On Hyper-Dual Pell Lorentzian Vectors

Original Research Article

Abstract

In this paper, we introduce hyper-dual Pell Lorentzian vectors in the three-dimensional Lorentzian space and formalize the hyper-dual Lorentzian scalar product. A primary contribution of this research is the analytical derivation of the hyper-dual Lorentzian angle, which characterizes the relative orientation and displacement between hyper-dual Pell Lorentzian vectors. We classify several geometric configurations such as parallelism, intersection, and orthogonality for dual and hyper-dual components of the inner product. This study offers significant potential for applications within hyper-dual Lorentzian space.

Keywords: hyper-dual Pell number; hyper-dual Pell-Lucas number; hyper-dual Pell Lorentzian vector; hyper-dual Lorentzian angle.

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1 Introduction

The study of dual numbers, first introduced by Clifford and later developed through the pioneering work of Kotelnikov (1895) and Study (1901), has provided a robust mathematical framework for representing rigid body displacements and kinematics. The dual number system allows for the integration of translational and rotational components of motion into a single algebraic entity. Over the decades, this approach has found extensive applications in screw calculus, mechanism theory, and spatial kinematics (Fischer, 1998; Veldkamp, 1976). Beyond classical mechanics, dual numbers have also surfaced in modern theoretical physics, particularly in supersymmetric mechanics and gauge invariance theories (Frydryszak, 2005; Wald, 1987).

In recent years, the necessity for higher-order derivative calculations in computational fluid dynamics and multidisciplinary design optimization led to the generalization of dual numbers into hyper-dual numbers (Fike and Alonso, 2011a,b). Unlike standard dual numbers, hyper-dual numbers enable the computation of exact second derivatives without the risk of step-size errors or subtractive cancellation inherent in numerical differentiation. This algebraic advantage has been successfully applied to the equations of motion for rigid bodies Cohen and Shoham (2017) and has further extended the boundaries of hyper-dual split quaternions in the context of rigid body motion within Lorentzian spaces (Aslan et al., 2020).

Simultaneously, the exploration of special number sequences such as Fibonacci, Lucas, and Pell numbers has remained a focal point of interest in discrete mathematics and number theory (Horadam, 1961; Koshy, 2017; Verner and Hoggatt, 1969). The Pell and Pell-Lucas sequences are renowned for their deep connections to the silver ratio and their numerous identities in mathematical physics (Abd-Elhameed and Napoli, 2023; Bicknell, 1975; Horadam, 1971). Modern research has begun to bridge the gap between these algebraic sequences and hyper-complex number systems, leading to the definition of Pell quaternions and polynomials Koshy (2014).

Recent literature has highlighted the synergy between hyper-dual vectors and these classical sequences. Specifically, the integration of hyper-dual vectors with Pell and Pell-Lucas numbers has opened new avenues for defining angles and geometric properties in specialized vector spaces (Babadağ and Atasoy, 2024). However, the extension of these concepts into Lorentzian space remains a fertile ground for investigation.

In this paper, we aim to synthesize these two domains by investigating hyper-dual Lorentzian vectors through the lens of Pell and Pell-Lucas numbers. We define the fundamental algebraic structures of these vectors and explore the concept of hyper-dual angles within the Lorentzian framework. By utilizing the unique properties of Pell sequences, we provide new identities and geometric interpretations that enhance the understanding of hyper-dual motion and orientation in semi-Euclidean spaces.

Numerous authors have focused on introducing and investigating various generalizations and modifications of the classical Fibonacci and Lucas sequences. These studies often aim to generalize these sequences into broader classes, which are subsequently utilized to compute certain radicals in reduced forms and explore new algebraic identities.

According to Panwar (2021), the generalized k-Fibonacci sequence, denoted by $\{F_{k,n}\}$, is defined by the following second-order linear recurrence relation:

$$F_{k,n} = pkF_{k,n-1} + qF_{k,n-2}, \quad n \geq 2$$

subject to the initial conditions:

$$F_{k,0} = a \quad \text{and} \quad F_{k,1} = b$$

where p, q, k, a, and b are real or complex parameters. This general framework allows for the derivation of several well-known sequences as special cases

If we set the parameters as $a = 0, b = 1, k = 2$, and $p = q = 1$, the sequence $\{F_{k,n}\}$ reduces to the standard Pell sequence. If we set the parameters as $a = 2, b = 2, k = 2$, and $p = q = 1$, the Pell-Lucas sequence is obtained.

The Pell and Pell-Lucas sequences are two fundamental integer sequences that arise from the same second-order linear recurrence relation but are distinguished by their respective initial conditions. These sequences play a vital role in number theory, particularly in the approximation of the silver ratio and the solutions to Pell's equation

The sequence of Pell numbers, denoted by $\{P_n\}_{n\geq 0}$, is characterized by the following terms:

$$0, 1, 2, 5, 12, 29, 70, 169, 408, 985, 2378, \ldots$$

Mathematically, the Pell sequence is defined by the recurrence relation:

$$P_n = 2P_{n-1} + P_{n-2}, \quad \text{for } n \geq 2 \tag{1.1}$$

with the initial values:

$$P_0 = 0 \quad \text{and} \quad P_1 = 1.$$

Parallel to the Pell sequence, the Pell-Lucas (or often referred to as companion Pell) sequence, denoted by $\{Q_n\}_{n\geq 0}$, is composed of the following integers:

$$2, 2, 6, 14, 34, 82, 198, 478, 1154, 2786, 6726, \ldots$$

This sequence follows the same recursive rule as the Pell sequence:

$$Q_n = 2Q_{n-1} + Q_{n-2}, \quad \text{for } n \geq 2$$

However, it is defined by the unique initial conditions:

$$Q_0 = 2 \quad \text{and} \quad Q_1 = 2.$$

The recursive nature of the Pell and Pell-Lucas sequences can be further analyzed through their characteristic equation. The associated characteristic equation is given by:

$$x^2 - 2x - 1 = 0$$

The roots of this quadratic equation are found to be:

$$\alpha = 1 + \sqrt{2} \quad \text{and} \quad \beta = 1 - \sqrt{2}$$

where α is known as the *silver ratio*. These roots satisfy the following algebraic properties:

$$\alpha - \beta = 2\sqrt{2}, \quad \alpha\beta = -1, \quad \text{and} \quad \alpha + \beta = 2.$$

Utilizing these characteristic roots, the Binet formulas for the Pell (P_n) and Pell-Lucas (Q_n) sequences are expressed as follows (Abd-Elhameed and Napoli, 2023; Bicknell, 1975; Horadam, 1971; Koshy, 2014):

$$P_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \tag{1.2}$$

and

$$Q_n = \alpha^n + \beta^n. \tag{1.3}$$

These formulas allow for the extension of the sequences to negative indices, yielding the following parity relations:

$$P_{-n} = (-1)^{n+1} P_n \quad \text{and} \quad Q_{-n} = (-1)^n Q_n.$$

Furthermore, the structural symmetry of these sequences leads to several significant identities that are essential for simplifying expressions involving hyper-dual Lorentzian vectors:

$$P_{n+1}P_m + P_nP_{m-1} = P_{m+n} \tag{1.4}$$

$$Q_{n+1} - Q_{n-1} = 2Q_n \tag{1.5}$$

$$Q_{n+2} + Q_{n-2} = 6Q_n \tag{1.6}$$

These identities, particularly the relations between Q_n and its neighbors, provide a robust computational basis for the derivation of hyper-dual Lorentzian angles.

The set of dual numbers, $\mathbb{D}$, is an associative algebra introduced by Clifford, defined as:

$$\mathbb{D} = \{d = a + \varepsilon a^* \mid a, a^* \in \mathbb{R},\, \varepsilon^2 = 0,\, \varepsilon \neq 0\}$$

where ε represents the dual unit. Dual numbers are particularly effective in representing line geometry and dual-angle kinematics in Euclidean space.

Generalizing the dual number concept, the set of hyper-dual numbers, $\widetilde{\mathbb{D}}$, is constructed by introducing a secondary dual unit, ε^*. This set can be represented in four-component form as (Fike and Alonso, 2011a,b; Cohen and Shoham, 2017):

$$\widetilde{\mathbb{D}} = \{\gamma = \gamma_0 + \gamma_1\varepsilon + \gamma_2\varepsilon^* + \gamma_3\varepsilon\varepsilon^* \mid \gamma_0, \gamma_1, \gamma_2, \gamma_3 \in \mathbb{R}\}$$

Alternatively, $\widetilde{\mathbb{D}}$ can be viewed as an extension of the dual number system:

$$\widetilde{\mathbb{D}} = \{\gamma = d + \varepsilon^* d^* \mid d, d^* \in \mathbb{D}\}$$

The hyper-dual units ε, ε^*, and their product $\varepsilon\varepsilon^*$ satisfy the following fundamental properties:

$$(\varepsilon)^2 = (\varepsilon^*)^2 = 0, \quad \varepsilon \neq \varepsilon^* \neq 0, \quad \text{and} \quad \varepsilon\varepsilon^* = \varepsilon^*\varepsilon.$$

Due to these properties, $\widetilde{\mathbb{D}}$ forms a commutative and associative algebra over both the field of real numbers $\mathbb{R}$ and the ring of dual numbers $\mathbb{D}$. The inclusion of the $\varepsilon\varepsilon^*$ term is crucial, as it allows for the exact calculation of second-order derivatives and the representation of complex geometric rotations without numerical error.

Let $f(x)$ be a sufficiently differentiable real-valued function. Its extension to a hyper-dual argument $\gamma = x_0 + x_1\varepsilon + x_2\varepsilon^* + x_3\varepsilon\varepsilon^*$ is given by the following Taylor expansion:

$$f(\gamma) = f(x_0) + x_1 f'(x_0)\varepsilon + x_2 f'(x_0)\varepsilon^* + \big(x_3 f'(x_0) + x_1x_2 f''(x_0)\big)\,\varepsilon\varepsilon^*.$$

A significant application of this expansion is the definition of the hyper-dual square root. For a hyper-dual number γ with a positive real part ($\gamma_0 > 0$), the square root is defined as:

$$\sqrt{\gamma} = \sqrt{\gamma_0} + \frac{\gamma_1}{2\sqrt{\gamma_0}}\varepsilon + \frac{\gamma_2}{2\sqrt{\gamma_0}}\varepsilon^* + \left(\frac{\gamma_3}{2\sqrt{\gamma_0}} - \frac{\gamma_1\gamma_2}{4\gamma_0\sqrt{\gamma_0}}\right)\varepsilon\varepsilon^* \tag{1.7}$$

This formulation is essential for normalizing hyper-dual vectors and defining unit magnitudes in non-Euclidean spaces.

A hyper-dual Lorentzian vector $\vec{\gamma}$ is defined as a vector whose components are hyper-dual numbers, or equivalently, as a combination of four Lorentzian vectors:

$$\vec{\gamma} = \vec{\gamma}_0 + \vec{\gamma}_1\varepsilon + \vec{\gamma}_2\varepsilon^* + \vec{\gamma}_3\varepsilon\varepsilon^*.$$

This can also be expressed in terms of dual Lorentzian vectors $\vec{d}$ and $\vec{d^*}$ as $\vec{\gamma} = \vec{d} + \varepsilon^*\vec{d^*}$.

The hyper-dual Lorentzian scalar product (inner product) of two vectors $\vec{\gamma}$ and $\vec{\delta}$ is defined by distributing the Lorentzian metric over the hyper-dual units:

$$\begin{aligned}\langle\vec{\gamma},\vec{\delta}\rangle_{HDL} =& \langle\vec{\gamma}_0,\vec{\delta}_0\rangle_L + \Big(\langle\vec{\gamma}_0,\vec{\delta}_1\rangle_L + \langle\vec{\gamma}_1,\vec{\delta}_0\rangle_L\Big)\varepsilon + \Big(\langle\vec{\gamma}_0,\vec{\delta}_2\rangle_L + \langle\vec{\gamma}_2,\vec{\delta}_0\rangle_L\Big)\varepsilon^* \\ &+ \Big(\langle\vec{\gamma}_0,\vec{\delta}_3\rangle_L + \langle\vec{\gamma}_1,\vec{\delta}_2\rangle_L + \langle\vec{\gamma}_2,\vec{\delta}_1\rangle_L + \langle\vec{\gamma}_3,\vec{\delta}_0\rangle_L\Big)\varepsilon\varepsilon^* \end{aligned} \tag{1.8}$$

where $\langle\cdot,\cdot\rangle_L$ denotes the standard Lorentzian inner product in $\mathbb{R}^3_1$. For any vectors $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$, this is computed as:

$$\langle\mathbf{u},\mathbf{v}\rangle_L = -u_1v_1 + u_2v_2 + u_3v_3.$$

The Lorentzian norm of the real part is then given by $\|\vec{\gamma}_0\|_L = \sqrt{|\langle\vec{\gamma}_0,\vec{\gamma}_0\rangle_L|}$. Depending on the sign of the inner product, these hyper-dual vectors are classified as spacelike, timelike, or lightlike (null), mirroring the causal structure of Lorentzian space.

The concept of a hyper-dual angle provides a sophisticated method for describing the relative orientation and displacement between hyper-dual vectors in Lorentzian space. Let $\vec{\gamma}$ and $\vec{\delta}$ be unit hyper-dual Lorentzian vectors. Their Lorentzian scalar product is fundamentally related to the hyper-dual Lorentzian angle Φ through the following trigonometric representation (Aslan et al., 2020):

$$\langle\vec{\gamma},\vec{\delta}\rangle_{HDL} = \cos\Phi.$$

By decomposing the hyper-dual angle $\Phi = \phi + \varepsilon\psi$, where ϕ and ψ represent the dual Lorentzian and standard Lorentzian angles respectively, the expression can be expanded using the Taylor series for hyper-dual functions:

$$\begin{aligned}\langle\vec{\gamma},\vec{\delta}\rangle_{HDL} &= \cos(\phi + \varepsilon^*\phi^*) \\ &= \cos\phi - \varepsilon^*\phi^*\sin\phi \\ &= (\cos\psi - \varepsilon\psi^*\sin\psi) - \varepsilon^*\phi^*(\sin\psi + \varepsilon\psi^*\cos\psi).\end{aligned} \tag{1.9}$$

Here, ψ represents the real Lorentzian angle between the vectors primary parts. ψ^*, ϕ, and ϕ^* account for the dual and hyper-dual shifts which correspond to the relative distance and twist between the lines in 3D Lorentzian space.

The magnitude of a hyper-dual Lorentzian vector $\vec{\gamma}$ is derived from the hyper-dual inner product and the expansion of the square root function. For a vector with a non-null real part ($\|\vec{\gamma}_0\|_L \neq 0$), the hyper-dual Lorentzian norm is defined as follows:

$$\|\vec{\gamma}\|_{HDL} = \|\vec{\gamma}_0\|_L + \frac{\langle\vec{\gamma}_0,\vec{\gamma}_1\rangle_L}{\|\vec{\gamma}_0\|_L}\varepsilon + \frac{\langle\vec{\gamma}_0,\vec{\gamma}_2\rangle_L}{\|\vec{\gamma}_0\|_L}\varepsilon^* + \left(\frac{\langle\vec{\gamma}_0,\vec{\gamma}_3\rangle_L + \langle\vec{\gamma}_1,\vec{\gamma}_2\rangle_L}{\|\vec{\gamma}_0\|_L} - \frac{\langle\vec{\gamma}_0,\vec{\gamma}_1\rangle_L\langle\vec{\gamma}_0,\vec{\gamma}_2\rangle_L}{\|\vec{\gamma}_0\|_L^3}\right)\varepsilon\varepsilon^*.$$

A vector $\vec{\gamma}$ is classified as unit hyper-dual Lorentzian vector if $\|\vec{\gamma}\|_{HDL} = 1$. This condition implies that the real part is a unit Lorentzian vector ($\|\vec{\gamma}_0\|_L = 1$) and the following orthogonality constraints are satisfied:

$$\langle\vec{\gamma}_0,\vec{\gamma}_1\rangle_L = \langle\vec{\gamma}_0,\vec{\gamma}_2\rangle_L = \langle\vec{\gamma}_0,\vec{\gamma}_3\rangle_L + \langle\vec{\gamma}_1,\vec{\gamma}_2\rangle_L = 0.$$

These constraints ensure that the dual and hyper-dual components are orthogonal to the spine of the vector in the Minkowski sense, mirroring the properties of dual unit vectors in line geometry.

In the following sections, we will demonstrate how these vectors and angles exert specific geometric properties that facilitate the study of motion and orientation in semi-Euclidean environments.

2 Hyper-dual Pell and Pell-Lucas Numbers

In this section, we formally introduce the hyper-dual Pell and hyper-dual Pell-Lucas numbers. We then proceed to establish their fundamental recurrence relations, algebraic identities, and essential properties within the hyper-dual number system.

Definition 2.1 (Babadağ and Atasoy (2024)). *The n^{th} hyper-dual Pell number, denoted by HP_n, and the n^{th} hyper-dual Pell-Lucas number, denoted by HQ_n, are defined as the following algebraic combinations of classical Pell and Pell-Lucas sequences:*

$$\mathrm{HP}_n = \mathrm{P}_n + \mathrm{P}_{n+1}\varepsilon + \mathrm{P}_{n+2}\varepsilon^* + \mathrm{P}_{n+3}\varepsilon\varepsilon^* \tag{2.1}$$

and

$$\mathrm{HQ}_n = \mathrm{Q}_n + \mathrm{Q}_{n+1}\varepsilon + \mathrm{Q}_{n+2}\varepsilon^* + \mathrm{Q}_{n+3}\varepsilon\varepsilon^* \tag{2.2}$$

where P_n and Q_n represent the n^{th} terms of the classical Pell and Pell-Lucas sequences, respectively. The units $\varepsilon, \varepsilon^*$, and $\varepsilon\varepsilon^*$ satisfy the hyper-dual axioms ($\varepsilon^2 = (\varepsilon^*)^2 = 0$).

These definition embed the discrete dynamics of the Pell recurrence into the four-dimensional hyper-dual space. By construction, each component of a hyper-dual Pell number is a successive term of the underlying Pell sequence, which ensures that the recurrence $x_n = 2x_{n-1} + x_{n-2}$ is preserved throughout the hyper-dual structure.

To illustrate the progression of these sequences, the first few terms of the hyper-dual Pell and Pell-Lucas sequences are computed using their respective definitions:

$\mathrm{HP}_1 = 1 + 2\varepsilon + 5\varepsilon^* + 12\varepsilon\varepsilon^*$, $\mathrm{HP}_2 = 2 + 5\varepsilon + 12\varepsilon^* + 29\varepsilon\varepsilon^*$

$\mathrm{HQ}_1 = 2 + 6\varepsilon + 14\varepsilon^* + 34\varepsilon\varepsilon^*$, $\mathrm{HQ}_2 = 6 + 14\varepsilon + 34\varepsilon^* + 82\varepsilon\varepsilon^*$.

The closed-form expressions for these sequences, which facilitate the calculation of the n^{th} term without iteration, are provided by the following theorem.

Theorem 2.2 (Babadağ and Atasoy (2024)). *The Binet-like formulas for the hyper-dual Pell and hyper-dual Pell-Lucas numbers are given, respectively, by:*

$$\mathrm{HP}_n = \frac{\varphi^n \underline{\varphi} - \psi^n \underline{\psi}}{\varphi - \psi} \tag{2.3}$$

and

$$\mathrm{HQ}_n = \varphi^n \underline{\varphi} + \psi^n \underline{\psi} \tag{2.4}$$

where the hyper-dual coefficients $\underline{\varphi}$ and $\underline{\psi}$ are defined as:

$$\underline{\varphi} = 1 + \varphi\varepsilon + \varphi^2\varepsilon^* + \varphi^3\varepsilon\varepsilon^*, \quad \underline{\psi} = 1 + \psi\varepsilon + \psi^2\varepsilon^* + \psi^3\varepsilon\varepsilon^*. \tag{2.5}$$

Here, $\varphi = 1 + \sqrt{2}$ and $\psi = 1 - \sqrt{2}$ are the roots of the characteristic equation $x^2 - 2x - 1 = 0$.

Proof. We begin by substituting the classical Binet formula for Pell numbers, $P_n = \frac{\varphi^n - \psi^n}{\varphi - \psi}$, into the definition of the hyper-dual Pell sequence (2.1):

$$\begin{aligned}
\mathrm{HP}_n &= \mathrm{P}_n + \mathrm{P}_{n+1}\varepsilon + \mathrm{P}_{n+2}\varepsilon^* + \mathrm{P}_{n+3}\varepsilon\varepsilon^* \\
&= \frac{\varphi^n - \psi^n}{\varphi - \psi} + \frac{\varphi^{n+1} - \psi^{n+1}}{\varphi - \psi}\varepsilon + \frac{\varphi^{n+2} - \psi^{n+2}}{\varphi - \psi}\varepsilon^* + \frac{\varphi^{n+3} - \psi^{n+3}}{\varphi - \psi}\varepsilon\varepsilon^* \\
&= \frac{\varphi^n(1 + \varphi\varepsilon + \varphi^2\varepsilon^* + \varphi^3\varepsilon\varepsilon^*) - \psi^n(1 + \psi\varepsilon + \psi^2\varepsilon^* + \psi^3\varepsilon\varepsilon^*)}{\varphi - \psi} \\
&= \frac{\varphi^n \underline{\varphi} - \psi^n \underline{\psi}}{\varphi - \psi}.
\end{aligned}$$

Similarly, by applying the Binet formula for Pell-Lucas numbers, $Q_n = \varphi^n + \psi^n$, to the definition (2.2), we obtain:

$$\begin{aligned}
\mathrm{HQ}_n &= \mathrm{Q}_n + \mathrm{Q}_{n+1}\varepsilon + \mathrm{Q}_{n+2}\varepsilon^* + \mathrm{Q}_{n+3}\varepsilon\varepsilon^* \\
&= (\varphi^n + \psi^n) + \left(\varphi^{n+1} + \psi^{n+1}\right)\varepsilon + \left(\varphi^{n+2} + \psi^{n+2}\right)\varepsilon^* + \left(\varphi^{n+3} + \psi^{n+3}\right)\varepsilon\varepsilon^* \\
&= \varphi^n(1 + \varphi\varepsilon + \varphi^2\varepsilon^* + \varphi^3\varepsilon\varepsilon^*) + \psi^n(1 + \psi\varepsilon + \psi^2\varepsilon^* + \psi^3\varepsilon\varepsilon^*) \\
&= \varphi^n\underline{\varphi} + \psi^n\underline{\psi}.
\end{aligned}$$

This completes the proof. □

Theorem 2.3. *Let the n^{th} terms of the hyper-dual Pell sequence $\{\mathrm{HP}_n\}$. For $n \geq 0$ and $m \geq 1$, the following equations hold:*

$$2\mathrm{HP}_{n-1} + \mathrm{HP}_{n-2} = \mathrm{HP}_n \tag{2.6}$$

$$\mathrm{HP}_n - \mathrm{HP}_{n+1}\varepsilon - \mathrm{HP}_{n+2}\varepsilon^* - \mathrm{HP}_{n+3}\varepsilon\varepsilon^* = \mathrm{P}_n - 2\mathrm{P}_{n+3}\varepsilon\varepsilon^* \tag{2.7}$$

$$\mathrm{HP}_n\mathrm{HP}_m + \mathrm{HP}_{n+1}\mathrm{HP}_{m+1} = \mathrm{P}_{n+m+1} + 2\mathrm{P}_{n+m+2}\varepsilon + 2\mathrm{P}_{n+m+3}\varepsilon^* + 2\mathrm{P}_{n+m+4}\varepsilon\varepsilon^*. \tag{2.8}$$

Proof. (2.6): By using the n^{th} and $(n+1)^{th}$ terms in Equations (1.1) and (2.1), we have

$$\begin{aligned}
2\mathrm{HP}_{n-1} + \mathrm{HP}_{n-2} &= 2(\mathrm{P}_{n-1} + \mathrm{P}_n\varepsilon + \mathrm{P}_{n+1}\varepsilon^* + \mathrm{P}_{n+2}\varepsilon\varepsilon^*) + (\mathrm{P}_{n-2} + \mathrm{P}_{n-1}\varepsilon + \mathrm{P}_n\varepsilon^* + \mathrm{P}_{n+1}\varepsilon\varepsilon^*) \\
&= 2\mathrm{P}_{n-1} + \mathrm{P}_{n-2} + (2\mathrm{P}_n + \mathrm{P}_{n-1})\varepsilon + (2\mathrm{P}_{n+1} + \mathrm{P}_n)\varepsilon^* + (2\mathrm{P}_{n+2} + \mathrm{P}_{n+1})\varepsilon\varepsilon^* \\
&= \mathrm{P}_n + \mathrm{P}_{n+1}\varepsilon + \mathrm{P}_{n+2}\varepsilon^* + \mathrm{P}_{n+3}\varepsilon\varepsilon^* \\
&= \mathrm{HP}_n
\end{aligned}$$

(2.7): By Equation (2.1), we directly obtain

$$\begin{aligned}
\mathrm{HP}_n - \mathrm{HP}_{n+1}\varepsilon - \mathrm{HP}_{n+2}\varepsilon^* - \mathrm{HP}_{n+3}\varepsilon\varepsilon^* &= (\mathrm{P}_n + \mathrm{P}_{n+1}\varepsilon + \mathrm{P}_{n+2}\varepsilon^* + \mathrm{P}_{n+3}\varepsilon\varepsilon^*) \\
&- (\mathrm{P}_{n+1} + \mathrm{P}_{n+2}\varepsilon + \mathrm{P}_{n+3}\varepsilon^* + \mathrm{P}_{n+4}\varepsilon\varepsilon^*)\varepsilon \\
&- (\mathrm{P}_{n+2} + \mathrm{P}_{n+3}\varepsilon + \mathrm{P}_{n+4}\varepsilon^* + \mathrm{P}_{n+5}\varepsilon\varepsilon^*)\varepsilon^* \\
&- (\mathrm{P}_{n+3} + \mathrm{P}_{n+4}\varepsilon + \mathrm{P}_{n+5}\varepsilon^* + \mathrm{P}_{n+6}\varepsilon\varepsilon^*)\varepsilon\varepsilon^* \\
&= \mathrm{P}_n + \mathrm{P}_{n+1}\varepsilon + \mathrm{P}_{n+2}\varepsilon^* + \mathrm{P}_{n+3}\varepsilon\varepsilon^* \\
&- \mathrm{P}_{n+1}\varepsilon - \mathrm{P}_{n+2}\varepsilon^2 - \mathrm{P}_{n+3}\varepsilon\varepsilon^* - \mathrm{P}_{n+4}\varepsilon^2\varepsilon^* \\
&- \mathrm{P}_{n+2}\varepsilon^* - \mathrm{P}_{n+3}\varepsilon\varepsilon^* - \mathrm{P}_{n+4}\left(\varepsilon^*\right)^2 - \mathrm{P}_{n+5}\varepsilon\left(\varepsilon^*\right)^2 \\
&- \mathrm{P}_{n+3}\varepsilon\varepsilon^* - \mathrm{P}_{n+4}\left(\varepsilon\right)^2\varepsilon^* - \mathrm{P}_{n+5}\varepsilon\left(\varepsilon^*\right)^2 - \mathrm{P}_{n+6}\varepsilon^2\left(\varepsilon^*\right)^2 \\
&= \mathrm{P}_n - 2\mathrm{P}_{n+3}\varepsilon\varepsilon^*
\end{aligned}$$

(2.8): From (1.1) and (1.4), the following well-known identities:

$$\begin{aligned}
\mathrm{P}_n\mathrm{P}_m + \mathrm{P}_{n+1}\mathrm{P}_{m+1} &= \mathrm{P}_{n+m+1}, \\
\mathrm{P}_n\mathrm{P}_{m+1} + \mathrm{P}_{n+1}\mathrm{P}_{m+2} &= \mathrm{P}_{n+1}\mathrm{P}_m + \mathrm{P}_{n+2}\mathrm{P}_{m+1} = \mathrm{P}_{n+m+2}, \\
\mathrm{P}_n\mathrm{P}_{m+2} + \mathrm{P}_{n+1}\mathrm{P}_{m+3} &= \mathrm{P}_{n+2}\mathrm{P}_m + \mathrm{P}_{n+3}\mathrm{P}_{m+1} = \mathrm{P}_{n+m+3}, \\
\mathrm{P}_n\mathrm{P}_{m+3} + \mathrm{P}_{n+1}\mathrm{P}_{m+4} &= \mathrm{P}_{n+3}\mathrm{P}_m + \mathrm{P}_{n+4}\mathrm{P}_{m+1} = \mathrm{P}_{n+m+4}.
\end{aligned}$$

Using (2.1), we obtain

$$\begin{aligned}
\mathrm{HP}_n\mathrm{HP}_m + \mathrm{HP}_{n+1}\mathrm{HP}_{m+1} &= \mathrm{P}_n\mathrm{P}_m + \mathrm{P}_{n+1}\mathrm{P}_{m+1} \\
&+ \left(\mathrm{P}_n\mathrm{P}_{m+1} + \mathrm{P}_{n+1}\mathrm{P}_m + \mathrm{P}_{n+1}\mathrm{P}_{m+2} + \mathrm{P}_{n+2}\mathrm{P}_{m+1}\right)\varepsilon \\
&+ \left(\mathrm{P}_n\mathrm{P}_{m+2} + \mathrm{P}_{n+2}\mathrm{P}_m + \mathrm{P}_{n+1}\mathrm{P}_{m+3} + \mathrm{P}_{n+3}\mathrm{P}_{m+1}\right)\varepsilon^* \\
&+ \left(\mathrm{P}_n\mathrm{P}_{m+3} + \mathrm{P}_{n+3}\mathrm{P}_m + \mathrm{P}_{n+1}\mathrm{P}_{m+4} + \mathrm{P}_{n+4}\mathrm{P}_{m+1}\right)\varepsilon\varepsilon^*
\end{aligned}$$

Applying the Pell identities listed above, we obtain

$$\mathrm{HP}_n\mathrm{HP}_m + \mathrm{HP}_{n+1}\mathrm{HP}_{m+1} = \mathrm{P}_{n+m+1} + 2\mathrm{P}_{n+m+2}\varepsilon + 2\mathrm{P}_{n+m+3}\varepsilon^* + 2\mathrm{P}_{n+m+4}\varepsilon\varepsilon^*.$$

This completes the proof. □

3 Hyper-dual Pell Lorentzian Vectors and Angles

In this section, we extend the concept of Pell sequences to vector spaces over the hyper-dual Lorentzian domain. We define the hyper-dual Pell Lorentzian vectors and investigate their geometric properties, specifically focusing on their inner product identities.

Definition 3.1. *The n^{th} hyper-dual Pell Lorentzian vector, denoted by $\overrightarrow{\mathrm{HP}}_n$, is defined as:*

$$\overrightarrow{\mathrm{HP}}_n = \vec{\mathrm{P}}_n + \vec{\mathrm{P}}_{n+1}\varepsilon + \vec{\mathrm{P}}_{n+2}\varepsilon^* + \vec{\mathrm{P}}_{n+3}\varepsilon\varepsilon^*$$

where $\vec{\mathrm{P}}_n = (\mathrm{P}_n, \mathrm{P}_{n+1}, \mathrm{P}_{n+2})$ is the n^{th} Pell Lorentzian vector in Lorentzian 3-space. This vector can be partitioned into dual Pell Lorentzian components $\vec{\mathbb{P}}_n$ and $\vec{\mathbb{P}}^*_n$ as:

$$\begin{aligned}\overrightarrow{\mathrm{HP}}_n &= (\vec{\mathrm{P}}_n + \vec{\mathrm{P}}_{n+1}\varepsilon) + (\vec{\mathrm{P}}_{n+2} + \vec{\mathrm{P}}_{n+3}\varepsilon)\varepsilon^* \\ &= \vec{\mathbb{P}}_n + \varepsilon^*\vec{\mathbb{P}}^*_n.\end{aligned}$$

The following theorem establishes the closed-form expression for the Lorentzian scalar product of these vectors, which is fundamental for defining the hyper-dual Pell Lorentzian angle.

Theorem 3.2. *The hyper-dual Lorentzian (HDL) scalar product of the hyper-dual Pell Lorentzian vectors $\overrightarrow{\mathrm{HP}}_n$ and $\overrightarrow{\mathrm{HP}}_m$ is given by:*

$$\begin{aligned}\langle\overrightarrow{\mathrm{HP}}_n, \overrightarrow{\mathrm{HP}}_m\rangle_{HDL} = &\left(\frac{7\mathrm{Q}_{n+m+2}}{8} - \frac{\mathrm{Q}_{n+m}}{4} + \frac{(-1)^m\mathrm{Q}_{n-m}}{8}\right) \\ &+ \left(\frac{7\mathrm{Q}_{n+m+3}}{4} - \frac{\mathrm{Q}_{n+m+1}}{2} + \frac{(-1)^m\mathrm{Q}_{n-m}}{4}\right)\varepsilon \\ &+ \left(\frac{7\mathrm{Q}_{n+m+4}}{4} - \frac{\mathrm{Q}_{n+m+2}}{2} + \frac{3(-1)^m\mathrm{Q}_{n-m}}{4}\right)\varepsilon^* \\ &+ (\frac{7\mathrm{Q}_{n+m+5}}{2} - \mathrm{Q}_{n+m+3} + \frac{(-1)^m(-\mathrm{Q}_{n-m-3} + \mathrm{Q}_{n-m-1} - \mathrm{Q}_{n-m+1} + \mathrm{Q}_{n-m+3})}{8})\varepsilon\varepsilon^*.\end{aligned} \tag{3.1}$$

Proof. By applying the definition of the hyper-dual Lorentzian inner product from (1.8), we have:

$$\begin{aligned}\langle\overrightarrow{\mathrm{HP}}_n, \overrightarrow{\mathrm{HP}}_m\rangle_{HDL} = &\langle\vec{\mathrm{P}}_n, \vec{\mathrm{P}}_m\rangle_L + \Big(\langle\vec{\mathrm{P}}_n, \vec{\mathrm{P}}_{m+1}\rangle_L + \langle\vec{\mathrm{P}}_{n+1}, \vec{\mathrm{P}}_m\rangle_L\Big)\varepsilon \\ &+ \Big(\langle\vec{\mathrm{P}}_n, \vec{\mathrm{P}}_{m+2}\rangle_L + \langle\vec{\mathrm{P}}_{n+2}, \vec{\mathrm{P}}_m\rangle_L\Big)\varepsilon^* \\ &+ \Big(\langle\vec{\mathrm{P}}_n, \vec{\mathrm{P}}_{m+3}\rangle_L + \langle\vec{\mathrm{P}}_{n+1}, \vec{\mathrm{P}}_{m+2}\rangle_L + \langle\vec{\mathrm{P}}_{n+2}, \vec{\mathrm{P}}_{m+1}\rangle_L + \langle\vec{\mathrm{P}}_{n+3}, \vec{\mathrm{P}}_m\rangle_L\Big)\varepsilon\varepsilon^*.\end{aligned}$$

To evaluate the real part $\langle\vec{\mathrm{P}}_n, \vec{\mathrm{P}}_m\rangle_L$, we utilize the Lorentzian metric $(-1, 1, 1)$ and Binet's formula for Pell numbers:

$$\begin{aligned}
\langle \vec{\mathrm{P}}_n, \vec{\mathrm{P}}_m \rangle_L &= -\mathrm{P}_n\mathrm{P}_m + \mathrm{P}_{n+1}\mathrm{P}_{m+1} + \mathrm{P}_{n+2}\mathrm{P}_{m+2} \\
&= -\left(\frac{\varphi^n - \psi^n}{\varphi - \psi}\right)\left(\frac{\varphi^m - \psi^m}{\varphi - \psi}\right) + \left(\frac{\varphi^{n+1} - \psi^{n+1}}{\varphi - \psi}\right)\left(\frac{\varphi^{m+1} - \psi^{m+1}}{\varphi - \psi}\right) \\
&+ \left(\frac{\varphi^{n+2} - \psi^{n+2}}{\varphi - \psi}\right)\left(\frac{\varphi^{m+2} - \psi^{m+2}}{\varphi - \psi}\right) \\
&= -\frac{\varphi^{n+m} + \psi^{n+m}}{(\varphi - \psi)^2} + \frac{\varphi^{n+m+2} + \psi^{n+m+2}}{(\varphi - \psi)^2} \\
&+ \frac{\varphi^{n+m+4} + \psi^{n+m+4}}{(\varphi - \psi)^2} + \frac{(\varphi^n\psi^m + \varphi^m\psi^n)\,\varphi^{-m}\psi^{-m}}{(\varphi - \psi)^2\ \varphi^{-m}\psi^{-m}} \\
&= \frac{1}{8}\left(-\mathrm{Q}_{n+m} + \mathrm{Q}_{n+m+2} + \mathrm{Q}_{n+m+4} + (-1)^m \mathrm{Q}_{n-m}\right).
\end{aligned}$$

Applying the recurrence relation $\mathrm{Q}_{n+m+4} = 6\mathrm{Q}_{n+m+2} - \mathrm{Q}_{n+m}$, we obtain:

$$\langle \vec{\mathrm{P}}_n, \vec{\mathrm{P}}_m \rangle_L = \frac{7\mathrm{Q}_{n+m+2}}{8} - \frac{\mathrm{Q}_{n+m}}{4} + \frac{(-1)^m \mathrm{Q}_{n-m}}{8}.$$

Similarly,

$$\begin{aligned}
\langle \vec{\mathrm{P}}_n, \vec{\mathrm{P}}_{m+1} \rangle_L &= \frac{7\mathrm{Q}_{n+m+3}}{8} - \frac{\mathrm{Q}_{n+m+1}}{4} - \frac{(-1)^m \mathrm{Q}_{n-m-1}}{8}, \\
\langle \vec{\mathrm{P}}_{n+1}, \vec{\mathrm{P}}_m \rangle_L &= \frac{7\mathrm{Q}_{n+m+3}}{8} - \frac{\mathrm{Q}_{n+m+1}}{4} + \frac{(-1)^m \mathrm{Q}_{n-m+1}}{8}, \\
\langle \vec{\mathrm{P}}_n, \vec{\mathrm{P}}_{m+2} \rangle_L &= \frac{7\mathrm{Q}_{n+m+4}}{8} - \frac{\mathrm{Q}_{n+m+2}}{4} + \frac{(-1)^m \mathrm{Q}_{n-m-2}}{8}, \\
\langle \vec{\mathrm{P}}_{n+2}, \vec{\mathrm{P}}_m \rangle_L &= \frac{7\mathrm{Q}_{n+m+4}}{8} - \frac{\mathrm{Q}_{n+m+2}}{4} + \frac{(-1)^m \mathrm{Q}_{n-m+2}}{8}, \\
\langle \vec{\mathrm{P}}_n, \vec{\mathrm{P}}_{m+3} \rangle_L &= \frac{7\mathrm{Q}_{n+m+5}}{8} - \frac{\mathrm{Q}_{n+m+3}}{4} - \frac{(-1)^m \mathrm{Q}_{n-m-3}}{8}, \\
\langle \vec{\mathrm{P}}_{n+1}, \vec{\mathrm{P}}_{m+2} \rangle_L &= \frac{7\mathrm{Q}_{n+m+5}}{8} - \frac{\mathrm{Q}_{n+m+3}}{4} + \frac{(-1)^m \mathrm{Q}_{n-m-1}}{8}, \\
\langle \vec{\mathrm{P}}_{n+2}, \vec{\mathrm{P}}_{m+1} \rangle_L &= \frac{7\mathrm{Q}_{n+m+5}}{8} - \frac{\mathrm{Q}_{n+m+3}}{4} - \frac{(-1)^m \mathrm{Q}_{n-m+1}}{8}, \\
\langle \vec{\mathrm{P}}_{n+3}, \vec{\mathrm{P}}_m \rangle_L &= \frac{7\mathrm{Q}_{n+m+5}}{8} - \frac{\mathrm{Q}_{n+m+3}}{4} + \frac{(-1)^m \mathrm{Q}_{n-m+3}}{8}.
\end{aligned}$$

By performing calculations for the $\varepsilon, \varepsilon^*$, and $\varepsilon\varepsilon^*$ components and simplifying using (1.5) and (1.6), we arrive at the result in (3.1).

□

Example 3.1. *To demonstrate the application of Theorem 3.2, we consider the first two hyper-dual Pell Lorentzian vectors. Let* $\overrightarrow{\mathrm{HP}}_1$ *and* $\overrightarrow{\mathrm{HP}}_0$ *be defined by their components in* $\mathbb{R}^3_1$ *as follows:*

$$\begin{aligned}
\overrightarrow{\mathrm{HP}}_1 &= (1,2,5) + (2,5,12)\varepsilon + (5,12,29)\varepsilon^* + (12,29,70)\varepsilon\varepsilon^* \\
\overrightarrow{\mathrm{HP}}_0 &= (0,1,2) + (1,2,5)\varepsilon + (2,5,12)\varepsilon^* + (5,12,29)\varepsilon\varepsilon^*.
\end{aligned}$$

The Lorentzian scalar product of $\overrightarrow{\mathrm{HP}}_1$ *and* $\overrightarrow{\mathrm{HP}}_0$ *are*

$$\begin{aligned}
\langle \overrightarrow{\mathrm{HP}}_1, \overrightarrow{\mathrm{HP}}_0 \rangle_{HDL} &= \frac{7\mathrm{Q}_3 - \mathrm{Q}_1}{8} + \frac{7\mathrm{Q}_4 - 2\mathrm{Q}_2 + \mathrm{Q}_1}{4}\varepsilon + \frac{7\mathrm{Q}_5 - 2\mathrm{Q}_3 + 3\mathrm{Q}_1}{4}\varepsilon^* \\
&+ \frac{28\mathrm{Q}_6 - 8\mathrm{Q}_4 - \mathrm{Q}_{-2} + \mathrm{Q}_0 - \mathrm{Q}_2 + \mathrm{Q}_4}{8}\varepsilon\varepsilon^* \\
&= 12 + 57\varepsilon + 138\varepsilon^* + 662\varepsilon\varepsilon^*.
\end{aligned}$$

By the other hand

$$\langle \overrightarrow{\mathrm{HP}}_1, \overrightarrow{\mathrm{HP}}_0\rangle_{HDL} = \langle \vec{\mathrm{P}}_1, \vec{\mathrm{P}}_0\rangle_L + \Big(\langle \vec{\mathrm{P}}_1, \vec{\mathrm{P}}_1\rangle_L + \langle \vec{\mathrm{P}}_2, \vec{\mathrm{P}}_0\rangle_L\Big)\varepsilon + \Big(\langle \vec{\mathrm{P}}_1, \vec{\mathrm{P}}_2\rangle_L + \langle \vec{\mathrm{P}}_3, \vec{\mathrm{P}}_0\rangle_L\Big)\varepsilon^*$$
$$+ \Big(\langle \vec{\mathrm{P}}_1, \vec{\mathrm{P}}_3\rangle_L + \langle \vec{\mathrm{P}}_2, \vec{\mathrm{P}}_2\rangle_L + \langle \vec{\mathrm{P}}_3, \vec{\mathrm{P}}_1\rangle_L + \langle \vec{\mathrm{P}}_4, \vec{\mathrm{P}}_0\rangle_L\Big)\varepsilon\varepsilon^*$$
$$= 12 + (28+29)\varepsilon + (68+70)\varepsilon^* + (164+165+164+169)\varepsilon\varepsilon^*$$
$$= 12 + 57\varepsilon + 138\varepsilon^* + 662\varepsilon\varepsilon^*.$$

The outcomes align consistently with the anticipated expectations.

Corollary 3.3. *The norm of the n^{th} hyper-dual Pell Lorentzian vector $\overrightarrow{\mathrm{HP}}_n$ is given by the following expression in terms of Pell-Lucas numbers:*

$$\|\overrightarrow{\mathrm{HP}}_n\|^2_{HDL} = \langle \overrightarrow{\mathrm{HP}}_n, \overrightarrow{\mathrm{HP}}_n\rangle_{HDL}$$
$$= \frac{7\mathrm{Q}_{2n+2}}{8} - \frac{\mathrm{Q}_{2n}}{4} - \frac{(-1)^n}{4} + (\frac{7\mathrm{Q}_{2n+3}}{4} - \frac{\mathrm{Q}_{2n+1}}{2} - \frac{(-1)^n}{2})\varepsilon$$
$$+ (\frac{7\mathrm{Q}_{2n+4}}{4} - \frac{\mathrm{Q}_{2n+2}}{2} - \frac{3(-1)^n}{2})\varepsilon^* + (\frac{7\mathrm{Q}_{2n+5}}{2} - \mathrm{Q}_{2n+3} - 3(-1)^n)\varepsilon\varepsilon^*. \tag{3.2}$$

Proof. The proof follows directly from Theorem 3.2. By setting $m = n$ in the general inner product formula (3.1), the indices of the Pell-Lucas terms reduce to $n + m = 2n$ and $n - m = 0$. □

Example 3.2. *Let us calculate the hyper-dual Lorentzian norm of the hyper-dual Pell Lorentzian vector $\overrightarrow{\mathrm{HP}}_1$, defined as:*

$$\overrightarrow{\mathrm{HP}}_1 = (1,2,5) + (2,5,12)\varepsilon + (5,12,29)\varepsilon^* + (12,29,70)\varepsilon\varepsilon^*.$$

If we take $n = 1$ in (3.2) *and use* (1.7), *then we will get*

$$\|\overrightarrow{\mathrm{HP}}_1\|_{HDL} = \sqrt{\frac{7\mathrm{Q}_4}{8} - \frac{\mathrm{Q}_2}{4} + \frac{1}{4} + (\frac{7\mathrm{Q}_5}{4} - \frac{\mathrm{Q}_3}{2} + \frac{1}{2})\varepsilon + (\frac{7\mathrm{Q}_6}{4} - \frac{\mathrm{Q}_4}{2} + \frac{3}{2})\varepsilon^* + (\frac{7\mathrm{Q}_7}{2} - \mathrm{Q}_5 + 3)\varepsilon\varepsilon^*}$$
$$= \sqrt{\frac{57}{2} + 137\varepsilon + 331\varepsilon^* + 1594\varepsilon\varepsilon^*}$$
$$= \frac{57}{\sqrt{114}} + \frac{137}{\sqrt{114}}\varepsilon + \frac{331}{\sqrt{114}}\varepsilon^* + \frac{45511}{57\sqrt{114}}\varepsilon\varepsilon^*.$$

Such calculations are essential when investigating the unit vector properties and associated Pell Lorentzian angles in the hyper-dual Lorentzian framework. From (1.9) and (3.1), the following cases can be given for the scalar product of hyper-dual Pell vectors $\overrightarrow{\mathrm{HP}}_n$ and $\overrightarrow{\mathrm{HP}}_m$.

Case 3.3. *Suppose the real part of the hyper-dual Lorentzian angle satisfies $\cos\phi = 0$ with $\phi^* \neq 0$. This condition implies that the principal Lorentzian angle is $\psi = \frac{\pi}{2}$ and the dual part is $\psi^* = 0$. Consequently, the hyper-dual Lorentzian scalar product from* (3.1) *reduces to:*

$$\langle \overrightarrow{\mathrm{HP}}_n, \overrightarrow{\mathrm{HP}}_m\rangle_{HDL} = -\varepsilon^*\phi^* = (\frac{7\mathrm{Q}_{n+m+4}}{4} - \frac{\mathrm{Q}_{n+m+2}}{2} + \frac{3(-1)^m\mathrm{Q}_{n-m}}{4})\varepsilon^*$$
$$+(\frac{7\mathrm{Q}_{n+m+5}}{2} - \mathrm{Q}_{n+m+3} + \frac{(-1)^m(-\mathrm{Q}_{n-m-3} + \mathrm{Q}_{n-m-1} - \mathrm{Q}_{n-m+1} + \mathrm{Q}_{n-m+3})}{8})\varepsilon\varepsilon^*.$$

By isolating the coefficients of the hyper-dual units, we derive the dual part of the Lorentzian angle ϕ^* as a dual number:

$$\phi^* = -(\frac{7\mathrm{Q}_{n+m+4}}{4} - \frac{\mathrm{Q}_{n+m+2}}{2} + \frac{3(-1)^m\mathrm{Q}_{n-m}}{4})$$
$$-(\frac{7\mathrm{Q}_{n+m+5}}{2} - \mathrm{Q}_{n+m+3} + \frac{(-1)^m(-\mathrm{Q}_{n-m-3} + \mathrm{Q}_{n-m-1} - \mathrm{Q}_{n-m+1} + \mathrm{Q}_{n-m+3})}{8})\varepsilon.$$

Geometrically, this configuration indicates that the corresponding dual lines d_1 and d_2 in the hyper-dual Lorentzian space are orthogonal (perpendicular) yet non-intersecting.

Case 3.4. *Assume that $\phi^* = 0$ while $\phi \neq 0$. Under these constraints, the hyper-dual Lorentzian scalar product from* (1.9) *simplifies to the cosine of the dual Lorentzian angle. By substituting the hyper-dual Pell identity from* (3.1)*, we obtain:*

$$\langle \overrightarrow{\mathrm{HP}}_n, \overrightarrow{\mathrm{HP}}_m \rangle_{HDL} = \cos\phi = \left(\frac{7\mathrm{Q}_{n+m+2}}{8} - \frac{\mathrm{Q}_{n+m}}{4} + \frac{(-1)^m \mathrm{Q}_{n-m}}{8} \right) + \left(\frac{7\mathrm{Q}_{n+m+3}}{4} - \frac{\mathrm{Q}_{n+m+1}}{2} + \frac{(-1)^m \mathrm{Q}_{n-m}}{4} \right) \varepsilon.$$

Consequently, the dual Lorentzian angle ϕ can be explicitly expressed using the inverse cosine function over the dual numbers:

$$\phi = \arccos\left[\left(\frac{7\mathrm{Q}_{n+m+2}}{8} - \frac{\mathrm{Q}_{n+m}}{4} + \frac{(-1)^m \mathrm{Q}_{n-m}}{8} \right) + \left(\frac{7\mathrm{Q}_{n+m+3}}{4} - \frac{\mathrm{Q}_{n+m+1}}{2} + \frac{(-1)^m \mathrm{Q}_{n-m}}{4} \right) \varepsilon \right].$$

Geometrically, the condition $\phi^* = 0$ implies that the shortest distance between the two corresponding dual lines d_1 and d_2 is zero. Therefore, in the hyper-dual Lorentzian space, these lines are intersect each other at a single point.

Case 3.5. *Assume the conditions $\cos\phi = 0$ and $\phi^* = 0$ are simultaneously satisfied. Then, $\psi = \frac{\pi}{2}$ and ψ^*. Consequently, the hyper-dual Lorentzian scalar product becomes identically zero:*

$$\langle \overrightarrow{\mathrm{HP}}_n, \overrightarrow{\mathrm{HP}}_m \rangle_{HDL} = 0.$$

In this case, the corresponding dual lines d_1 and d_2 are not only orthogonal in the principal sense but also possess a zero shortest distance. Geometrically, this indicates that the dual lines intersect each other at a right angle within the hyper-dual Lorentzian space.

Case 3.6. *Assume that both the principal part and the dual part of the Lorentzian angle vanish, such that $\phi = 0$ and $\phi^* = 0$. Under these specific conditions, the hyper-dual Lorentzian scalar product yields unity:*

$$\langle \overrightarrow{\mathrm{HP}}_n, \overrightarrow{\mathrm{HP}}_m \rangle_{HDL} = 1.$$

In this case, the corresponding dual lines d_1 and d_2 are parallel within the hyper-dual Lorentzian space.

4 Conclusions

In this research, we have given the algebraic some properties of the hyper-dual Pell and Pell-Lucas sequences. Furthermore, we extended these concepts into the realm of non-Euclidean geometry by defining hyper-dual Pell Lorentzian vectors within the Lorentzian 3-space. A significant contribution of this work is the establishment of a closed-form identity for the hyper-dual Lorentzian scalar product, which facilitates the calculation of the hyper-dual Lorentzian angle between two hyper-dual Pell Lorentzian vectors.

Future studies could investigate other generalized hyper-dual Fibonacci-type Lorentzian vectors, expanding the scope of understanding within this mathematical framework. We anticipate that the results presented herein will serve as a catalyst for further exploration into the geometry of hyper-dual Lorentzian spaces, bridging the gap between number theory and modern geometric analysis.

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Author(s) hereby declare that NO generative AI technologies such as Large Language Models (ChatGPT, COPILOT, etc) and text-to-image generators have been used during writing or editing of manuscripts.

Competing Interests

Authors have declared that no competing interests exist.

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