Cyclotomic Orbit Structures and Actions of Subgroups of $GL(d, q)$

Abstract
Cyclotomic cosets are classical algebraic objects that arise naturally in modular arithmetic, finite field theory, and coding theory, especially in the study of polynomial factorization and cyclic codes. Traditionally, they are defined arithmetically through the formation of sets of the form $C_i = \{i \cdot p^j \bmod n \mid j \geq 0\}$, and their study has largely remained within that framework. However, this approach does not fully capture the group-action structure inherent in their formation. In particular, the interpretation of cyclotomic cosets as orbits under subgroup actions has not been formally developed in a way that extends to general linear groups over finite fields. This creates a theoretical gap between classical cyclotomy and the broader framework of orbit theory and linear group actions. This paper addresses this gap by introducing the concept of cyclotomic actions of subgroups of $GL(d, q)$. This idea extends to the natural action of subgroups of $GL(d, q)$ on the vector space $\mathbb{F}_q^d$, thereby defining a generalized cyclotomic orbit structure in higher dimensions. It is established that

$$\mathrm{Orb}_{\langle p \rangle}(i) = \{p^j \cdot i \mid j \geq 0\} = \{i \cdot p^j \bmod n \mid j \geq 0\}, \implies C_i = \mathrm{Orb}_{\langle p \rangle}(i),$$

showing that classical cyclotomic cosets are precisely orbit structures arising from cyclic subgroup actions. This formulation places classical cyclotomic cosets within a wider orbit-theoretic setting and provides a new perspective for studying cyclotomic behavior through linear actions, contributing to the further study of orbit structures, invariants, and their advanced applications in finite fields, and group theory.

Keywords: *Cyclotomic actions; Cyclotomic cosets; group actions; orbits, general linear groups,$GL(d, q)$; finite fields.*

1 Introduction

Cyclotomic cosets are fundamental algebraic structures that arise in finite field theory and coding theory, particularly in the factorization of polynomials of the form $x^n - 1$ over $\mathbb{F}_q$ and in the construction of cyclic and constacyclic codes [2, 5, 6, 10]. According to Ongili et al.[11], for a prime p with $\gcd(p, n) = 1$, a cyclotomic coset modulo n is defined as

$$C_i = \{i \cdot p^j \bmod n \mid j \geq 0\}, \quad i \in \mathbb{Z}_n.$$

These cosets form a partition of $\mathbb{Z}_n$, are finite, and exhibit periodicity determined by the smallest integer m such that $p^m \equiv 1 \pmod{n}$ [11]. Their structural properties have been widely applied in the enumeration and construction of cyclic codes over finite fields, particularly for specific prime fields $\mathrm{GF}(p)$ [11], where the number of cyclotomic cosets directly determines the number of irreducible factors of $x^n - 1$ and hence the number of cyclic codes [10, 11, 7, 8, 9, 13, 14]. From the perspective of group theory [12, 3, 4, 1], let G be a group acting on a set X. The orbit of an element $x \in X$ under the action of G is defined by

$$\mathrm{Orb}_G(x) = \{g \cdot x \mid g \in G\},$$

which partitions X into disjoint equivalence classes and satisfies the orbit-stabilizer relation

$$|\,\mathrm{Orb}_G(x)| = [G : \mathrm{Stab}_G(x)].$$

A comparison of these constructions reveals a strong structural similarity that both cyclotomic cosets and orbits are generated through repeated application of a transformation, both partition the underlying set, and both exhibit periodic behavior. In particular, the mapping $x \mapsto px \bmod n$ induces a cyclic subgroup $\langle p \rangle$, under which each cyclotomic coset can be expressed as

$$C_i = \{p^j \cdot i \mid j \geq 0\} = \mathrm{Orb}_{\langle p \rangle}(i),$$

establishing cyclotomic cosets as special cases of orbits under cyclic subgroup actions. This correspondence motivates the extension of cyclotomic structures beyond the classical one-dimensional setting. In particular, by considering the natural action of subgroups $H \leq GL(d, q)$ on the vector space $\mathbb{F}_q^d$, cyclotomic cosets admit a natural generalization to higher-dimensional orbit structures, forming the basis for cyclotomic actions in linear groups.

1.1 Definitions

This section presents the fundamental concepts and structures that form the basis of the results developed in this paper.

1. **Finite Field.** A finite field $\mathbb{F}_q$ is a field consisting of a finite number of elements, where $q = p^m$ for some prime p and positive integer m. The operations of addition and multiplication are defined such that $\mathbb{F}_q$ forms a commutative field with identity.

2. **General Linear Group.** The general linear group of degree d over $\mathbb{F}_q$, denoted by $GL(d, q)$, is the group of all invertible $d \times d$ matrices with entries in $\mathbb{F}_q$, under matrix multiplication:
$$GL(d, q) = \{A \in M_{d \times d}(\mathbb{F}_q) \mid \det(A) \neq 0\}.$$

3. **Group Action.** Let G be a group and X a non-empty set. A (left) group action of G on X is a function
$$G \times X \to X, \quad (g, x) \mapsto g \cdot x,$$
satisfying the following properties for all $g, h \in G$ and $x \in X$:
$$e \cdot x = x, \quad (gh) \cdot x = g \cdot (h \cdot x),$$
where e is the identity element of G.

4. **Orbit.** Let G act on a set X. The orbit of an element $x \in X$ under G is defined as
$$\mathrm{Orb}_G(x) = \{g \cdot x \mid g \in G\}.$$
The set X is partitioned into disjoint orbits under this action.

5. **Stabilizer.** Let G act on X. The stabilizer of an element $x \in X$ is defined as
$$\mathrm{Stab}_G(x) = \{g \in G \mid g \cdot x = x\}.$$
It is a subgroup of G.

6. **Orbit-Stabilizer Theorem.** If G is a finite group acting on a set X, then for any $x \in X$,
$$|\mathrm{Orb}_G(x)| = [G : \mathrm{Stab}_G(x)] = \frac{|G|}{|\mathrm{Stab}_G(x)|}.$$

7. **Cyclotomic Coset.** Let n be a positive integer and p a prime such that $\gcd(p, n) = 1$. The cyclotomic coset modulo n containing $i \in \mathbb{Z}_n$ is defined as
$$C_i = \{i \cdot p^j \bmod n \mid j \geq 0\}.$$
Each cyclotomic coset is finite and the collection of all such cosets forms a partition of $\mathbb{Z}_n$.

8. **Order of an Element Modulo n.** Let p be relatively prime to n. The order of p modulo n, denoted by $\mathrm{ord}_n(p)$, is the smallest positive integer m such that
$$p^m \equiv 1 \pmod{n}.$$
This value determines the size of cyclotomic cosets.

9. **Cyclic Subgroup.** Let G be a group and $g \in G$. The cyclic subgroup generated by g is defined as
$$\langle g \rangle = \{g^k \mid k \in \mathbb{Z}\}.$$

10. **Natural Action of $GL(d, q)$.** The group $GL(d, q)$ acts naturally on the vector space $\mathbb{F}_q^d$ by matrix multiplication:
$$A \cdot v = Av, \quad \forall A \in GL(d, q),\ v \in \mathbb{F}_q^d.$$

2 Results and Discussions

This section uses mathematical proofs, and examples to establish that classical cyclotomic cosets can be reinterpreted as orbits under cyclic subgroup actions, and that this extends naturally to subgroup actions in $GL(d, q)$.

2.1 Cyclotomic Cosets as Orbits under Cyclic Subgroup Actions

Lemma 2.1. *Let n be a positive integer and let p be a prime such that $\gcd(p, n) = 1$. Then multiplication by p modulo n defines a group action of the cyclic subgroup $\langle p \rangle$ on $\mathbb{Z}_n$.*

Proof. :
Since $\gcd(p, n) = 1$, the residue class of p modulo n is invertible in $\mathbb{Z}_n$. Hence p belongs to the multiplicative group of units modulo n, and so the set

$$\langle p \rangle = \{p^j \bmod n \mid j \geq 0\}$$

forms a cyclic subgroup under multiplication modulo n.

Define a map

$$\langle p \rangle \times \mathbb{Z}_n \to \mathbb{Z}_n$$

by

$$(p^j, i) \mapsto p^j \cdot i := ip^j \bmod n,$$

for all $i \in \mathbb{Z}_n$ and all $p^j \in \langle p \rangle$.

It remains to show that this map satisfies the axioms of a group action;

Identity property: Let $1 \in \langle p \rangle$ be the identity element. Then for every $i \in \mathbb{Z}_n$,

$$1 \cdot i = i \cdot 1 \bmod n = i \bmod n = i.$$

Thus the identity element fixes every element of $\mathbb{Z}_n$.

Compatibility property: Let $p^a, p^b \in \langle p \rangle$. Then for every $i \in \mathbb{Z}_n$,

$$p^a \cdot (p^b \cdot i) = p^a \cdot (ip^b \bmod n) = (ip^b)p^a \bmod n = ip^{a+b} \bmod n.$$

On the other hand,

$$(p^a p^b) \cdot i = p^{a+b} \cdot i = ip^{a+b} \bmod n.$$

Hence

$$p^a \cdot (p^b \cdot i) = (p^a p^b) \cdot i.$$

Therefore both axioms of a group action are satisfied. It follows that multiplication by p modulo n defines a group action of $\langle p \rangle$ on $\mathbb{Z}_n$. □

Theorem 2.1. *Let n be a positive integer and let p be a prime such that $\gcd(p, n) = 1$. Let $\mathbb{Z}_n = \{0, 1, 2, \ldots, n-1\}$, and let $\langle p \rangle$ denote the cyclic subgroup generated by p under multiplication modulo n. Then the cyclotomic coset $C_i = \{ip^j \bmod n \mid j \geq 0\}$ containing $i \in \mathbb{Z}_n$ is precisely the orbit of i under the action of $\langle p \rangle$ on $\mathbb{Z}_n$. That is,*

$$C_i = \mathrm{Orb}_{\langle p \rangle}(i).$$

Proof. :
Since $\gcd(p, n) = 1$, the residue class of p modulo n is a unit in $\mathbb{Z}_n$. Hence multiplication by p modulo n defines a permutation of $\mathbb{Z}_n$, and the set

$$\langle p \rangle = \{p^j \bmod n \mid j \in \mathbb{Z}_{\geq 0}\}$$

forms a cyclic subgroup of the multiplicative group of units modulo n.

Define an action of $\langle p \rangle$ on $\mathbb{Z}_n$ by

$$(p^j, i) \longmapsto p^j \cdot i := ip^j \bmod n,$$

for all $i \in \mathbb{Z}_n$ and all $p^j \in \langle p \rangle$.

It follows from Lemma 2.1 that $\langle p \rangle$ acts on $\mathbb{Z}_n$.

Now, by definition, the orbit of $i \in \mathbb{Z}_n$ under this action is

$$\mathrm{Orb}_{\langle p \rangle}(i) = \{g \cdot i \mid g \in \langle p \rangle\}.$$

Since every element $g \in \langle p \rangle$ has the form $p^j \bmod n$ for some $j \geq 0$, it follows that

$$\mathrm{Orb}_{\langle p \rangle}(i) = \{p^j \cdot i \mid j \geq 0\} = \{ip^j \bmod n \mid j \geq 0\}.$$

But the right-hand side is exactly the definition of the cyclotomic coset C_i. Therefore

$$C_i = \mathrm{Orb}_{\langle p \rangle}(i).$$

Hence every classical cyclotomic coset is an orbit under the cyclic subgroup generated by multiplication by p modulo n. □

2.2 Consequences of Orbit Interpretation and Derived Properties

Remark 2.2. *The reinterpretation is achieved by replacing the usual arithmetic description,*

$$C_i = \{ip^j \bmod n \mid j \geq 0\},$$

with a group-action description. In the arithmetic viewpoint, successive powers of p are repeatedly multiplied by i modulo n. In the orbit-theoretic viewpoint, the same process is regarded as the action of the cyclic subgroup $\langle p \rangle$ on the element i. Thus no new set is created; rather, the same object is viewed through a different structural framework.

Corollary 2.3. *The cyclotomic cosets modulo n form a partition of $\mathbb{Z}_n$.*

Proof. By the theorem 2.1,

$$C_i = \mathrm{Orb}_{\langle p \rangle}(i) \qquad \text{for every } i \in \mathbb{Z}_n.$$

Thus it is enough to show that the family of orbits of the action of $\langle p \rangle$ on $\mathbb{Z}_n$ forms a partition of $\mathbb{Z}_n$.

First, every element of $\mathbb{Z}_n$ belongs to some orbit. Indeed, let $i \in \mathbb{Z}_n$. Since the identity element $1 \in \langle p \rangle$, it follows that

$$i = 1 \cdot i \in \mathrm{Orb}_{\langle p \rangle}(i) = C_i.$$

Hence

$$\mathbb{Z}_n \subseteq \bigcup_{i \in \mathbb{Z}_n} C_i.$$

The reverse inclusion is immediate, since each $C_i \subseteq \mathbb{Z}_n$. Therefore

$$\mathbb{Z}_n = \bigcup_{i \in \mathbb{Z}_n} C_i.$$

It remains to show that distinct cyclotomic cosets are disjoint. Let $C_i \cap C_j \neq \varnothing$. Then there exists some $x \in \mathbb{Z}_n$ such that

$$x \in C_i \quad \text{and} \quad x \in C_j.$$

Since $C_i = \mathrm{Orb}_{\langle p \rangle}(i)$ and $C_j = \mathrm{Orb}_{\langle p \rangle}(j)$, there exist integers $a, b \geq 0$ such that

$$x \equiv ip^a \pmod{n} \qquad \text{and} \qquad x \equiv jp^b \pmod{n}.$$

Thus

$$ip^a \equiv jp^b \pmod{n}.$$

Because $\gcd(p, n) = 1$, the element p is invertible modulo n, so p^a and p^b are also invertible modulo n. Hence there exists an integer $c \geq 0$ such that

$$j \equiv ip^c \pmod{n},$$

which shows that $j \in C_i$. Now let $y \in C_j$. Then for some integer $t \geq 0$,

$$y \equiv jp^t \pmod{n}.$$

Substituting $j \equiv ip^c \pmod{n}$, one obtains

$$y \equiv (ip^c)p^t = ip^{c+t} \pmod{n},$$

so $y \in C_i$. Therefore $C_j \subseteq C_i$. By symmetry, $C_i \subseteq C_j$, and hence

$$C_i = C_j.$$

Thus any two cyclotomic cosets are either equal or disjoint, and their union is all of $\mathbb{Z}_n$. Therefore the cyclotomic cosets modulo n form a partition of $\mathbb{Z}_n$.
This is true from the fact that since cyclotomic cosets are now identified with orbits under the action of $\langle p \rangle$ on $\mathbb{Z}_n$ by Theorem 2.1, the result follows from the general fact that orbits of a group action partition the underlying set. □

Corollary 2.4. *Every orbit under the action of* $\langle p \rangle$, $\mathrm{Orb}_{\langle p \rangle}(i)$, *is finite*

Proof. The subgroup $\langle p \rangle$ is finite modulo n, because the multiplicative group of units modulo n is finite. Since $C_i = \mathrm{Orb}_{\langle p \rangle}(i)$, each cyclotomic coset is finite. Hence every orbit under the action of $\langle p \rangle$ is finite. □

Corollary 2.5. *If* $\mathrm{Stab}_{\langle p \rangle}(i)$ *denotes the stabilizer of* i*, then*

$$|C_i| = \left[\langle p \rangle : \mathrm{Stab}_{\langle p \rangle}(i)\right].$$

Proof. This is an immediate consequence of the Orbit-Stabilizer Theorem together with the identity

$$C_i = \mathrm{Orb}_{\langle p \rangle}(i).$$

Hence,

$$|C_i| = \left[\langle p \rangle : \mathrm{Stab}_{\langle p \rangle}(i)\right] = \frac{|\langle p \rangle|}{|\,\mathrm{Stab}_{\langle p \rangle}(i)|}$$

□

Example 2.6. *Consider $n = 7$ and $p = 2$. Since $\gcd(2, 7) = 1$, the action is valid. The subgroup generated by 2 modulo 7 is*

$$\langle 2 \rangle = \{1, 2, 4\},$$

because

$$2^1 \equiv 2 \pmod 7, \qquad 2^2 \equiv 4 \pmod 7, \qquad 2^3 \equiv 8 \equiv 1 \pmod 7.$$

Take $i = 1$. Then the cyclotomic coset containing 1 is

$$C_1 = \{1 \cdot 2^j \bmod 7 \mid j \geq 0\} = \{1, 2, 4\}.$$

Now compute the orbit of 1 under $\langle 2 \rangle$:

$$\mathrm{Orb}_{\langle 2 \rangle}(1) = \{1 \cdot 1, 1 \cdot 2, 1 \cdot 4\} \bmod 7 = \{1, 2, 4\}.$$

Hence

$$C_1 = \mathrm{Orb}_{\langle 2 \rangle}(1).$$

Similarly, for $i = 3$,

$$C_3 = \{3 \cdot 2^j \bmod 7 \mid j \geq 0\} = \{3, 6, 5\}.$$

Also,

$$\mathrm{Orb}_{\langle 2 \rangle}(3) = \{3 \cdot 1, 3 \cdot 2, 3 \cdot 4\} \bmod 7 = \{3, 6, 5\}.$$

Thus the reinterpretation holds concretely in this case.

Example 2.7. *Consider $n = 15$ and $p = 2$. Since $\gcd(2, 15) = 1$, multiplication by powers of 2 modulo 15 defines an action. The powers of 2 modulo 15 are*

$$2^1 \equiv 2, \quad 2^2 \equiv 4, \quad 2^3 \equiv 8, \quad 2^4 \equiv 16 \equiv 1 \pmod{15}.$$

Hence

$$\langle 2 \rangle = \{1, 2, 4, 8\}.$$

Take $i = 1$. Then

$$C_1 = \{1, 2, 4, 8\}.$$

The orbit is

$$\mathrm{Orb}_{\langle 2 \rangle}(1) = \{1 \cdot 1, 1 \cdot 2, 1 \cdot 4, 1 \cdot 8\} \bmod 15 = \{1, 2, 4, 8\}.$$

Take $i = 3$. Then

$$C_3 = \{3 \cdot 2^j \bmod 15 \mid j \geq 0\} = \{3, 6, 12, 9\}.$$

The corresponding orbit is

$$\mathrm{Orb}_{\langle 2 \rangle}(3) = \{3 \cdot 1, 3 \cdot 2, 3 \cdot 4, 3 \cdot 8\} \bmod 15 = \{3, 6, 12, 9\}.$$

Again,

$$C_3 = \mathrm{Orb}_{\langle 2 \rangle}(3).$$

2.3 Cyclotomic Orbit Structures under Subgroups of $GL(d, q)$

This section extends the idea from the one-dimensional setting $\mathbb{Z}_n$ to higher-dimensional vector spaces over finite fields.

Definition 2.8. *Let $H \leq GL(d, q)$ act naturally on $\mathbb{F}_q^d$ by*

$$h \cdot v = hv, \qquad h \in H,\ v \in \mathbb{F}_q^d.$$

For any $v \in \mathbb{F}_q^d$, define

$$C_H(v) = \{h \cdot v \mid h \in H\}.$$

So that $C_H(v)$ is the cyclotomic orbit of v under H.

Theorem 2.9. *Let $H \leq GL(d, q)$. Then the family of sets*

$$\{C_H(v) \mid v \in \mathbb{F}_q^d\}$$

forms a partition of $\mathbb{F}_q^d$. Moreover, each $C_H(v)$ is finite and

$$|C_H(v)| = [H : \mathrm{Stab}_H(v)].$$

Proof. :
Since $H \leq GL(d, q)$, each $h \in H$ is an invertible linear transformation on $\mathbb{F}_q^d$. Hence the rule

$$h \cdot v = hv$$

defines a group action of H on $\mathbb{F}_q^d$. Indeed, if I is the identity matrix, then

$$I \cdot v = Iv = v$$

for all $v \in \mathbb{F}_q^d$, and if $h_1, h_2 \in H$, then

$$h_1 \cdot (h_2 \cdot v) = h_1(h_2 v) = (h_1 h_2)v = (h_1 h_2) \cdot v.$$

Therefore H acts on $\mathbb{F}_q^d$.

By definition, $C_H(v)$ is the orbit of v under this action. Since orbits of a group action partition the underlying set, the family

$$\{C_H(v) \mid v \in \mathbb{F}_q^d\}$$

partitions $\mathbb{F}_q^d$.

Because $\mathbb{F}_q^d$ is finite and H is a finite subgroup of $GL(d, q)$, every orbit $C_H(v)$ is finite. Finally, by the Orbit-Stabilizer Theorem,

$$|C_H(v)| = |\,\mathrm{Orb}_H(v)| = [H : \mathrm{Stab}_H(v)].$$

□

Example 2.10. *Let $q = 3$, $d = 2$, and let*

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in GL(2,3).$$

Let $H = \langle A \rangle \leq GL(2,3)$. Consider the vector

$$v = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Then

$$Av = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = v.$$

Hence

$$C_H(v) = \{v\}.$$

Now take

$$w = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Then

$$Aw = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad A^2 w = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \qquad A^3 = I.$$

Thus

$$C_H(w) = \left\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right\}.$$

This shows how the higher-dimensional analogue of a cyclotomic coset arises as an orbit under a subgroup of $GL(d,q)$.

Remark 2.11. *The classical cyclotomic coset is therefore recovered when the acting subgroup is cyclic and the action is given by repeated multiplication in a one-dimensional setting. The extension to $GL(d,q)$ replaces scalar multiplication by linear transformation, while preserving the essential orbit structure.*

3 Conclusion and Recommendations

3.1 Conclusion

This study establishes a structural reinterpretation of classical cyclotomic cosets within the framework of group actions. It showns that for a prime p with $\gcd(p,n) = 1$, each cyclotomic coset

$$C_i = \{ip^j \bmod n \mid j \geq 0\}$$

is precisely the orbit of the element $i \in \mathbb{Z}_n$ under the action of the cyclic subgroup $\langle p \rangle$. This result demonstrates that cyclotomic cosets are not merely analogous to orbits, but are exactly orbit structures arising from a well-defined group action. Building on this observation, the concept of cyclotomic actions of subgroups of $GL(d,q)$ is introduced. Under this framework, classical cyclotomic cosets appear as special cases of orbit structures, while their higher-dimensional analogues arise through the natural action of subgroups $H \leq GL(d,q)$ on the vector space $\mathbb{F}_q^d$. This establishes a direct and rigorous connection between cyclotomy and orbit theory, thereby providing a unified perspective that links modular arithmetic, group actions, and linear algebra over finite fields. The results presented form a foundational basis for the study of cyclotomic orbit structures in linear groups.

3.2 Recommendations

The framework developed in this study opens several directions for further investigation;

(i) A detailed classification of cyclotomic orbits under various subgroups of $GL(d, q)$, including cyclic, diagonal, and more general matrix groups, would provide deeper insight into their structure and distribution.

(ii) The exploration of connections between cyclotomic orbit structures and cycle index theory, particularly in relation to permutation representations of linear groups.

(iii) Potential applications in coding theory, especially in the construction of new classes of linear and cyclic codes.

(iv) The extension of the present framework to other classes of groups, such as projective linear groups $PGL(d, q)$, which may yield richer algebraic structures and broader applications in both pure and applied mathematics.

Competing Interests
Author has declared no competing interest.

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