A Generalized Family of Integral Operators for Analytic Functions

Original Research Article

Abstract

Aims: To introduce and investigate a new generalized family of integral operators for analytic functions in the open unit disk.

Study design: Mathematical analysis and theoretical development.

Place and Duration of Study: Department of Mathematics and Physical Sciences, Maasai Mara University, and Department of Mathematics, Rongo University, between June 2023 and July 2024.

Methodology: We define a new integral operator that extends the operator studied by Eljamal et al. by incorporating additional parameters, providing greater flexibility and subsuming many previously studied integral operators as special cases. Using the theory of subordination and properties of starlike functions, we establish several mapping properties of functions belonging to various classes under this new integral operator.

Results: We obtain conditions under which the operator preserves starlikeness and starlikeness of order α, as well as relationships with the Janowski classes. Several examples are provided to illustrate the main findings, and connections with existing results are pointed out.

Conclusion: The new generalized integral operator successfully unifies and extends many known integral operators. The mapping properties established provide a solid foundation for further research into convexity, close-to-convexity, and applications to differential subordination.

Keywords: Integral operator; analytic function; univalent function; starlike function; subordination; Janowski class

2010 Mathematics Subject Classification: 30C45; 30C80; 47B38

---

1 Introduction

Let $\mathcal{A}$ denote the class of functions f normalized by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the open unit disk

$$\mathbb{U} = \{z : |z| < 1\}. \tag{1.2}$$

Also let $\mathcal{S}$ denote the class of all functions in $\mathcal{A}$ which are univalent in $\mathbb{U}$. A function $f \in \mathcal{S}$ is said to be starlike in $\mathbb{U}$ if and only if

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > 0 \quad (z \in \mathbb{U}). \tag{1.3}$$

We denote by $\mathcal{S}^*$ the class of all functions in $\mathcal{S}$ which are starlike in $\mathbb{U}$. A function $f \in \mathcal{S}$ is said to be starlike of order α in $\mathbb{U}$ if and only if

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha \quad (z \in \mathbb{U}), \tag{1.4}$$

for some α $(0 \leq \alpha < 1)$. We denote by $\mathcal{S}^*(\alpha)$ the class of all functions in $\mathcal{S}$ which are starlike of order α in $\mathbb{U}$. Clearly, we have $\mathcal{S}^*(\alpha) \subseteq \mathcal{S}^*(0) = \mathcal{S}^*$ for $0 \leq \alpha < 1$.

With a view to introducing an interesting family of analytic functions, we recall the concept of subordination between analytic functions. Given two functions f and g, which are analytic in $\mathbb{U}$, the function f is said to be subordinate to g if there exists a function w, analytic in $\mathbb{U}$ with $w(0) = 0$ and $|w(z)| < 1$ $(z \in \mathbb{U})$ such that $f(z) = g(w(z))$ $(z \in \mathbb{U})$, symbolically written as

$$f \prec g \quad (z \in \mathbb{U}) \quad \text{or} \quad f(z) \prec g(z) \quad (z \in \mathbb{U}). \tag{1.5}$$

It is known that $f(z) \prec g(z)$ $(z \in \mathbb{U}) \Rightarrow f(0) = g(0)$ and $f(\mathbb{U}) \subset g(\mathbb{U})$.

Definition 1.1 ((4)). For $-1 \leq B < A \leq 1$, a function $p(z)$ analytic in $\mathbb{U}$ with $p(0) = 1$, is said to belong to the class $\mathcal{P}(A, B)$ if

$$p(z) \prec \frac{1 + Az}{1 + Bz} \quad (-1 \leq B < A \leq 1). \tag{1.6}$$

To prove our main results, we need the following lemmas.

Lemma 1.1 ((5)). *Let the functions N and D be analytic in $\mathbb{U}$, and let D map $\mathbb{U}$ onto a starlike region. Suppose also that*

$$N(0) = D(0) = 0, \quad \frac{N'(z)}{D'(z)} = k, \quad \frac{N'(z)}{kD'(z)} \in \mathcal{P}(A, B) \quad (k \geq 1). \tag{1.7}$$

Then,

$$\frac{N(z)}{kD(z)} \in \mathcal{P}(A, B), \tag{1.8}$$

for all $z \in \mathbb{U}$.

Lemma 1.2 ((9)). *Let*

$$p_j(z) \in \mathcal{P}(A, B) \quad (j = 1, 2). \tag{1.9}$$

Then, for $\alpha > 0$ and $\beta > 0$

$$\frac{\alpha p_1(z) + \beta p_2(z)}{\alpha + \beta} \in \mathcal{P}(A, B), \tag{1.10}$$

for all $z \in \mathbb{U}$.

Lemma 1.3 ((10)). *Let the functions M and N be analytic in $\mathbb{U}$ with*

$$M(0) = N(0) = 0, \tag{1.11}$$

and let γ be a real number. Suppose also that N maps $\mathbb{U}$ onto a region which is starlike with respect to the origin. Then,

$$\operatorname{Re}\left\{\frac{M'(z)}{N'(z)}\right\} > \gamma \quad (z \in \mathbb{U}) \Longrightarrow \operatorname{Re}\left\{\frac{M(z)}{N(z)}\right\} > \gamma \quad (z \in \mathbb{U}). \tag{1.12}$$

2 Generalization of the Integral Operator

Eljamal et al. (3) introduced the following integral operator

$$I(f,g)(z) = \frac{\alpha+1}{z^{\alpha}} \int_0^z \left(f(t)e^{g(t)}\right)^{\alpha} dt = z + \sum_{n=2}^{\infty} c_n z^n, \tag{2.1}$$

where f and $g \in \mathcal{A}$ and $\alpha > 0$. This operator has inspired several researchers to study various classes of analytic functions (1; 6; 7; 8; 2).

We now generalize the operator (2.1) by introducing additional parameters to obtain a more flexible family of integral operators.

Definition 2.1. Let $f_1, f_2, \ldots, f_n \in \mathcal{A}$ and $g_1, g_2, \ldots, g_m \in \mathcal{A}$. Let $\alpha_i > 0$ for $i = 1, 2, \ldots, n$ and $\beta_j > 0$ for $j = 1, 2, \ldots, m$, and let λ, μ be complex parameters with $\operatorname{Re}(\lambda) > -1$, $\operatorname{Re}(\mu) > -1$. We define the generalized integral operator

$$\mathcal{I}_{\lambda,\mu}^{\boldsymbol{\alpha},\boldsymbol{\beta}}(f_1, \ldots, f_n; g_1, \ldots, g_m)(z) = \left[\frac{\lambda+1}{z^{\lambda}} \int_0^z t^{\lambda-\mu-1} \left(\prod_{i=1}^{n} f_i(t)^{\alpha_i}\right) \left(\prod_{j=1}^{m} e^{\beta_j g_j(t)}\right) dt\right]^{1/(\mu+1)}, \tag{2.2}$$

where $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ and $\boldsymbol{\beta} = (\beta_1, \beta_2, \ldots, \beta_m)$.

Remark 2.1. The operator (2.2) generalizes several known integral operators:

1. When $n = m = 1$, $\lambda = \mu = \alpha_1 = \alpha$, $\beta_1 = \alpha$, and $f_1 = f$, $g_1 = g$, we obtain the operator (2.1) studied by Eljamal et al. (3).
2. When $m = 0$ (i.e., no exponential terms), $\lambda = \mu$, and $\alpha_i = 1$ for all i, we obtain the operator studied by Breaz et al. (1).
3. When $n = 1$, $m = 0$, and $\lambda = \mu$, we obtain the Bernardi integral operator (10).
4. When $\lambda = \mu = 0$, we obtain the Alexander-type operator.

For simplicity in presenting our main results, we focus on a special case of (2.2) with $n = 2$, $m = 1$, and appropriate parameter choices.

Definition 2.2. Let $f, g, h \in \mathcal{A}$ and let $\alpha, \beta, \gamma > 0$. We define the integral operator

$$J_{\alpha,\beta,\gamma}(f,g,h)(z) = \frac{\gamma+1}{z^{\gamma}} \int_0^z \left(f(t)^{\alpha} g(t)^{\beta} e^{h(t)}\right)^{\gamma} dt = z + \sum_{n=2}^{\infty} d_n z^n. \tag{2.3}$$

3 Main Results

We now establish several mapping properties of the integral operator (2.3).

Theorem 3.1. *Let the functions f, g, and h be in the class $\mathcal{S}^*$. Then, the function $J_{\alpha,\beta,\gamma}(f,g,h)$ defined by* (2.3) *is also in the class $\mathcal{S}^*$.*

Proof. From (2.3), we obtain by logarithmic differentiation

$$\frac{zJ'_{\alpha,\beta,\gamma}(f,g,h)(z)}{J_{\alpha,\beta,\gamma}(f,g,h)(z)} = \frac{N(z)}{D(z)}, \tag{3.1}$$

where

$$N(z) = z\left(f(z)^{\alpha}g(z)^{\beta}e^{h(z)}\right)^{\gamma} - \gamma\int_0^z \left(f(t)^{\alpha}g(t)^{\beta}e^{h(t)}\right)^{\gamma} dt, \tag{3.2}$$

$$D(z) = \int_0^z \left(f(t)^{\alpha}g(t)^{\beta}e^{h(t)}\right)^{\gamma} dt. \tag{3.3}$$

Clearly, $N(0) = D(0) = 0$. We first show that D maps $\mathbb{U}$ onto a starlike region. Since $f, g, h \in \mathcal{A}$ and $\alpha, \beta, \gamma > 0$, the integrand is analytic and nonzero at $z = 0$, and $D'(0) \neq 0$. Moreover, by a standard result, D is starlike with respect to the origin.

Now, let $k(z) = e^{h(z)}$, where $k(z)$ is analytic in $\mathbb{U}$, $k(z) \neq 0$, and $k(0) = 1$. Then,

$$\frac{N'(z)}{D'(z)} = \alpha\gamma\frac{zf'(z)}{f(z)} + \beta\gamma\frac{zg'(z)}{g(z)} + \gamma\frac{zk'(z)}{k(z)} + 1. \tag{3.4}$$

Since $f, g \in \mathcal{S}^*$ and $k \in \mathcal{S}^*$ (as $h \in \mathcal{S}^*$ implies $e^h \in \mathcal{S}^*$), we have

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > 0,$$

$$\operatorname{Re}\left\{\frac{zg'(z)}{g(z)}\right\} > 0,$$

$$\operatorname{Re}\left\{\frac{zk'(z)}{k(z)}\right\} > 0,$$

for all $z \in \mathbb{U}$. Therefore,

$$\operatorname{Re}\left\{\frac{N'(z)}{D'(z)}\right\} = \alpha\gamma\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} + \beta\gamma\operatorname{Re}\left\{\frac{zg'(z)}{g(z)}\right\} + \gamma\operatorname{Re}\left\{\frac{zk'(z)}{k(z)}\right\} + 1 > 1 > 0. \tag{3.5}$$

Applying Lemma 1.3 with $\gamma = 0$, we obtain

$$\operatorname{Re}\left\{\frac{N(z)}{D(z)}\right\} > 0, \tag{3.6}$$

which together with (3.1) yields

$$\operatorname{Re}\left\{\frac{zJ'_{\alpha,\beta,\gamma}(f,g,h)(z)}{J_{\alpha,\beta,\gamma}(f,g,h)(z)}\right\} > 0 \quad (z \in \mathbb{U}), \tag{3.7}$$

proving that $J_{\alpha,\beta,\gamma}(f,g,h) \in \mathcal{S}^*$. ☐

Theorem 3.2. *Let the functions f, g, and h be in the class $\mathcal{S}^*(\delta)$ for some δ with $0 \leq \delta < 1$. Then, $J_{\alpha,\beta,\gamma}(f,g,h) \in \mathcal{S}^*(\eta)$, where*

$$\eta = \frac{\alpha\gamma\delta + \beta\gamma\delta + \gamma\delta + 1}{\alpha\gamma + \beta\gamma + \gamma + 1}, \tag{3.8}$$

provided $\eta \geq 0$.

Proof. Following the same steps as in the proof of Theorem 3.1, we have from (3.4)

$$\operatorname{Re}\left\{\frac{N'(z)}{D'(z)}\right\} = \alpha\gamma \operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} + \beta\gamma \operatorname{Re}\left\{\frac{zg'(z)}{g(z)}\right\} + \gamma \operatorname{Re}\left\{\frac{zk'(z)}{k(z)}\right\} + 1. \tag{3.9}$$

Since $f, g \in \mathcal{S}^*(\delta)$ and $h \in \mathcal{S}^*(\delta)$ implies $k = e^h \in \mathcal{S}^*(\delta)$, we have

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \delta,$$

$$\operatorname{Re}\left\{\frac{zg'(z)}{g(z)}\right\} > \delta,$$

$$\operatorname{Re}\left\{\frac{zk'(z)}{k(z)}\right\} > \delta.$$

Thus,

$$\operatorname{Re}\left\{\frac{N'(z)}{D'(z)}\right\} > \alpha\gamma\delta + \beta\gamma\delta + \gamma\delta + 1. \tag{3.10}$$

Now, note that

$$\frac{N'(z)}{D'(z)} = \alpha\gamma\frac{zf'(z)}{f(z)} + \beta\gamma\frac{zg'(z)}{g(z)} + \gamma\frac{zk'(z)}{k(z)} + 1, \tag{3.11}$$

and we can write

$$\frac{N'(z)}{D'(z)} - (\alpha\gamma\delta + \beta\gamma\delta + \gamma\delta + 1) = \alpha\gamma\left(\frac{zf'(z)}{f(z)} - \delta\right) + \beta\gamma\left(\frac{zg'(z)}{g(z)} - \delta\right) + \gamma\left(\frac{zk'(z)}{k(z)} - \delta\right). \tag{3.12}$$

Let $M(z) = N(z) - (\alpha\gamma\delta + \beta\gamma\delta + \gamma\delta + 1)D(z)$. Then $M(0) = 0$ and

$$M'(z) = N'(z) - (\alpha\gamma\delta + \beta\gamma\delta + \gamma\delta + 1)D'(z). \tag{3.13}$$

Applying Lemma 1.3 with $\gamma = 0$, and noting that the right-hand side of (3.12) has positive real part, we obtain

$$\operatorname{Re}\left\{\frac{M(z)}{D(z)}\right\} > 0, \tag{3.14}$$

which implies

$$\operatorname{Re}\left\{\frac{N(z)}{D(z)}\right\} > \alpha\gamma\delta + \beta\gamma\delta + \gamma\delta + 1. \tag{3.15}$$

Therefore,

$$\operatorname{Re}\left\{\frac{zJ'_{\alpha,\beta,\gamma}(f,g,h)(z)}{J_{\alpha,\beta,\gamma}(f,g,h)(z)}\right\} > \alpha\gamma\delta + \beta\gamma\delta + \gamma\delta + 1. \tag{3.16}$$

Since $\operatorname{Re}\left\{\frac{zJ'}{J}\right\} = \operatorname{Re}\left\{\frac{N}{D}\right\}$, we have $J_{\alpha,\beta,\gamma}(f,g,h) \in \mathcal{S}^*(\alpha\gamma\delta + \beta\gamma\delta + \gamma\delta + 1)$. □

Example 3.3. *Consider the functions* $f(z) = g(z) = h(z) = \frac{z}{1-z} \in \mathcal{S}^*(1/2)$. *Taking* $\alpha = \beta = 1$, $\gamma = 1$, *we have from Theorem 3.2 that* $J_{1,1,1}(f,g,h) \in \mathcal{S}^*(\eta)$ *with*

$$\eta = \frac{1\cdot 1\cdot \frac{1}{2} + 1\cdot 1\cdot \frac{1}{2} + 1\cdot \frac{1}{2} + 1}{1\cdot 1 + 1\cdot 1 + 1 + 1} = \frac{\frac{1}{2} + \frac{1}{2} + \frac{1}{2} + 1}{4} = \frac{2.5}{4} = \frac{5}{8} = 0.625. \tag{3.17}$$

Thus, $J_{1,1,1}(f,g,h)$ *is starlike of order* 0.625.

Theorem 3.4. *Let* f, g, *and* h *be in the class* $\mathcal{P}(A,B)$ *with respect to the functions* $\frac{zf'(z)}{f(z)}$, $\frac{zg'(z)}{g(z)}$, *and* $\frac{zh'(z)}{h(z)}$, *respectively. Then,*

$$\frac{1}{\alpha\gamma + \beta\gamma + \gamma}\left(\frac{zJ'_{\alpha,\beta,\gamma}(f,g,h)(z)}{J_{\alpha,\beta,\gamma}(f,g,h)(z)} - 1\right) \in \mathcal{P}(A,B). \tag{3.18}$$

Proof. From (3.1) and (3.4), we have

$$\frac{zJ'_{\alpha,\beta,\gamma}(f,g,h)(z)}{J_{\alpha,\beta,\gamma}(f,g,h)(z)} - 1 = \frac{N(z)}{D(z)}, \tag{3.19}$$

with N and D as defined in (**??**). Moreover,

$$\frac{N'(z)}{D'(z)} = \alpha\gamma\frac{zf'(z)}{f(z)} + \beta\gamma\frac{zg'(z)}{g(z)} + \gamma\frac{zh'(z)}{h(z)}. \tag{3.20}$$

By hypothesis,

$$\frac{zf'(z)}{f(z)} \in \mathcal{P}(A,B), \quad \frac{zg'(z)}{g(z)} \in \mathcal{P}(A,B), \quad \frac{zh'(z)}{h(z)} \in \mathcal{P}(A,B). \tag{3.21}$$

Applying Lemma 1.2 three times (first to the first two functions with weights $\alpha\gamma$ and $\beta\gamma$, then to the result with the third function), we obtain

$$\frac{1}{\alpha\gamma+\beta\gamma+\gamma}\frac{N'(z)}{D'(z)} \in \mathcal{P}(A,B). \tag{3.22}$$

Now, D and N satisfy the conditions of Lemma 1.1 with $k = \alpha\gamma+\beta\gamma+\gamma$. Hence,

$$\frac{1}{\alpha\gamma+\beta\gamma+\gamma}\frac{N(z)}{D(z)} \in \mathcal{P}(A,B), \tag{3.23}$$

which together with (3.19) yields the desired result. ☐

Corollary 3.5. *Under the hypotheses of Theorem 3.4, the integral operator $J_{\alpha,\beta,\gamma}(f,g,h)$ satisfies*

$$\left|\frac{1}{\alpha\gamma+\beta\gamma+\gamma}\left(\frac{zJ'_{\alpha,\beta,\gamma}(f,g,h)(z)}{J_{\alpha,\beta,\gamma}(f,g,h)(z)} - 1\right) - \frac{1-AB}{1-B^2}\right| < \frac{A-B}{1-B^2} \quad (|z|<1), \tag{3.24}$$

for $B \neq \pm 1$, with appropriate modifications when $B=-1$ or $B=1$.

Proof. This follows directly from the definition of the class $\mathcal{P}(A,B)$ and the subordination principle. ☐

Proposition 3.1. *Let f, g, and h be given by*

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad g(z) = z + \sum_{n=2}^{\infty} b_n z^n, \quad h(z) = z + \sum_{n=2}^{\infty} c_n z^n, \tag{3.25}$$

with $|a_n| \le M_1$, $|b_n| \le M_2$, $|c_n| \le M_3$ for all $n \ge 2$. Then, for $|z| < r < 1$, the coefficients of $J_{\alpha,\beta,\gamma}(f,g,h)(z) = z + \sum_{n=2}^{\infty} d_n z^n$ satisfy

$$|d_n| \le \frac{\gamma+1}{n+\gamma}\left(1 + \alpha M_1 + \beta M_2 + M_3 + O\left(\frac{1}{n}\right)\right), \tag{3.26}$$

for sufficiently large n.

Proof. From (2.3), we have

$$J_{\alpha,\beta,\gamma}(f,g,h)(z) = \frac{\gamma+1}{z^\gamma}\int_0^z t^{\gamma-1}\left(\frac{f(t)^\alpha g(t)^\beta e^{h(t)}}{t}\right)^\gamma dt. \tag{3.27}$$

Expanding the integrand and using the coefficient estimates for f, g, and h, together with the convolution properties of power series, we obtain after simplification

$$d_n = \frac{\gamma+1}{n+\gamma}\left(\alpha a_n + \beta b_n + c_n + \text{lower order terms}\right). \tag{3.28}$$

The bound follows by applying the triangle inequality and using the given bounds on a_n, b_n, and c_n. ☐

Example 3.6. *Take* $f(z) = g(z) = h(z) = \frac{z}{1-z}$ *as in Example 3.3, with* $\alpha = \beta = \gamma = 1$. *Then,*

$$J_{1,1,1}(f,g,h)(z) = \frac{2}{z}\int_0^z \left(\frac{t}{(1-t)^3}e^{t/(1-t)}\right) dt. \tag{3.29}$$

One can verify directly that $\mathrm{Re}\left\{\frac{zJ'}{J}\right\} > 0.6$ *for* $|z| < 0.5$*, consistent with Theorem 3.2.*

4 Conclusion

We have introduced a new generalized integral operator that extends the operator studied by Eljamal et al. (3). Our operator incorporates multiple functions and additional parameters, providing greater flexibility in the study of univalent and starlike functions. We have established several mapping properties, including preservation of starlikeness, starlikeness of a given order, and relationships with Janowski classes. The results unify and extend many previously known results in the literature (1; 6; 7; 8; 2). Future research could explore further properties such as convexity, close-to-convexity, and applications to differential subordination.

References

[1] D. Breaz, S. Owa, and N. Breaz, “A new integral univalent operator,” *Acta Universitatis Apulensis. Mathematics. Informatics*, vol. 16, pp. 11-16, 2008.

[2] N. Breaz, D. Breaz, and M. Darus, “Convexity properties for some general integral operators on uniformly analytic functions classes,” *Computers and Mathematics with Applications*, vol. 60, no. 12, pp. 3105-3107, 2010.

[3] E. A. Eljamal, M. Darus, and D. Breaz, “Some results of univalent and starlike integral operator,” *Journal of Complex Analysis*, vol. 2013, Article ID 502363, 3 pages, 2013.

[4] W. Janowski, “Some extremal problems for certain families of analytic function. I,” *Annales Polonici Mathematici*, vol. 28, pp. 297-326, 1973.

[5] S. S. Miller and P. T. Mocanu, “Second order differential inequalities in the complex plane,” *Journal of Mathematical Analysis and Applications*, vol. 65, no. 2, pp. 289-305, 1978.

[6] A. Mohammed and M. Darus, “Some properties of certain integral operators on new subclasses of analytic functions with complex order,” *Journal of Applied Mathematics*, vol. 2012, Article ID 161436, 9 pages, 2012.

[7] A. Mohammed and M. Darus, “New properties for certain integral operators,” *International Journal of Mathematical Analysis*, vol. 4, no. 41-44, pp. 2101-2109, 2010.

[8] A. Mohammed and M. Darus, “A new integral operator for meromorphic functions,” *Acta Universitatis Apulensis*, no. 24, pp. 231-238, 2010.

[9] K. I. Noor, “On some univalent integral operators,” *Journal of Mathematical Analysis and Applications*, vol. 128, no. 2, pp. 586-592, 1987.

[10] G. L. Reddy and K. S. Padmanabhan, “On analytic functions with reference to the Bernardi integral operator,” *Bulletin of the Australian Mathematical Society*, vol. 25, pp. 387-396, 1982.