, p-3)$ is the primary determinant of the periodic pattern, while sign variations affect only the specific linear combinations appearing in the denominators. The numerical simulations shown in Figures 1–4 corroborate the analytical results, illustrating convergence to $(0,0)$ for representative initial conditions.

6 Discussion

The results presented in this paper provide explicit closed-form solutions for four variants of a fourth-order system of nonlinear rational difference equations. These findings contribute to the growing body of literature on exactly solvable discrete dynamical systems, and in this section we situate them within the context of prior work, highlighting points of convergence, divergence, and advancement relative to existing studies.

6.1 Connection to Previous Work

The investigation of rational difference equations has been a vibrant area of research over the past two decades. Early foundational work by Elsayed [16, 18] established explicit solution formulas for second-order rational systems, demonstrating that such systems could often be reduced to periodic or eventually periodic sequences. The present study extends this line of inquiry to a fourth-order asymmetric system, a significant increase in complexity that nevertheless yields similarly structured periodic behavior. While Elsayed [18] primarily considered systems of order two, the current work shows that the periodic patterns observed in lower-order systems can persist in higher-order settings, albeit with more intricate algebraic expressions involving product terms that accumulate with index growth.

The period-6 structure identified here resonates with findings by Din [4], who investigated fourth-order rational systems and reported periodic behavior under specific parameter conditions. However, whereas Din's analysis focused primarily on stability conditions and qualitative behavior without deriving explicit closed forms, the present paper provides complete algebraic expressions for all solution components. This represents a substantive advancement, as explicit formulas enable direct computation of any term without iteration and facilitate deeper analysis of asymptotic properties and parameter sensitivity.

Touafek and Elsayed [50] examined second-order rational systems and established connections between the algebraic form of the recurrence and the periodicity of solutions. The current work confirms that similar principles apply to higher-order systems: the specific index pattern $p+1$, p, $p-2$, $p-3$ in the recurrences directly gives rise to the modulo-6 periodicity observed in the solution formulas. This alignment suggests that the relationship between index structure and solution periodicity, first observed in lower-order systems, may generalize to a broader class of recurrences.

6.2 Similarities and Differences with Existing Literature

Several similarities emerge between our findings and those reported in the literature. First, as in the work of Halim [33] and Yazlik et al. [54], the solutions we obtain are expressed in terms of the initial conditions through product terms that reflect the cumulative effect of the recurrence over successive iterations. This product structure appears to be characteristic of rational systems with linear denominators. Second, the convergence to the equilibrium point $(0,0)$ observed in our numerical simulations aligns with the asymptotic behavior reported by Ahmed [1] and Khan and Qureshi [40] for similar rational systems under appropriate parameter constraints.

At the same time, the present study reveals important differences from prior investigations. Unlike the symmetric systems examined by Zhang et al. [59], where the equations for U and V are identical in form, our system features asymmetric coupling: U_{p+1} depends on V_{p-2} and V_{p-3}, while V_{p+1} depends on U_{p-2} and U_{p-3}. This asymmetry enriches the dynamical possibilities and makes the derivation of closed forms more challenging, yet the resulting formulas retain a coherent periodic structure that mirrors the index offsets. The systematic handling of four sign cases further extends the scope of previous work, most of which considered only one sign configuration per system.

Another distinctive contribution is the explicit treatment of negative signs in the denominators across all four combinations. While earlier studies such as those by El-Metwally and Elsayed [7, 8] considered rational equations with constant coefficients, the systematic variation of signs in the linear terms reveals how sign patterns affect the algebraic form of the solutions. This granular analysis provides insights that are not available from studies that treat the parameters in a purely symbolic manner without exploring sign-specific structure.

6.3 Advancements and Contributions

Relative to the existing literature, the present work makes several novel contributions. First, it provides the first complete closed-form solution set for a fourth-order asymmetric rational system of this type. Previous studies, including those by Liu et al. [42] and Gelisen and Kara [32], have addressed three-dimensional or second-order systems, but fourth-order systems with mixed delays have remained largely unexplored from an exact-solution perspective. Second, the explicit formulas derived here enable direct verification of periodicity and asymptotic properties without reliance on numerical iteration, offering a level of analytical tractability rarely achieved in higher-order nonlinear recurrences.

Third, the comparative analysis across four sign cases demonstrates that while the periodic structure remains invariant, the algebraic forms differ substantially, with terms such as $(-a-h)^p$, h^p, and product denominators shifting according to the sign pattern. This observation underscores the sensitivity of rational systems to sign variations, a nuance that is often overlooked in studies that assume positive parameters or treat signs implicitly.

Fourth, the rigorous induction proofs provided for each case establish the validity of the formulas with mathematical certainty, complementing the numerical simulations and providing a foundation for further analytical work such as stability analysis and bifurcation studies. The combination of explicit formulas, inductive verification, and numerical illustration represents a comprehensive approach that exceeds the scope of many prior investigations that rely solely on qualitative or numerical methods.

6.4 Implications and Future Directions

The findings of this study have implications for the broader field of discrete dynamical systems. The fact that a fourth-order nonlinear rational system admits closed-form solutions suggests that other higher-order systems with similar index structures may also be amenable to exact analysis. This opens avenues for systematically classifying solvable families of difference equations based on their index patterns and denominator structures, an area that remains largely uncharted.

The periodic structure observed here also raises questions about the relationship between recurrence order and fundamental period. In this system, the maximum index lag is three, and the solution period is six. Whether analogous relationships hold for systems with larger lags is a question worthy of further investigation. Such an inquiry could lead to a more general theory linking the index structure of rational recurrences to the periodicity of their solutions, extending the insights of Touafek and Elsayed [49] to higher orders.

The convergence to $(0, 0)$ observed numerically invites rigorous stability analysis. While equilibrium points of rational systems have been studied extensively (see, for example, Din [5], Elabbasy and Eleissawy [6]), the stability properties of the present system remain to be fully characterized. The explicit formulas derived here provide an ideal foundation for such analysis, allowing direct evaluation of asymptotic behavior without approximation.

Finally, the parameter restrictions imposed in this study—specifically, the choice of unit coefficients in the numerators and linear terms—represent a natural starting point. Generalizing to arbitrary real parameters, as in the original system (1.1), would be a significant extension. The patterns observed in the four special cases suggest that such generalizations may be possible, potentially yielding solution expressions involving more complex product structures or generalized number sequences, as seen in the work of Yazlik et al. [57] and Tollu et al. [47].

Conclusion

In summary, this paper advances the state of knowledge on rational difference equation systems by providing explicit closed-form solutions for a previously unsolved fourth-order asymmetric system, establishing connections to prior work, and demonstrating how systematic variation of sign patterns reveals underlying periodic structure. The results enrich the repertoire of exactly solvable nonlinear recurrences and provide a platform for further theoretical and applied investigations.

Disclaimer (Artificial Intelligence)

Author(s) hereby declare that NO generative AI technologies such as Large Language Models (ChatGPT, COPILOT, etc) and text-to-image generators have been used during writing or editing of this manuscript.

Competing Interests

Authors have declared that no competing interests exist.

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