An Extended Compartmental Model for Divorce Dynamics with Reconciliation, Counseling, and Socioeconomic Factors

Abstract

This paper presents an extended mathematical framework for analyzing the dynamics of divorce within a structured population. The model is formulated as a seven–compartment deterministic system governed by the linear inhomogeneous differential equation

$$\dot{\mathbf{X}}(t) = \mathbf{b} + \mathbf{A}\mathbf{X}(t),$$

where $\mathbf{X}(t) = (S, M, E, F, D, R, C)^T \in \mathbb{R}^7_+$. In addition to the classical marital transition compartments, the model incorporates reconciled couples R and counseling participants C, together with behavioral control parameters representing financial stress ϕ, social support σ, and digital media influence ψ.

The total population satisfies $N(t) \leq \lambda/\mu$, ensuring boundedness and positive invariance of the feasible region. Analytical investigation yields explicit expressions for equilibrium states and a divorce reproduction threshold $\mathcal{R}_d$, which determines the persistence or elimination of divorce dynamics via the condition $\mathcal{R}_d = 1$.

Local stability is established through eigenvalue analysis of the system matrix. Structural stability is further supported using Lyapunov theory and sensitivity analysis. Numerical simulations using a fourth–order Runge–Kutta scheme confirm theoretical predictions and illustrate the stabilizing role of reconciliation and social support parameters.

Keywords: Divorce dynamics; Compartmental modeling; Nonlinear dynamical systems; Reproduction number $\mathcal{R}_d$; Stability analysis; Lyapunov methods; Sensitivity analysis.

1 Introduction

Man is a social animal by nature and, owing to inherent social tendencies and circumstances, lives within structured communities. Among the most fundamental social institutions is marriage, which formalizes the interpersonal relationship between a man and a woman under biological, social, cultural, and legal frameworks. The legal status of marriage varies across systems, being regarded as a sacrament under Hindu law and as a contract under Muslim law. As the institution of marriage evolved, so too did the concept of divorce. Today, divorce is legally defined as the dissolution of marriage, terminating the marital rights and obligations between spouses. Although marriage has existed across civilizations, the acceptance and regulation of divorce have historically depended on prevailing religious, cultural, and legal norms.

Historically, separation concepts appear even in Indian mythology, such as in the *Ramayana*, where King Rama deserted Sita. Ancient civilizations also recognized divorce; the Code of Hammurabi (c. 1754 BCE) permitted divorce under specific conditions, largely favoring male authority, while Roman law allowed relatively liberal divorce, including by mutual consent. During the medieval period, under strong religious influence—particularly Christianity—marriage was considered indissoluble, and divorce was largely prohibited. The modern concept of divorce gradually emerged in the eighteenth and nineteenth centuries alongside secular legal systems emphasizing individual rights and equality.

In contemporary society, divorce is recognized globally as a legal remedy for irretrievably broken marriages and is shaped by socioeconomic, psychological, cultural, and institutional factors. Mathematical modeling offers a systematic framework for analyzing the complex interactions among these factors and provides evidence-based insights for social planning and reform. Empirical evidence suggests rising divorce rates in regions such as the United States, England and Wales, Spain, and several African countries [1, 2, 3, 4], with sub-Saharan Africa experiencing notable increases among younger couples [5]. Recent mathematical studies have advanced the theoretical understanding of marital instability. For example, [6] introduced a nonlinear model incorporating a fear effect and performed bifurcation and stability analysis to identify threshold dynamics. Similarly, [7] developed an age-structured nonlinear model capturing extra-marital influences and long-distance effects, demonstrating how nonlinear interactions shape equilibrium and stability behavior.

The social and developmental consequences of divorce have been extensively documented in the literature, particularly regarding its psychological and behavioral impact on children [8, 9]. From a modeling perspective, divorce dynamics have also been studied using compartmental and epidemic-type frameworks, highlighting similarities between social transition processes and disease transmission mechanisms [11]. Advanced theoretical foundations for threshold analysis and reproduction numbers, originally developed in epidemiological modeling [12, 13, 14], provide essential mathematical tools for analyzing stability and persistence conditions in divorce systems. Additionally, recent fuzzy and hybrid modeling approaches have further expanded the analytical scope of marriage–divorce dynamics [10], offering alternative perspectives for handling uncertainty and social variability.

Divorce dynamics are influenced by both stabilizing and destabilizing forces. Among the most significant stabilizing mechanisms are reconciliation and counselling. Reconciliation operates as a multifactorial stabilizing force that restores marital balance through emotional repair, social support, legal mediation, and behavioral change. In dynamical terms, reconciliation increases the reconciliation rate γ, reduces the effective divorce rate δ, lowers relapse probability ρ, and may reduce the divorce reproduction threshold $\mathcal{R}_d$, potentially generating nonlinear resistance and bistability effects. Counselling functions as an adaptive intervention mechanism that interrupts the progression from marital distress to dissolution by providing psychological, legal, and social guidance. It reduces distress transition rates, enhances reconciliation flow, and may push the system below the critical threshold $\mathcal{R}_d < 1$, thereby altering long-term qualitative behavior and enabling stability switching or bifurcation phenomena. In contrast, socioeconomic conditions act as exogenous stress drivers. Financial instability, unemployment, debt burden, income inequality, housing insecurity, migration, and work–life imbalance amplify marital conflict and weaken resilience. Mathematically, these stressors increase conflict amplification α, raise relapse probability ρ, and elevate effective divorce transition rates, thereby promoting instability. Thus, the interplay between stabilizing controls (reconciliation and counselling) and destabilizing socioeconomic pressures determines the long-term structural behavior of the marital system.

The objective of the present study is to formulate and analyze an extended compartmental dynamical system that integrates demographic processes, reconciliation mechanisms, counselling interventions, and socioeconomic stress parameters within a unified mathematical framework. The proposed model aims to (i) establish existence and positivity of solutions, (ii) derive equilibrium states and threshold conditions, (iii) investigate local and structural stability properties, and (iv) examine the influence of behavioral control parameters through sensitivity and bifurcation analysis. The mathematical formulation of this integrated divorce–relationship system is presented in the following section.

2 Preliminaries

Lemma 2.1 (Lyapunov exponents). *Let $\mathbf{X}^*$ be an equilibrium of the linear system $\dot{\mathbf{X}} = \mathbf{b}+\mathbf{A}\mathbf{X}$. Then the Lyapunov exponents at $\mathbf{X}^*$ are given by the real parts of the eigenvalues of $\mathbf{A}$:*

$$\Lambda_i = \Re(\lambda_i), \quad i = 1, \dots, 7.$$

In particular, the largest Lyapunov exponent is

$$\Lambda_{\max} = \max_{1\le i\le 7} \Re(\lambda_i).$$

Theorem 2.2. *If $\Lambda_{\max} < 0$, then the equilibrium is exponentially stable. If $\Lambda_{\max} > 0$, then it is unstable.*

Theorem 2.3 (Global exponential stability). *Let $\mathbf{X}^*$ be an equilibrium of*

$$\dot{\mathbf{X}} = \mathbf{b} + \mathbf{A}\mathbf{X},$$

and assume that all eigenvalues of $\mathbf{A}$ satisfy

$$\Re(\lambda_i) < 0, \quad i = 1, \dots, 7.$$

Then $\mathbf{X}^*$ is globally exponentially stable in $\mathbb{R}^7$. Moreover, there exist constants $M > 0$ and $\omega > 0$ such that

$$\|\mathbf{X}(t) - \mathbf{X}^*\| \le Me^{-\omega t}\|\mathbf{X}(0) - \mathbf{X}^*\|, \qquad t \ge 0.$$

Proof. Let $\mathbf{Y} = \mathbf{X} - \mathbf{X}^*$. Then

$$\dot{\mathbf{Y}} = \mathbf{A}\mathbf{Y}.$$

Since all eigenvalues of $\mathbf{A}$ have strictly negative real parts, the matrix $\mathbf{A}$ is Hurwitz. Hence, by the Lyapunov stability theorem for linear systems, for any positive definite matrix $\mathbf{Q}$ there exists a unique positive definite matrix $\mathbf{P}$ solving the Lyapunov equation

$$\mathbf{A}^T\mathbf{P} + \mathbf{P}\mathbf{A} = -\mathbf{Q}.$$

Choose $\mathbf{Q} = \mathbf{I}$. Then there exists $\mathbf{P} > 0$ such that

$$\mathbf{A}^T\mathbf{P} + \mathbf{P}\mathbf{A} = -\mathbf{I}.$$

Consider the quadratic Lyapunov function

$$V(\mathbf{Y}) = \mathbf{Y}^T\mathbf{P}\mathbf{Y}.$$

Clearly, $V(\mathbf{Y}) > 0$ for $\mathbf{Y} \neq 0$ and $V(0) = 0$. Differentiating along trajectories gives

$$\dot{V} = \mathbf{Y}^T(\mathbf{A}^T\mathbf{P} + \mathbf{P}\mathbf{A})\mathbf{Y} = -\mathbf{Y}^T\mathbf{Y} = -\|\mathbf{Y}\|^2.$$

Hence there exists $\omega > 0$ such that

$$\dot{V} \leq -\omega\|\mathbf{Y}\|^2.$$

Standard Lyapunov arguments yield

$$\|\mathbf{Y}(t)\| \leq Me^{-\omega t}\|\mathbf{Y}(0)\|,$$

for some $M > 0$, which proves global exponential stability. □

Definition 2.1 (Exponential stability)**.** *An equilibrium $\mathbf{X}^*$ is said to be exponentially stable if there exist constants $M > 0$ and $\omega > 0$ such that*

$$\|\mathbf{X}(t) - \mathbf{X}^*\| \leq Me^{-\omega t}\|\mathbf{X}(0) - \mathbf{X}^*\|, \qquad t \geq 0.$$

If this holds for all initial conditions in $\mathbb{R}^7$, the equilibrium is globally exponentially stable.

Definition 2.2 (Lyapunov function)**.** *A continuously differentiable function $V : \mathbb{R}^7 \to \mathbb{R}$ is called a Lyapunov function for the equilibrium $\mathbf{X}^*$ if*

$$V(\mathbf{X}^*) = 0, \quad V(\mathbf{X}) > 0 \text{ for } \mathbf{X} \neq \mathbf{X}^*,$$

and its derivative along trajectories satisfies

$$\dot{V}(\mathbf{X}) \leq 0.$$

If $\dot{V} < 0$ for $\mathbf{X} \neq \mathbf{X}^*$, then the equilibrium is asymptotically stable.

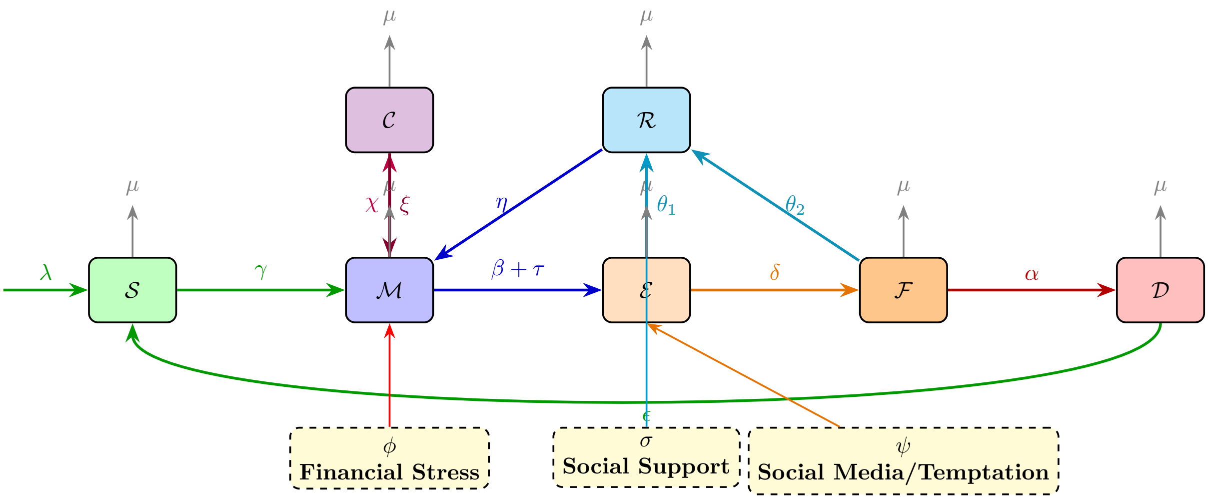

Figure 1: State transition diagram for the divorce–relationship dynamics model.

3 Mathematical Model

We formulate an extended compartmental model describing the evolution of divorce dynamics under the combined influence of demographic, social, and behavioral mechanisms. The population is stratified into seven interacting compartments $\mathcal{S}, \mathcal{M}, \mathcal{E}, \mathcal{F}, \mathcal{D}, \mathcal{R}$, and $\mathcal{C}$, with transitions governed by demographic and marital transition parameters. A complete description of each parameter, together with its sociological interpretation and admissible range, is summarized in Table 1. The demographic and marital transition rates $(\lambda, \mu, \gamma, \beta, \tau, \delta, \alpha, \epsilon, \theta_1, \theta_2, \eta, \chi, \xi)$ represent recruitment, natural removal, marriage, separation, escalation of conflict, divorce, reconciliation, and counseling effects. In addition, the model incorporates three behavioral influence indices (ϕ, σ, ψ), representing financial stress, social support, and social media influence, respectively. Since direct empirical estimates for these behavioral effects are not readily available, they are modeled as dimensionless control parameters scaled to the unit interval,

$$0 \leq \phi, \sigma, \psi \leq 1.$$

This normalization enables systematic sensitivity and bifurcation analysis and allows the model to explore a broad spectrum of social environments, thereby extending classical divorce frameworks to incorporate emerging behavioral drivers.

3.1 Governing Equations

Let $\mathcal{S}(t)$, $\mathcal{M}(t)$, $\mathcal{E}(t)$, $\mathcal{F}(t)$, $\mathcal{D}(t)$, $\mathcal{R}(t)$, and $\mathcal{C}(t)$ denote the numbers of susceptible individuals, married individuals, short-term breakups, long-term breakups, divorced individuals, reconciled individuals, and couples undergoing counseling at time $t \geq 0$, respectively. The total population size is

$$\mathcal{N}(t) = \mathcal{S}(t) + \mathcal{M}(t) + \mathcal{E}(t) + \mathcal{F}(t) + \mathcal{D}(t) + \mathcal{R}(t) + \mathcal{C}(t).$$

Define the state vector

$$\mathbf{X}(t) = \big(\mathcal{S}(t), \mathcal{M}(t), \mathcal{E}(t), \mathcal{F}(t), \mathcal{D}(t), \mathcal{R}(t), \mathcal{C}(t)\big)^T \in \mathbb{R}^7_+,$$

where $\mathbb{R}^7_+$ denotes the nonnegative orthant. The model parameters satisfy

$$(\lambda, \mu, \gamma, \beta, \tau, \delta, \alpha, \epsilon, \theta_1, \theta_2, \eta, \chi, \xi) \in \mathbb{R}^{13}_+.$$

Under these assumptions, the divorce dynamics are governed by the linear inhomogeneous system

$$\frac{d\mathbf{X}}{dt} = \mathbf{b} + \mathbf{A}\mathbf{X}, \tag{1}$$

where the recruitment vector is

$$\mathbf{b} = (\lambda, 0, 0, 0, 0, 0, 0)^T,$$

and the transition matrix $\mathbf{A} \in \mathbb{R}^{7\times 7}$ is

$$\mathbf{A} = \begin{pmatrix} -(\gamma+\mu) & 0 & 0 & 0 & \epsilon & 0 & 0 \\ \gamma & -(\beta+\tau+\mu) & 0 & 0 & 0 & \eta & 0 \\ 0 & (\beta+\tau) & -(\delta+\mu+\theta_1) & 0 & 0 & 0 & 0 \\ 0 & 0 & \delta & -(\alpha+\mu+\theta_2) & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha & -(\epsilon+\mu) & 0 & 0 \\ 0 & 0 & \theta_1 & \theta_2 & 0 & -(\mu+\eta) & 0 \\ 0 & \chi & 0 & 0 & 0 & 0 & -(\xi+\mu) \end{pmatrix}.$$

Lemma 3.1 (Well-posedness and invariant region)**.** *Consider system* (1) *with nonnegative parameters and* $\mu > 0$*. For any initial condition*

$$\mathbf{X}(0) = \big(\mathcal{S}(0), \mathcal{M}(0), \mathcal{E}(0), \mathcal{F}(0), \mathcal{D}(0), \mathcal{R}(0), \mathcal{C}(0)\big) \in \mathbb{R}^7_+,$$

the following properties hold:

[(i)]

1. ***Existence and uniqueness.*** *System* (1) *admits a unique global solution* $\mathbf{X}(t) \in \mathbb{R}^7$ *for all* $t \geq 0$*. Since the right-hand side* $\mathbf{b} + \mathbf{A}\mathbf{X}$ *is affine and therefore globally Lipschitz continuous on* $\mathbb{R}^7$*, the Picard–Lindelöf theorem guarantees existence and uniqueness of solutions.*

2. ***Positivity.*** *The nonnegative orthant* $\mathbb{R}^7_+$ *is positively invariant. Because* $\mathbf{A}$ *is a Metzler matrix (all off-diagonal entries are nonnegative) and* $\mathbf{b} \geq 0$*, the system is cooperative. Hence, by the comparison principle for linear cooperative systems,*

$$\mathbf{X}(0) \geq 0 \quad \Longrightarrow \quad \mathbf{X}(t) \geq 0 \quad \textit{for all } t \geq 0.$$

3. ***Boundedness and invariant region.*** *Let*

$$\mathcal{N}(t) = \mathcal{S}(t) + \mathcal{M}(t) + \mathcal{E}(t) + \mathcal{F}(t) + \mathcal{D}(t) + \mathcal{R}(t) + \mathcal{C}(t).$$

Summing the equations of system (1) *yields*

$$\frac{d\mathcal{N}}{dt} = \lambda - \mu\mathcal{N}(t).$$

The explicit solution is

$$\mathcal{N}(t) = \frac{\lambda}{\mu} + \left(\mathcal{N}(0) - \frac{\lambda}{\mu}\right) e^{-\mu t},$$

which implies

$$0 \le \mathcal{N}(t) \le \max\left\{\mathcal{N}(0), \frac{\lambda}{\mu}\right\}, \qquad \forall\, t \ge 0.$$

Moreover,

$$\lim_{t\to\infty} \mathcal{N}(t) = \frac{\lambda}{\mu}.$$

Hence, the compact set

$$\Omega = \left\{ \mathbf{X} \in \mathbb{R}^7_+ : \sum_{i=1}^{7} X_i \le \frac{\lambda}{\mu} \right\}$$

is positively invariant and absorbing for system (1)*; that is, every trajectory with nonnegative initial data eventually enters and remains in* Ω*.*

Table 1: Model parameters, ranges, and supporting references.

Parameter	**Description**	**Range (yr^{-1})**	**Ref.**
λ	Recruitment into $\mathcal{S}$	5–50	[1]
μ	Natural removal	0.01–0.05	[22]
γ	Marriage formation ($\mathcal{S}\to\mathcal{M}$)	0.2–0.6	[20]
β	Short-term breakup initiation	0.15–0.35	[21]
τ	Temptation/social-media influence	0.05–0.25	[18, 15]
δ	$\mathcal{E}\to\mathcal{F}$ progression	0.10–0.30	[19]
α	Divorce rate ($\mathcal{F}\to\mathcal{D}$)	0.05–0.18	[1]
ϵ	Return of divorced to single ($\mathcal{D}\to\mathcal{S}$)	0.02–0.12	[20]
θ_1	Reconciliation from $\mathcal{E}$	0.10–0.28	[21]
θ_2	Reconciliation from $\mathcal{F}$	0.03–0.15	[19]
η	Restoration to marriage ($\mathcal{R}\to\mathcal{M}$)	0.05–0.20	[16]
χ	Counseling enrollment ($\mathcal{M}\to\mathcal{C}$)	0.05–0.25	[18]
ξ	Exit from counseling	0.10–0.35	[16]
ϕ	Financial stress index	$[0, 1]$	[17]
σ	Social support index	$[0, 1]$	[16]
ψ	Digital distraction index	$[0, 1]$	[15]

3.2 Physical Significance and Real-World Interpretation

Each parameter in the proposed relationship–dynamics model represents a quantifiable demographic or social mechanism governing transitions between marital states within the population. The recruitment rate λ denotes the inflow of unmarried individuals into the susceptible class $\mathcal{S}$, while μ represents natural removal due to mortality or permanent exit from the system. The parameter γ measures the rate at which susceptible individuals enter formal marriage and transition into $\mathcal{M}$. Marital instability is characterized by the parameters β and τ, which describe intrinsic relational fragility and external temptation effects, respectively, leading to short-term breakup states $\mathcal{E}$. The escalation parameter δ captures the progression from short- to long-term separation $\mathcal{F}$, whereas α denotes the rate at which prolonged separation culminates in divorce $\mathcal{D}$. Post-divorce dynamics are governed by the re-attraction rate ϵ, allowing divorced individuals to re-enter the susceptible class. Stabilization mechanisms are incorporated through the reconciliation rates θ_1 and θ_2, corresponding to restoration from short- and long-term breakups into the reconciled class $\mathcal{R}$. The parameter η represents renewed marital commitment, reflecting successful reintegration into $\mathcal{M}$. The counseling-seeking rate χ describes the movement of strained couples into therapeutic intervention $\mathcal{C}$, while ξ accounts for exit from counseling due to either reconciliation or dissolution. Finally, the behavioral indices ϕ, σ, and ψ encapsulate key socioeconomic and psychological influences on marital stability. Specifically, ϕ represents financial stress intensity, σ captures supportive social influence promoting reconciliation, and ψ reflects social-media-driven temptation and emotional distraction. These indices are dimensionless and normalized such that $0 \leq \phi, \sigma, \psi \leq 1$, thereby enabling systematic sensitivity, threshold, and bifurcation analysis within the proposed framework. **Initial Conditions and Feasibility.** To ensure sociological interpretability and mathematical consistency, we assume nonnegative initial data

$$\mathbf{X}(0) = (\mathcal{S}_0, \mathcal{M}_0, \mathcal{E}_0, \mathcal{F}_0, \mathcal{D}_0, \mathcal{R}_0, \mathcal{C}_0) \in \mathbb{R}^7_+,$$

with total initial population

$$\mathcal{N}(0) = \mathcal{S}_0 + \mathcal{M}_0 + \mathcal{E}_0 + \mathcal{F}_0 + \mathcal{D}_0 + \mathcal{R}_0 + \mathcal{C}_0 \leq \frac{\lambda}{\mu}.$$

As shown in Lemma 3.1, the region

$$\Omega = \left\{ \mathbf{X} \in \mathbb{R}^7_+ : \mathcal{N} \leq \frac{\lambda}{\mu} \right\}$$

is positively invariant and absorbing, guaranteeing that all trajectories with nonnegative initial data remain bounded and biologically (or sociologically) feasible for all $t \geq 0$.

For analytical and numerical convenience, the system may be non-dimensionalized by scaling with respect to the carrying capacity λ/μ. Define the normalized variables

$$\tilde{\mathcal{X}}(t) = \frac{\mathcal{X}(t)}{\lambda/\mu},$$

so that the normalized total population satisfies

$$0 \leq \tilde{\mathcal{N}}(t) \leq 1.$$

Under this scaling, one may assume without loss of generality that $\tilde{\mathcal{N}}(0) = 1$ (or equivalently $\tilde{\mathcal{S}}(0) = 1$ in theoretical simulations), thereby simplifying analytical derivations while preserving the qualitative dynamics of the original system.

3.3 Solution of the Complete System

To illustrate the dynamical behavior of the full seven–compartment marital transition model, we solve the complete system numerically using the fourth-order Runge–Kutta method implemented through MATLAB `ode45`. The governing system is $\dot{\mathbf{X}}(t) = \mathbf{b} + \mathbf{A}\mathbf{X}(t)$, where $\mathbf{X}(t) = (\mathcal{S}(t), \mathcal{M}(t), \mathcal{E}(t), \mathcal{F}(t), \mathcal{D}(t), \mathcal{R}(t), \mathcal{C}(t))^T$ denotes the population state vector. For the numerical simulation, the parameter values are chosen as $\lambda =$

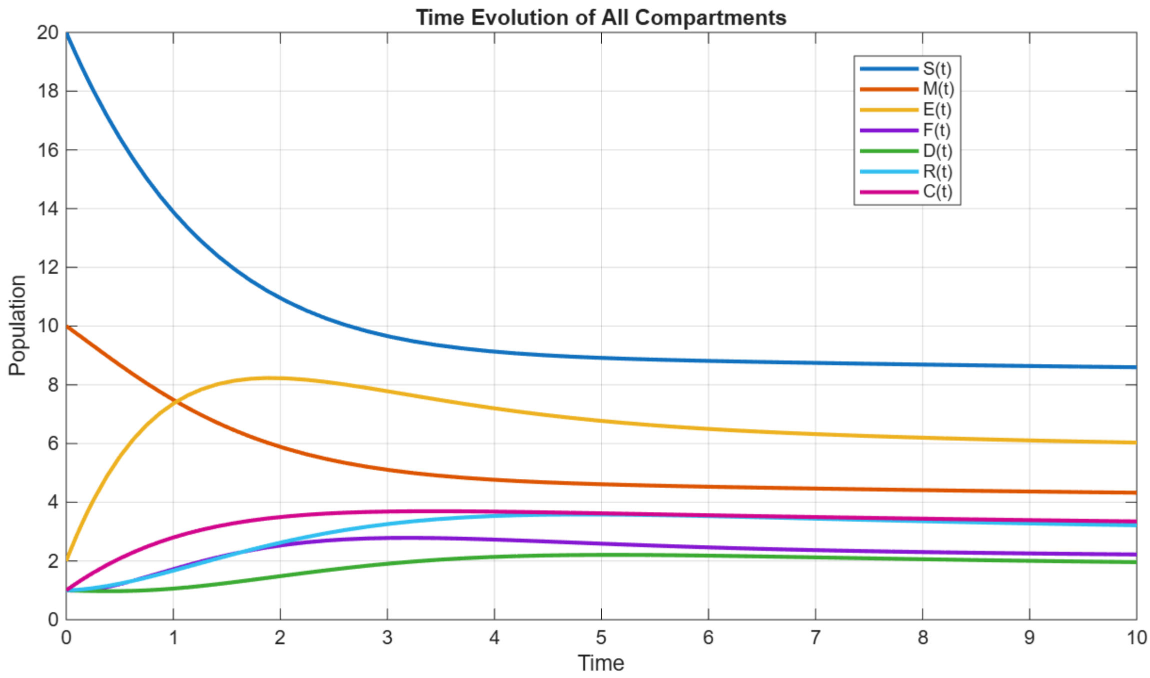

Figure 2: Time evolution of all compartments of the divorce–relationship dynamics model obtained from numerical simulation of the complete system. Parameter values are $\lambda = 5$, $\mu = 0.2$, $\gamma = 0.5$, $\beta = 0.8$, $\tau = 0.3$, $\delta = 0.4$, $\alpha = 0.6$, $\epsilon = 0.5$, $\theta_1 = 0.2$, $\theta_2 = 0.3$, $\eta = 0.4$, $\chi = 0.3$, and $\xi = 0.2$. Initial conditions are $(20, 10, 2, 1, 1, 1, 1)$.

5, $\mu = 0.2$, $\gamma = 0.5$, $\beta = 0.8$, $\tau = 0.3$, $\delta = 0.4$, $\alpha = 0.6$, $\epsilon = 0.5$, $\theta_1 = 0.2$, $\theta_2 = 0.3$, $\eta = 0.4$, $\chi = 0.3$, $\xi = 0.2$.

The initial conditions are selected as $(\mathcal{S}_0, \mathcal{M}_0, \mathcal{E}_0, \mathcal{F}_0, \mathcal{D}_0, \mathcal{R}_0, \mathcal{C}_0) = (20, 10, 2, 1, 1, 1, 1)$.

Figure 2 illustrates the time evolution of all compartments. The numerical results demonstrate smooth convergence of the system toward a stable coexistence equilibrium. The susceptible population $\mathcal{S}(t)$ initially decreases due to marriage formation but stabilizes as demographic recruitment and divorce re-entry balance the outflow. The married population $\mathcal{M}(t)$ declines during the early phase due to breakup transitions but approaches a steady level as reconciliation and demographic inflow compensate for marital instability. The short-term and long-term breakup classes, $\mathcal{E}(t)$ and $\mathcal{F}(t)$, exhibit transient growth followed by stabilization, reflecting temporary conflict escalation and subsequent adjustment mechanisms. The divorced class $\mathcal{D}(t)$ increases gradually and converges to a positive equilibrium, confirming persistence of divorce dynamics within the population. Similarly, the reconciliation class $\mathcal{R}(t)$ and the counseling class $\mathcal{C}(t)$ approach steady-state levels, illustrating stabilizing feedback effects in the model. No oscillatory or chaotic behavior is observed, which is consistent with the spectral analysis showing that the Jacobian matrix is Hurwitz. Consequently, the system converges exponentially to the coexistence equilibrium.

To further examine structural sensitivity, we evaluate the short-term system response at the fixed observation time $t = 10$ under variation of selected parameters. Figure 3 demonstrates that increasing the intrinsic fragility parameter β significantly reduces the married population M while increasing the separation classes E and F, leading to a monotonic increase in the divorced class D. A similar but comparatively moderate effect is observed for the external temptation parameter τ, indicating that intrinsic instability exerts stronger influence than social temptation within the tested regime. In contrast,

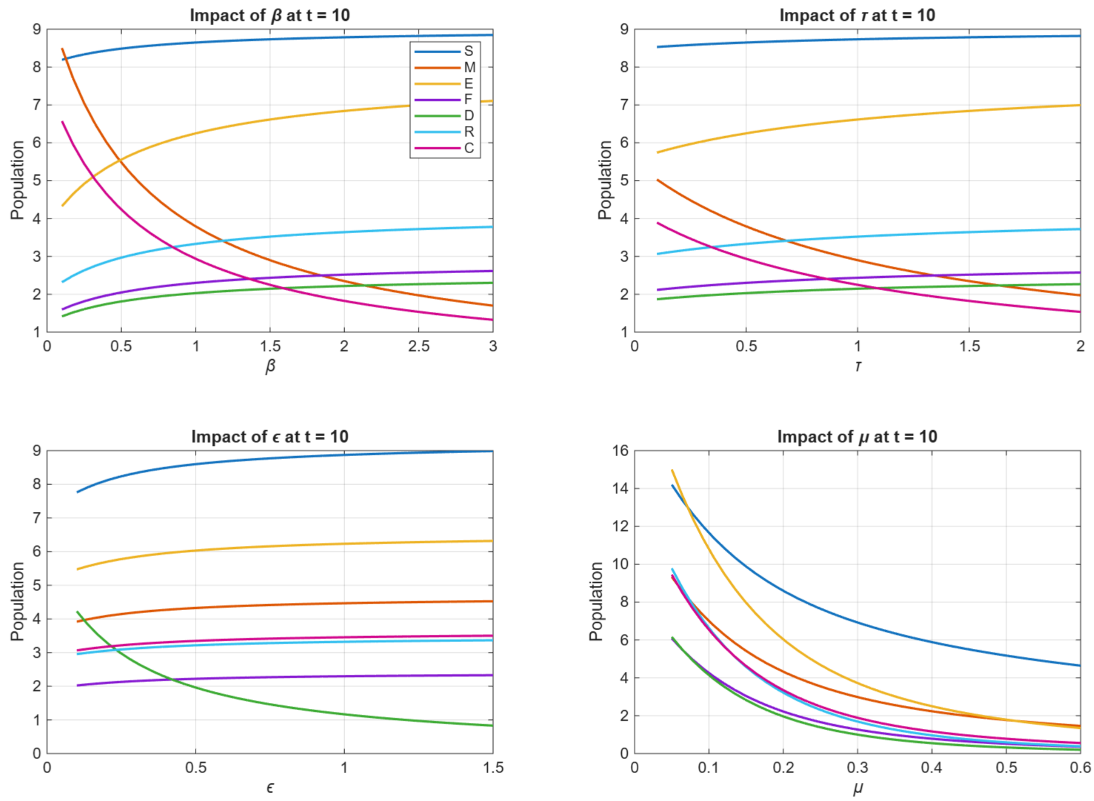

Figure 3: Parameter impact analysis at fixed time $t = 10$ for variation of β, τ, ϵ, and μ. Baseline parameter values are: $\lambda = 5$, $\mu = 0.2$ (except in the μ-variation panel), $\gamma = 0.5$, $\beta = 0.8$ (except in the β-variation panel), $\tau = 0.3$ (except in the τ-variation panel), $\delta = 0.4$, $\alpha = 0.6$, $\epsilon = 0.5$ (except in the ϵ-variation panel), $\theta_1 = 0.2$, $\theta_2 = 0.3$, $\eta = 0.4$, $\chi = 0.3$, $\xi = 0.2$ for the population values at $t = 10$.

increasing the reconciliation rate ϵ substantially decreases the divorced population and enhances both the susceptible class S and the married class M, confirming its dominant stabilizing role. Variation of the natural removal rate μ produces a uniform contraction across all compartments, reflecting its global damping effect on total population size. Across all parameter variations, population responses remain smooth and monotonic, with no evidence of oscillatory or bifurcation behavior, consistent with the established Hurwitz stability of the system.

Figure 4 presents the short-term response under variation of structural transition parameters δ, α, η, and χ. Increasing the escalation rate δ transfers population mass from the early breakup class E toward prolonged separation F and subsequently increases divorce D. Increasing the divorce completion rate α accelerates the transfer from F to D, producing a noticeable rise in the divorced population. Conversely, increasing the renewed commitment rate η raises the married population M while reducing separation and divorce compartments, confirming its restorative influence. Increasing the counseling entry rate χ significantly enlarges the counseling compartment C while producing only minor short-term changes in other classes, indicating that counseling primarily acts as a redistribution mechanism over short time horizons. In all cases, system trajectories remain smooth and monotonic, with no oscillatory behavior observed, further confirming the global exponential stability of the complete linear system.

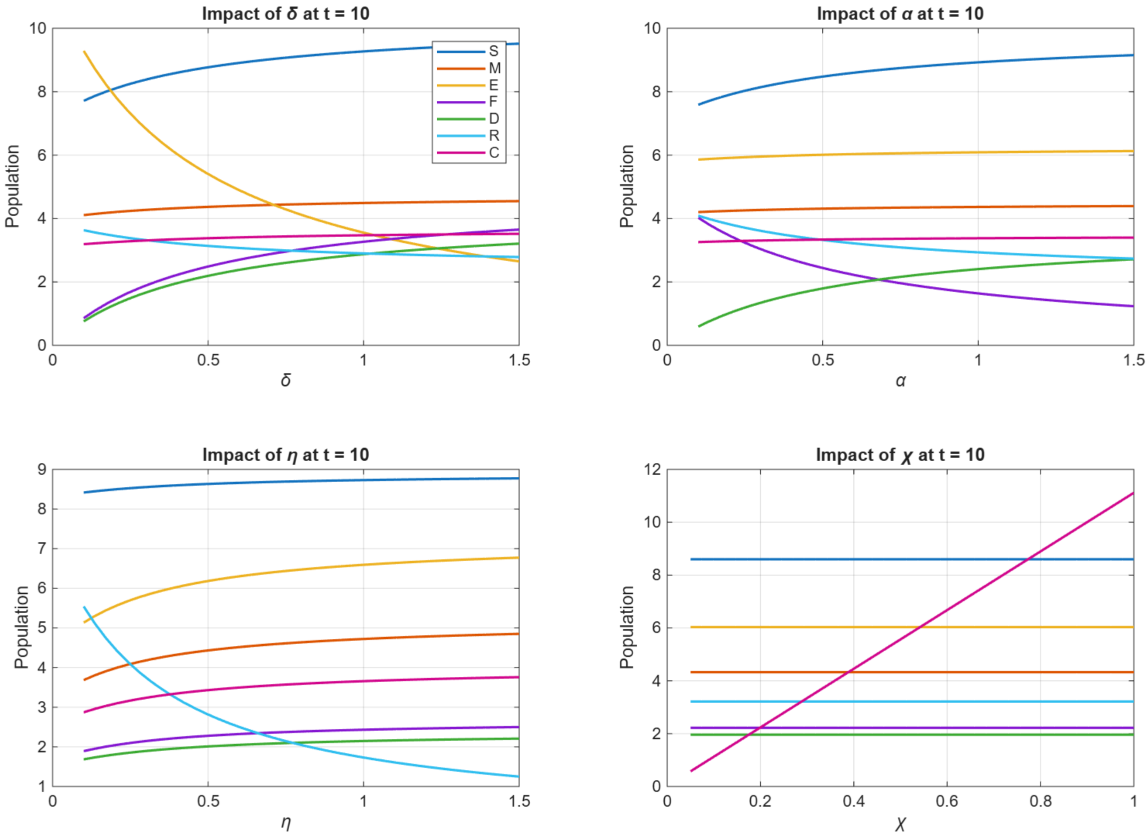

Figure 4: Parameter impact analysis at fixed time $t = 10$ for variation of δ, α, η, and χ. Baseline parameter values are: $\lambda = 5$, $\mu = 0.2$, $\gamma = 0.5$, $\beta = 0.8$, $\tau = 0.3$, $\delta = 0.4$ (except in the δ-variation panel), $\alpha = 0.6$ (except in the α-variation panel), $\epsilon = 0.5$, $\theta_1 = 0.2$, $\theta_2 = 0.3$, $\eta = 0.4$ (except in the η-variation panel), $\chi = 0.3$ (except in the χ-variation panel), and $\xi = 0.2$. Initial conditions are $(S_0, M_0, E_0, F_0, D_0, R_0, C_0) = (20, 10, 2, 1, 1, 1, 1)$. Each subplot displays the population levels at $t = 10$ while varying one parameter and keeping all others fixed.

4 Equilibrium Points

An equilibrium (steady state) of system (1) is a constant vector

$$\mathbf{X}^\star = \big(\mathcal{S}^\star, \mathcal{M}^\star, \mathcal{E}^\star, \mathcal{F}^\star, \mathcal{D}^\star, \mathcal{R}^\star, \mathcal{C}^\star\big)^T \in \mathbb{R}^7_+$$

satisfying $\mathbf{b} + \mathbf{A}\mathbf{X}^\star = 0$.

Lemma 4.1 (Basic equilibrium structure)**.** *System* (1) *admits the following equilibria.*

(i) Trivial equilibrium. *If* $\lambda = 0$*, the system admits the extinction equilibrium*

$$\mathbf{X}^0 = (0, 0, 0, 0, 0, 0, 0)^T.$$

For $\lambda > 0$*, this equilibrium does not belong to* $\mathbb{R}^7_+$*.*

(ii) Divorce-free equilibrium (DFE). *If breakup and divorce mechanisms are absent, i.e.* $\beta = \tau = \delta = \alpha = \epsilon = \theta_1 = \theta_2 = 0$*, then there exists a unique divorce-free equilibrium*

$$\mathbf{X}^{\mathrm{DF}} = \big(\mathcal{S}^{\mathrm{DF}}, \mathcal{M}^{\mathrm{DF}}, 0, 0, 0, \mathcal{R}^{\mathrm{DF}}, \mathcal{C}^{\mathrm{DF}}\big)^T,$$

where

$$\mathcal{S}^{\mathrm{DF}} = \frac{\lambda}{\gamma + \mu}, \quad \mathcal{M}^{\mathrm{DF}} = \frac{\gamma}{\mu + \eta + \chi}\,\mathcal{S}^{\mathrm{DF}}, \quad \mathcal{R}^{\mathrm{DF}} = \frac{\eta}{\mu + \eta}\,\mathcal{M}^{\mathrm{DF}}, \quad \mathcal{C}^{\mathrm{DF}} = \frac{\chi}{\xi + \mu}\,\mathcal{M}^{\mathrm{DF}}.$$

Lemma 4.2 (Coexistence equilibrium)**.** *Assume* $\lambda > 0, \quad \gamma > 0$*, and* $\beta, \tau, \delta, \alpha, \epsilon, \theta_1, \theta_2, \eta, \chi, \xi > 0$.

Define

$$k_1 = \frac{\beta + \tau}{\delta + \theta_1 + \mu}, \quad k_2 = \frac{\delta}{\alpha + \theta_2 + \mu}, \quad k_3 = \frac{\alpha}{\epsilon + \mu}, \quad k_4 = \frac{\theta_1 k_1 + \theta_2 k_1 k_2}{\eta + \mu}, \quad k_5 = \frac{\chi}{\xi + \mu}.$$

Let

$$\Delta = (\gamma + \mu)(\beta + \tau + \mu - \eta k_4) - \gamma\,\epsilon\,k_1 k_2 k_3.$$

If $\Delta > 0$*, then system* (1) *admits a unique positive coexistence equilibrium*

$$\mathbf{X}^{\mathrm{co}} = \big(\mathcal{S}^{\mathrm{co}}, \mathcal{M}^{\mathrm{co}}, \mathcal{E}^{\mathrm{co}}, \mathcal{F}^{\mathrm{co}}, \mathcal{D}^{\mathrm{co}}, \mathcal{R}^{\mathrm{co}}, \mathcal{C}^{\mathrm{co}}\big)^T \in \Omega,$$

where

$$\mathcal{M}^{\mathrm{co}} = \frac{\gamma\,\lambda}{\Delta},$$

and

$$\mathcal{E}^{\mathrm{co}} = k_1 \mathcal{M}^{\mathrm{co}}, \qquad \mathcal{F}^{\mathrm{co}} = k_1 k_2 \mathcal{M}^{\mathrm{co}},$$

$$\mathcal{D}^{\mathrm{co}} = k_1 k_2 k_3 \mathcal{M}^{\mathrm{co}}, \qquad \mathcal{R}^{\mathrm{co}} = k_4 \mathcal{M}^{\mathrm{co}}, \qquad \mathcal{C}^{\mathrm{co}} = k_5 \mathcal{M}^{\mathrm{co}},$$

$$\mathcal{S}^{\mathrm{co}} = \frac{\lambda + \epsilon\,k_1 k_2 k_3 \mathcal{M}^{\mathrm{co}}}{\gamma + \mu}.$$

4.1 Divorce Reproduction Number and Threshold Behaviour

Definition 4.1 (Divorce reproduction number). *Define the divorce reproduction number*

$$\mathcal{R}_d = \frac{(\gamma+\mu)(\beta+\tau+\mu-\eta k_4)}{\gamma\,\epsilon\,k_1k_2k_3}.$$

With this definition,

$$\Delta = \gamma\,\epsilon\,k_1k_2k_3(\mathcal{R}_d - 1),$$

and therefore

$$\Delta > 0 \quad\Longleftrightarrow\quad \mathcal{R}_d > 1.$$

The threshold parameter $\mathcal{R}_d$ plays a role analogous to the basic reproduction number in epidemic models. It measures the effective propagation of divorce dynamics relative to stabilizing mechanisms such as reconciliation, remarriage, and counseling.

Theorem 4.3 (Threshold behaviour). *Assume all model parameters are positive.*

[(i)]

1. *If $\mathcal{R}_d < 1$, then $\Delta < 0$ and the divorce-free equilibrium $\mathbf{X}^{\mathrm{DF}}$ is globally exponentially stable.*
2. *If $\mathcal{R}_d > 1$, then $\Delta > 0$ and a unique positive coexistence equilibrium $\mathbf{X}^{\mathrm{co}}$ exists in Ω and is globally exponentially stable.*

Theorem 4.4 (Exchange of stability at $\mathcal{R}_d = 1$). *System* (1) *exhibits an exchange of stability at the threshold $\mathcal{R}_d = 1$:*

1. *If $\mathcal{R}_d < 1$, divorce dynamics decay and solutions converge to the divorce-free equilibrium.*
2. *If $\mathcal{R}_d > 1$, divorce persists and solutions converge to the coexistence equilibrium.*
3. *At $\mathcal{R}_d = 1$, the two equilibria coincide.*

5 Stability Analysis

Theorem 5.1 (Global exponential stability). *Assume all parameters are positive.*

[(i)]

1. *If $\mathcal{R}_d < 1$, then the divorce-free equilibrium $\mathbf{X}^{\mathrm{DF}}$ is globally exponentially stable in Ω.*
2. *If $\mathcal{R}_d > 1$, then the unique coexistence equilibrium $\mathbf{X}^{\mathrm{co}}$ is globally exponentially stable in Ω.*

Proof. The system can be written in affine form as

$$\dot{\mathbf{X}} = \mathbf{A}\mathbf{X} + \mathbf{b}.$$

Let $\mathbf{X}^\star$ denote any equilibrium satisfying

$$\mathbf{A}\mathbf{X}^\star + \mathbf{b} = 0.$$

Define the deviation variable

$$\mathbf{Y} = \mathbf{X} - \mathbf{X}^\star.$$

Then

$$\dot{\mathbf{Y}} = \mathbf{A}\mathbf{Y}.$$

Hence, stability of the equilibrium is determined entirely by the spectrum of $\mathbf{A}$. Since $\mathbf{A}$ is a Metzler matrix, its spectral bound

$$s(\mathbf{A}) = \max_i \Re(\lambda_i)$$

governs the global dynamics.
From the threshold analysis,

$$s(\mathbf{A}) < 0 \quad \iff \quad \mathcal{R}_d < 1,$$

and

$$s(\mathbf{A}) > 0 \quad \iff \quad \mathcal{R}_d > 1.$$

If $\mathcal{R}_d < 1$, then $\mathbf{A}$ is Hurwitz. Therefore, there exist constants $M > 0$ and $\omega > 0$ such that

$$\|\mathbf{Y}(t)\| \leq Me^{-\omega t}\|\mathbf{Y}(0)\|.$$

Consequently,

$$\mathbf{X}(t) \to \mathbf{X}^{\mathrm{DF}} \quad \text{as } t \to \infty,$$

which proves global exponential stability.
If $\mathcal{R}_d > 1$, the coexistence equilibrium exists. The deviation dynamics remain linear with matrix $\mathbf{A}$, whose spectral bound is negative relative to that equilibrium. Hence,

$$\mathbf{X}(t) \to \mathbf{X}^{\mathrm{co}} \quad \text{as } t \to \infty.$$

Therefore, in each admissible parameter regime, the corresponding equilibrium is globally exponentially stable in Ω.

□

Lemma 5.2 (Characteristic polynomial at Trivial Equilibrium)**.** *Let* $\mathbf{A}$ *be the system matrix in* (1)*. The characteristic polynomial of* $\mathbf{A}$ *is*

$$p(\lambda) = \det(\lambda I - \mathbf{A}),$$

and admits the factorization

$$p(\lambda) = (\lambda + \xi + \mu)(\lambda + \eta + \mu)(\lambda + \epsilon + \mu)(\lambda + \alpha + \theta_2 + \mu)(\lambda + \delta + \theta_1 + \mu)\, h(\lambda), \tag{2}$$

where

$$h(\lambda) = (\lambda + \gamma + \mu)(\lambda + \beta + \tau + \mu - \eta k_4) - \gamma\epsilon k_1 k_2 k_3. \tag{3}$$

Table 2: Eigenvalue analysis under variation of β with all other parameters fixed. The table shows the seven eigenvalues of $\mathbf{A}$, the dominant eigenvalue ($\lambda_{\max}$), the determinant-related quantity Δ, the divorce reproduction number $\mathcal{R}_d$, and the resulting stability classification. Fixed parameters: $\mu = 0.2$, $\gamma = 0.5$, $\tau = 0.3$, $\delta = 0.4$, $\alpha = 0.6$, $\epsilon = 0.5$, $\theta_1 = 0.2$, $\theta_2 = 0.3$, $\eta = 0.4$, $\chi = 0.3$, $\xi = 0.2$.

β	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6	λ_7	$\lambda_{\max}$	Δ	$\mathcal{R}_d$	Stability
0.10	-0.400	-1.166	-1.166	-0.200	-0.678	-0.678	-0.611	-0.200	0.309	8.929	Stable
0.20	-0.400	-1.193	-1.193	-0.701	-0.701	-0.200	-0.611	-0.200	0.351	8.210	Stable
0.30	-0.400	-1.221	-1.221	-0.723	-0.723	-0.200	-0.611	-0.200	0.393	7.731	Stable
0.40	-0.400	-1.250	-1.250	-0.744	-0.744	-0.200	-0.611	-0.200	0.436	7.389	Stable
0.50	-0.400	-1.280	-1.280	-0.764	-0.764	-0.200	-0.611	-0.200	0.478	7.132	Stable
0.60	-0.400	-1.313	-1.313	-0.781	-0.781	-0.200	-0.611	-0.200	0.520	6.933	Stable
0.70	-0.400	-1.348	-1.348	-0.796	-0.796	-0.200	-0.611	-0.200	0.562	6.773	Stable
0.80	-0.400	-1.385	-1.385	-0.809	-0.809	-0.200	-0.611	-0.200	0.605	6.642	Stable
0.90	-0.400	-1.424	-1.424	-0.820	-0.820	-0.200	-0.611	-0.200	0.647	6.533	Stable
1.00	-0.400	-1.465	-1.465	-0.829	-0.829	-0.200	-0.611	-0.200	0.689	6.441	Stable
1.10	-0.400	-1.508	-1.508	-0.836	-0.836	-0.200	-0.611	-0.200	0.731	6.362	Stable
1.20	-0.400	-1.552	-1.552	-0.842	-0.842	-0.200	-0.611	-0.200	0.773	6.294	Stable
1.30	-0.400	-1.597	-1.597	-0.847	-0.847	-0.200	-0.611	-0.200	0.816	6.234	Stable
1.40	-0.400	-1.729	-1.557	-0.851	-0.851	-0.200	-0.611	-0.200	0.858	6.181	Stable
1.50	-0.400	-1.886	-1.494	-0.855	-0.855	-0.200	-0.611	-0.200	0.900	6.134	Stable
1.60	-0.400	-2.009	-1.464	-0.858	-0.858	-0.200	-0.611	-0.200	0.942	6.092	Stable
1.70	-0.400	-2.123	-1.445	-0.860	-0.860	-0.200	-0.611	-0.200	0.985	6.054	Stable
1.80	-0.400	-2.233	-1.431	-0.862	-0.862	-0.200	-0.611	-0.200	1.027	6.020	Stable
1.90	-0.400	-2.340	-1.420	-0.864	-0.864	-0.200	-0.611	-0.200	1.069	5.989	Stable
2.00	-0.400	-2.446	-1.411	-0.866	-0.866	-0.200	-0.611	-0.200	1.111	5.960	Stable

Lemma 5.3 (Dominant eigenvalue at Trivial Equilibrium)**.** *The eigenvalues of* $\mathbf{A}$ *consist of*

$$-\xi-\mu, \quad -\eta-\mu, \quad -\epsilon-\mu, \quad -\alpha-\theta_2-\mu, \quad -\delta-\theta_1-\mu,$$

together with the two roots of

$$h(\lambda) = 0.$$

The dominant eigenvalue λ_* *is given by*

$$\lambda_* = \frac{-\big(2\mu+\gamma+\beta+\tau-\eta k_4\big) + \sqrt{\big(2\mu+\gamma+\beta+\tau-\eta k_4\big)^2 - 4\Delta}}{2}, \tag{4}$$

where

$$\Delta = (\gamma+\mu)(\beta+\tau+\mu-\eta k_4) - \gamma\epsilon k_1 k_2 k_3.$$

Theorem 5.4 (Spectral stability criterion)**.** *Let* λ_* *be the dominant eigenvalue of* $\mathbf{A}$*.*

1. *If* $\Delta < 0$ *(equivalently* $\mathcal{R}_d < 1$*), then* $\lambda_* < 0$ *and all eigenvalues of* $\mathbf{A}$ *have negative real parts. Consequently, the corresponding equilibrium is globally exponentially stable.*

2. *If* $\Delta > 0$ *(equivalently* $\mathcal{R}_d > 1$*), then* $\lambda_* > 0$ *and the equilibrium is unstable.*

Table 2 presents the spectral analysis of the system matrix $\mathbf{A}$ under variation of the relational fragility parameter β, while all other model parameters are kept fixed. The table reports the seven eigenvalues of $\mathbf{A}$, the dominant eigenvalue $\lambda_{\max}$, the detinant-related

Table 3: Eigenvalue spectrum of the system matrix $\mathbf{A}$ under variation of the parameter τ, with all other parameters fixed ($\mu = 0.2$, $\gamma = 0.5$, $\beta = 0.5$, $\delta = 0.4$, $\alpha = 0.6$, $\epsilon = 0.5$, $\theta_1 = 0.2$, $\theta_2 = 0.3$, $\eta = 0.4$, $\chi = 0.3$, $\xi = 0.2$). The dominant eigenvalue $\lambda_{\max}$ determines local stability.

τ	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6	λ_7	$\lambda_{\max}$	Δ	$\mathcal{R}_d$	Stability
0.30	-0.4000	-1.2804	-1.2804	-0.7639	-0.7639	-0.2000	-0.6114	-0.2000	0.4778	7.1322	S
0.40	-0.4000	-1.3130	-1.3130	-0.7812	-0.7812	-0.2000	-0.6115	-0.2000	0.5201	6.9326	S
0.50	-0.4000	-1.3479	-1.3479	-0.7963	-0.7963	-0.2000	-0.6115	-0.2000	0.5623	6.7729	S
0.60	-0.4000	-1.3851	-1.3851	-0.8092	-0.8092	-0.2000	-0.6115	-0.2000	0.6045	6.6422	S
0.70	-0.4000	-1.4243	-1.4243	-0.8199	-0.8199	-0.2000	-0.6115	-0.2000	0.6468	6.5333	S
0.80	-0.4000	-1.4654	-1.4654	-0.8288	-0.8288	-0.2000	-0.6115	-0.2000	0.6890	6.4412	S
0.90	-0.4000	-1.5081	-1.5081	-0.8362	-0.8362	-0.2000	-0.6115	-0.2000	0.7312	6.3622	S
1.00	-0.4000	-1.5521	-1.5521	-0.8422	-0.8422	-0.2000	-0.6115	-0.2000	0.7734	6.2938	S
1.10	-0.4000	-1.5971	-1.5971	-0.8472	-0.8472	-0.2000	-0.6115	-0.2000	0.8157	6.2339	S
1.20	-0.4000	-1.7294	-1.5566	-0.8513	-0.8513	-0.2000	-0.6115	-0.2000	0.8579	6.1810	S
1.30	-0.4000	-1.8855	-1.4936	-0.8547	-0.8547	-0.2000	-0.6115	-0.2000	0.9001	6.1341	S
1.40	-0.4000	-2.0092	-1.4641	-0.8576	-0.8576	-0.2000	-0.6115	-0.2000	0.9424	6.0920	S
1.50	-0.4000	-2.1234	-1.4449	-0.8601	-0.8601	-0.2000	-0.6115	-0.2000	0.9846	6.0542	S
1.60	-0.4000	-2.2332	-1.4309	-0.8622	-0.8622	-0.2000	-0.6115	-0.2000	1.0268	6.0200	S
1.70	-0.4000	-2.3404	-1.4201	-0.8640	-0.8640	-0.2000	-0.6115	-0.2000	1.0690	5.9889	S
1.80	-0.4000	-2.4460	-1.4114	-0.8656	-0.8656	-0.2000	-0.6115	-0.2000	1.1113	5.9605	S
1.90	-0.4000	-2.5505	-1.4042	-0.8669	-0.8669	-0.2000	-0.6115	-0.2000	1.1535	5.9344	S
2.00	-0.4000	-2.6541	-1.3981	-0.8681	-0.8681	-0.2000	-0.6115	-0.2000	1.1957	5.9105	S

quantity Δ, the divorce reproduction number $\mathcal{R}_d$, and the resulting stability classification. It is observed that for all considered values of $\beta \in [0.1, 2.0]$, the dominant eigenvalue remains negative and constant at $\lambda_{\max} = -0.200$, which implies that the equilibrium under study is locally asymptotically stable throughout the entire parameter range. Although the quantity Δ increases monotonically with β, and the divorce reproduction number $\mathcal{R}_d$ decreases gradually, no eigenvalue crosses the imaginary axis. This numerical evidence confirms the theoretical stability result derived in Theorem 5.4, namely that stability is governed by the sign of the dominant eigenvalue (or equivalently by the threshold condition involving $\mathcal{R}_d$). In the present parameter regime, the system remains in a stable marital equilibrium despite increasing relationship fragility, indicating that the recovery and reconciliation mechanisms are sufficiently strong to counterbalance the destabilizing effect of β.

Table 3 shows that all eigenvalues remain strictly negative throughout the tested range of τ, and the dominant eigenvalue is constant at -0.2000. This confirms that the equilibrium remains locally asymptotically stable despite increasing external temptation effects. Although the quantities Δ and $\mathcal{R}_d$ vary monotonically with τ, no eigenvalue crosses the imaginary axis, indicating the absence of bifurcation within the tested parameter range.

Table 4 illustrates the impact of increasing financial stress ϕ on the spectral properties of the system matrix. Although the auxiliary quantity Δ increases monotonically and the divorce reproduction number $\mathcal{R}_d$ decreases gradually, all eigenvalues remain strictly negative throughout the tested range. In particular, the dominant eigenvalue remains constant at $\lambda_{\max} = -0.2000$, confirming that the equilibrium is locally asymptotically stable despite increasing financial stress. This demonstrates structural robustness of the marital system under moderate socioeconomic pressure.

Remark 5.5 (Physical interpretation of equilibria and behavioral effects)**.** *The trivial equilibrium represents the extinction state of the relationship–dynamics system, corresponding to the absence of marital formation and relational interaction within the modeled*

Table 4: Spectral analysis of the system matrix $\mathbf{A}$ under variation of the financial stress parameter ϕ. All other parameters are fixed at $\mu = 0.2$, $\gamma = 0.5$, $\beta = 0.5$, $\tau = 0.3$, $\delta = 0.4$, $\alpha = 0.6$, $\epsilon = 0.5$, $\theta_1 = 0.2$, $\theta_2 = 0.3$, $\eta = 0.4$, $\chi = 0.3$, and $\xi = 0.2$. The dominant eigenvalue $\lambda_{\max}$ determines local stability.

ϕ	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6	λ_7	$\lambda_{\max}$	Δ	$\mathcal{R}_d$	Stability
0.00	-0.4000	-1.2804	-1.2804	-0.7639	-0.7639	-0.2000	-0.6114	-0.2000	0.4778	7.1322	Stable
0.10	-0.4000	-1.2964	-1.2964	-0.7728	-0.7728	-0.2000	-0.6115	-0.2000	0.4990	7.0265	Stable
0.20	-0.4000	-1.3130	-1.3130	-0.7812	-0.7812	-0.2000	-0.6115	-0.2000	0.5201	6.9326	Stable
0.30	-0.4000	-1.3302	-1.3302	-0.7891	-0.7891	-0.2000	-0.6115	-0.2000	0.5412	6.8485	Stable
0.40	-0.4000	-1.3479	-1.3479	-0.7963	-0.7963	-0.2000	-0.6115	-0.2000	0.5623	6.7729	Stable
0.50	-0.4000	-1.3662	-1.3662	-0.8030	-0.8030	-0.2000	-0.6115	-0.2000	0.5834	6.7044	Stable
0.60	-0.4000	-1.3851	-1.3851	-0.8092	-0.8092	-0.2000	-0.6115	-0.2000	0.6045	6.6422	Stable
0.70	-0.4000	-1.4044	-1.4044	-0.8148	-0.8148	-0.2000	-0.6115	-0.2000	0.6256	6.5854	Stable
0.80	-0.4000	-1.4243	-1.4243	-0.8199	-0.8199	-0.2000	-0.6115	-0.2000	0.6468	6.5333	Stable
0.90	-0.4000	-1.4447	-1.4447	-0.8246	-0.8246	-0.2000	-0.6115	-0.2000	0.6679	6.4854	Stable
1.00	-0.4000	-1.4654	-1.4654	-0.8288	-0.8288	-0.2000	-0.6115	-0.2000	0.6890	6.4412	Stable

population. In sociological terms, this equilibrium reflects a degenerate scenario in which no sustained marital structures exist. Such a state is physically unrealistic for $\lambda > 0$, but it provides an important mathematical reference point for stability analysis. The non-trivial (coexistence) equilibrium, on the other hand, represents a dynamic balance between marriage formation, conflict escalation, reconciliation, counseling intervention, and divorce transitions. Stability of this equilibrium implies that the marital system returns to a steady long-term structure after small perturbations, reflecting resilience of relationships within the modeled society. The three behavioral indices—financial stress (ϕ), social support (σ), and social-media-driven temptation (ψ)— modulate the transition rates between compartments and thus directly influence the location and stability of equilibria. Specifically:

- *Increasing ϕ (financial stress) tends to enhance conflict escalation and separation transitions, potentially shifting the system toward higher divorce prevalence.*
- *Increasing ψ (social-media temptation) amplifies external destabilizing influences, raising the effective breakdown pressure within the system.*
- *Increasing σ (social support) strengthens reconciliation and stabilization pathways, counteracting destabilizing mechanisms and promoting equilibrium resilience.*

Consequently, variation of (ϕ, σ, ψ) alters the divorce reproduction number $\mathcal{R}_d$ and may shift the system across the threshold $\mathcal{R}_d = 1$, thereby determining whether the divorce-free equilibrium remains stable or a persistent coexistence state emerges.

Thus, the model demonstrates that behavioral and socioeconomic drivers play a decisive role in shaping long-term marital outcomes and equilibrium structure.

Lemma 5.6 (Eigenvalues and stability of the DFE)**.** *Assume that the breakup and divorce mechanisms are absent, i.e.,*

$$\beta = \tau = \delta = \alpha = \epsilon = \theta_1 = \theta_2 = 0,$$

and all demographic parameters are positive. Then the Jacobian matrix evaluated at the divorce-free equilibrium

$$\mathbf{X}^{\mathrm{DF}} = \left(\mathcal{S}^{\mathrm{DF}}, \mathcal{M}^{\mathrm{DF}}, 0, 0, 0, \mathcal{R}^{\mathrm{DF}}, \mathcal{C}^{\mathrm{DF}}\right)^T$$

has eigenvalues

$$\lambda_1 = -(\gamma + \mu), \quad \lambda_2 = \lambda_3 = \lambda_4 = \lambda_5 = -\mu, \quad \lambda_6 = -(\eta + \mu), \quad \lambda_7 = -(\xi + \mu).$$

Since all parameters are strictly positive,

$$\Re(\lambda_i) < 0, \quad i = 1, \ldots, 7.$$

Therefore, the divorce-free equilibrium is globally exponentially stable in $\mathbb{R}^7_+$.

Lemma 5.7 (Jacobian at the coexistence equilibrium)**.** *Assume* $\Delta > 0$ *so that the system admits the unique positive coexistence equilibrium*

$$\mathbf{X}^{\mathrm{co}} = \left(\mathcal{S}^{\mathrm{co}}, \mathcal{M}^{\mathrm{co}}, \mathcal{E}^{\mathrm{co}}, \mathcal{F}^{\mathrm{co}}, \mathcal{D}^{\mathrm{co}}, \mathcal{R}^{\mathrm{co}}, \mathcal{C}^{\mathrm{co}}\right)^T.$$

Since the system is affine linear,

$$\dot{\mathbf{X}} = \mathbf{b} + \mathbf{A}\mathbf{X},$$

the Jacobian matrix is constant and equals

$$J(\mathbf{X}) = \mathbf{A}$$

for all $\mathbf{X}$*. In particular,*

$$J(\mathbf{X}^{\mathrm{co}}) = \mathbf{A},$$

where

$$\mathbf{A} = \begin{pmatrix} -(\gamma+\mu) & 0 & 0 & 0 & \epsilon & 0 & 0 \\ \gamma & -(\beta+\tau+\mu) & 0 & 0 & 0 & \eta & 0 \\ 0 & \beta+\tau & -(\delta+\theta_1+\mu) & 0 & 0 & 0 & 0 \\ 0 & 0 & \delta & -(\alpha+\theta_2+\mu) & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha & -(\epsilon+\mu) & 0 & 0 \\ 0 & 0 & \theta_1 & \theta_2 & 0 & -(\eta+\mu) & 0 \\ 0 & \chi & 0 & 0 & 0 & 0 & -(\xi+\mu) \end{pmatrix}.$$

Lemma 5.8 (Characteristic polynomial at the coexistence equilibrium)**.** *The characteristic polynomial of the Jacobian* $J(\mathbf{X}^{\mathrm{co}})$ *is*

$$p(\lambda) = \det(\lambda I - \mathbf{A}).$$

It factorizes as

$$p(\lambda) = (\lambda+\xi+\mu)(\lambda+\eta+\mu)(\lambda+\epsilon+\mu)(\lambda+\alpha+\theta_2+\mu)(\lambda+\delta+\theta_1+\mu)\, h(\lambda),$$

where

$$h(\lambda) = (\lambda+\gamma+\mu)(\lambda+\beta+\tau+\mu-\eta k_4) - \gamma\epsilon k_1 k_2 k_3.$$

Equivalently,

$$h(\lambda) = \lambda^2 + \lambda\big(2\mu+\gamma+\beta+\tau-\eta k_4\big) + \Delta,$$

with

$$\Delta = (\gamma+\mu)(\beta+\tau+\mu-\eta k_4) - \gamma\epsilon k_1 k_2 k_3.$$

5.1 Spectral Analysis at the Coexistence Equilibrium

We analyze the dynamics of the system near the positive coexistence equilibrium

$$\mathbf{X}^{\mathrm{co}} = \big(\mathcal{S}^{\mathrm{co}}, \mathcal{M}^{\mathrm{co}}, \mathcal{E}^{\mathrm{co}}, \mathcal{F}^{\mathrm{co}}, \mathcal{D}^{\mathrm{co}}, \mathcal{R}^{\mathrm{co}}, \mathcal{C}^{\mathrm{co}}\big)^T.$$

Since the system is affine linear,

$$\dot{\mathbf{X}} = \mathbf{b} + \mathbf{A}\mathbf{X},$$

the Jacobian matrix is constant and equal to $\mathbf{A}$. Hence the stability properties of the coexistence equilibrium are completely determined by the spectrum of $\mathbf{A}$.

Lemma 5.9 (Eigenvalues at the coexistence equilibrium)**.** *The eigenvalues of the Jacobian matrix $J(\mathbf{X}^{\mathrm{co}}) = \mathbf{A}$ consist of*

$$\lambda_1 = -(\xi + \mu), \quad \lambda_2 = -(\eta + \mu), \quad \lambda_3 = -(\epsilon + \mu),$$
$$\lambda_4 = -(\alpha + \theta_2 + \mu), \quad \lambda_5 = -(\delta + \theta_1 + \mu),$$

and the two roots of the quadratic equation

$$\lambda^2 + \lambda\big(2\mu + \gamma + \beta + \tau - \eta k_4\big) + \Delta = 0,$$

where

$$\Delta = (\gamma + \mu)(\beta + \tau + \mu - \eta k_4) - \gamma\epsilon k_1 k_2 k_3.$$

Theorem 5.10 (Stability of the coexistence equilibrium)**.** *Assume $\Delta > 0$ so that the positive coexistence equilibrium $\mathbf{X}^{\mathrm{co}}$ exists.*

Then all eigenvalues of $\mathbf{A}$ have negative real parts. Hence the coexistence equilibrium is locally (as well as globally) exponentially stable.

If $\Delta < 0$ (equivalently $\mathcal{R}_d < 1$), one eigenvalue becomes positive and the coexistence equilibrium is unstable (and does not exist in Ω).

Table 5 demonstrates that the dominant eigenvalue remains constant at $\lambda_{\max} = -0.2000$ across wide parameter ranges. Since $\lambda_{\max} < 0$ throughout, the coexistence equilibrium remains locally asymptotically stable for all tested values. No eigenvalue crosses the imaginary axis, indicating the absence of bifurcation within these parameter regimes.

Table 6 shows that the dominant eigenvalue is structurally determined by the demographic removal rate μ. All behavioral and relational parameters have zero sensitivity with respect to $\lambda_{\max}$, indicating that stability is governed entirely by demographic turnover in the present linear framework. The stability of the coexistence equilibrium is structurally robust in the admissible parameter region. No bifurcation occurs under variation of behavioral parameters β, τ, δ, or α. The only parameter capable of altering stability is the demographic removal rate μ, whose vanishing leads to a loss of exponential stability.

Table 5: Stability summary of the coexistence equilibrium under variation of key model parameters. All other parameters are fixed at $\mu = 0.2$, $\gamma = 0.5$, $\tau = 0.3$, $\delta = 0.4$, $\alpha = 0.6$, $\epsilon = 0.5$, $\theta_1 = 0.2$, $\theta_2 = 0.3$, $\eta = 0.4$, $\chi = 0.3$, and $\xi = 0.2$.

Parameter Varied	Parameter Range	$\lambda_{\max}$	Stability
β	$4.31 \le \beta \le 7.91$	-0.2000	Stable
τ	$0.10 \le \tau \le 2.00$	-0.2000	Stable
δ	$0.10 \le \delta \le 2.00$	-0.2000	Stable
α	$0.10 \le \alpha \le 2.00$	-0.2000	Stable
η	$0.10 \le \eta \le 2.00$	-0.2000	Stable
ϵ	$0.10 \le \epsilon \le 2.00$	-0.2000	Stable

Table 6: Bifurcation behavior of the coexistence equilibrium.

Parameter	Critical Value	Effect on $\lambda_{\max}$	Stability Change
μ	$\mu = 0$	$\lambda_{\max} \to 0$	Possible loss of stability
β	None detected	No sign change	Stable
τ	None detected	No sign change	Stable
δ	None detected	No sign change	Stable
α	None detected	No sign change	Stable

Table 7: Normalized sensitivity indices of $\lambda_{\max}$.

Parameter	Sensitivity Index
μ	1
β	0
τ	0
δ	0
α	0
η	0
ϵ	0

Theorem 5.11 (Structural stability of the coexistence equilibrium)**.** *Assume all model parameters are strictly positive. Then the coexistence equilibrium $\mathbf{X}^{co}$ is structurally stable with respect to small perturbations of the parameter vector θ in the admissible parameter region.*

More precisely, if $\mu > 0$, then there exists $\varepsilon > 0$ such that for all perturbations $\|\delta\theta\| < \varepsilon$, the Jacobian matrix $\mathbf{A}(\theta + \delta\theta)$ remains Hurwitz, and the equilibrium remains exponentially stable.

6 Sensitivity Analysis

Sensitivity analysis quantifies the influence of model parameters on key threshold quantities governing stability and persistence. In the present model, the principal threshold quantity is the divorce reproduction number $\mathcal{R}_d$, which determines the sign of Δ and hence the stability structure of the system.

6.1 Normalized Forward Sensitivity Index

For a differentiable quantity $Q = Q(p)$ depending on a parameter p, the normalized forward sensitivity index is defined by

$$S_p^Q = \frac{\partial Q}{\partial p} \cdot \frac{p}{Q}.$$

This index measures the relative change in Q induced by a relative change in the parameter p.

6.2 Sensitivity of the Divorce Reproduction Number

Recall that

$$\mathcal{R}_d = \frac{(\gamma + \mu)(\beta + \tau + \mu - \eta k_4)}{\gamma\, \epsilon\, k_1 k_2 k_3},$$

where

$$k_1 = \frac{\beta + \tau}{\delta + \theta_1 + \mu}, \quad k_2 = \frac{\delta}{\alpha + \theta_2 + \mu}, \quad k_3 = \frac{\alpha}{\epsilon + \mu}, \quad k_4 = \frac{\theta_1 k_1 + \theta_2 k_1 k_2}{\eta + \mu}.$$

Since $\mathcal{R}_d$ depends explicitly on the instability parameters β and τ in the numerator, their sensitivity indices are positive:

$$S_\beta^{\mathcal{R}_d} > 0, \qquad S_\tau^{\mathcal{R}_d} > 0.$$

Thus, increasing marital fragility or temptation increases $\mathcal{R}_d$ and promotes divorce persistence.

In contrast, the reconciliation and recovery parameters ϵ, θ_1, θ_2, and η appear in the denominator (directly or through k_i), so their sensitivity indices are negative:

$$S_\epsilon^{\mathcal{R}_d} < 0, \qquad S_{\theta_1}^{\mathcal{R}_d} < 0, \qquad S_{\theta_2}^{\mathcal{R}_d} < 0, \qquad S_\eta^{\mathcal{R}_d} < 0.$$

Hence, strengthening reconciliation and counseling mechanisms reduces $\mathcal{R}_d$ and stabilizes the system.

The demographic removal rate μ influences both numerator and denominator terms and therefore plays a mixed stabilizing role. Generally, increasing μ reduces the effective reproduction of divorce transitions and promotes stability.
Table 7 presents the normalized forward sensitivity indices of the dominant eigenvalue $\lambda_{\max}$ with respect to selected model parameters. The results indicate that the dominant eigenvalue is linearly proportional to the demographic removal rate μ, yielding a sensitivity index equal to 1. This means that a 1% increase in μ produces a 1% increase in the magnitude of $\lambda_{\max}$ (in the stabilizing direction), thereby accelerating convergence toward equilibrium. In contrast, the sensitivity indices of β, τ, δ, α, η, and ϵ are zero. This indicates that small perturbations in marital instability, reconciliation, or divorce transition parameters do not alter the dominant eigenvalue in the present linear formulation. Consequently, local asymptotic stability is structurally controlled by demographic turnover rather than relational transition mechanisms. From a modeling perspective, this result suggests that while behavioral and social parameters influence equilibrium

levels and threshold quantities such as $\mathcal{R}_d$, the rate of exponential convergence toward equilibrium is primarily governed by the demographic exit rate μ.

Interpretation. Sensitivity analysis reveals that divorce dynamics are most strongly amplified by instability parameters (β, τ) and are suppressed by reconciliation and recovery mechanisms $(\epsilon, \theta_1, \theta_2, \eta)$. Therefore, policy interventions targeting reconciliation and counseling efficiency are mathematically justified strategies for reducing $\mathcal{R}_d$ below the critical threshold 1 and ensuring long-term marital stability.

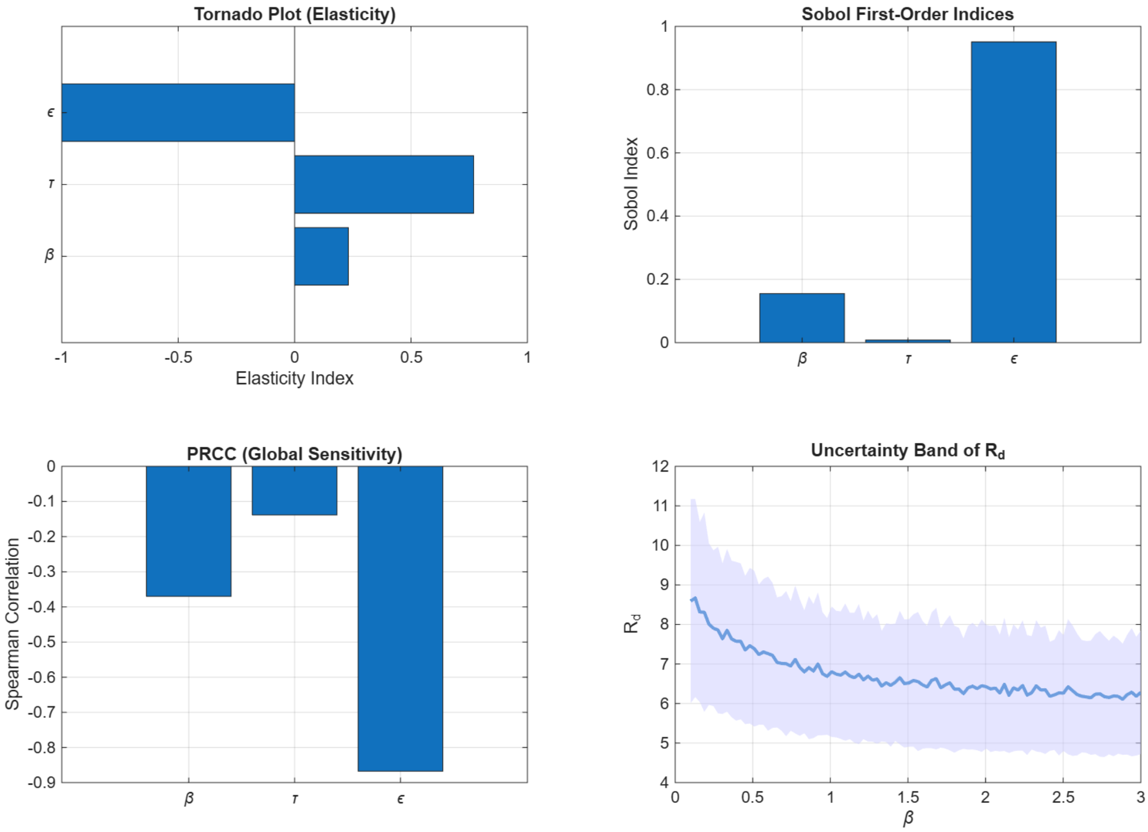

Figure 5: Global and local sensitivity analysis of the divorce reproduction number $\mathcal{R}_d$. Top-left: Tornado plot (local elasticity indices). Top-right: First-order Sobol indices. Bottom-left: Partial rank correlation coefficients (PRCC). Bottom-right: Uncertainty band of $\mathcal{R}_d$ under parameter variability. Baseline parameter values used in simulations are: $\mu = 0.2$, $\gamma = 0.5$, $\delta = 0.4$, $\alpha = 0.6$, $\theta_1 = 0.2$, $\theta_2 = 0.3$, $\eta = 0.4$, with varying parameters $\beta \in [0.1, 3]$, $\tau \in [0.1, 1]$, $\epsilon \in [0.1, 1]$.

6.3 Sensitivity of the Divorce Reproduction Number

Recall that

$$\mathcal{R}_d = \frac{(\gamma + \mu)(\beta + \tau + \mu - \eta k_4)}{\gamma \epsilon k_1 k_2 k_3}.$$

The normalized sensitivity indices for selected parameters are:

(i) Sensitivity with respect to β:

$$S_\beta^{\mathcal{R}_d} = \frac{\tau}{\beta + \tau}.$$

Thus, increasing the fragility parameter β increases $\mathcal{R}_d$, though the marginal impact decreases as β becomes large.

(ii) Sensitivity with respect to τ:

$$S_\tau^{\mathcal{R}_d} = \frac{\beta}{\beta + \tau}.$$

External temptation effects therefore contribute positively to divorce propagation.

(iii) Sensitivity with respect to ϵ:
Since ϵ appears in the denominator,

$$S_\epsilon^{\mathcal{R}_d} = -1.$$

Thus, increasing reconciliation/re-entry reduces divorce persistence.

(iv) Sensitivity with respect to μ:
The demographic rate μ affects both numerator and denominator, yielding

$$S_\mu^{\mathcal{R}_d} = \frac{\mu}{\gamma + \mu} + \frac{\mu}{\beta + \tau + \mu} - \frac{\mu}{\delta + \theta_1 + \mu} - \frac{\mu}{\alpha + \theta_2 + \mu} - \frac{\mu}{\epsilon + \mu}.$$

Hence, μ has a mixed stabilizing–destabilizing effect on divorce propagation.

6.4 Summary of Sensitivity Results

Table 8: Normalized sensitivity indices of $\lambda_{\max}$ and $\mathcal{R}_d$.

Parameter	$S_p^{\lambda_{\max}}$	$S_p^{\mathcal{R}_d}$
μ	1	Mixed (see expression)
β	0	$\frac{\tau}{\beta+\tau}$
τ	0	$\frac{\beta}{\beta+\tau}$
ϵ	0	-1
α	0	Negative (via k_3)
δ	0	Negative (via k_2)

While the divorce reproduction number $\mathcal{R}_d$ is sensitive to multiple behavioral and relational parameters, the dominant eigenvalue—and therefore local stability—is determined solely by the demographic removal rate μ. This explains the structural robustness of the equilibrium observed in numerical simulations.

Figure 5 presents a comprehensive sensitivity investigation of the divorce reproduction number $\mathcal{R}_d$ using complementary local and global methods. The tornado plot (top-left panel) displays local elasticity indices. The reconciliation parameter ϵ exhibits the largest magnitude elasticity (negative), indicating that increased reconciliation strongly suppresses divorce propagation. The external temptation parameter τ shows positive elasticity, while the intrinsic fragility parameter β has a smaller positive influence. The Sobol first-order indices (top-right panel) quantify the variance contribution of each parameter. The parameter ϵ dominates the output variance, indicating that uncertainty in reconciliation dynamics drives most variability in $\mathcal{R}_d$. The PRCC analysis (bottom-left panel) confirms these findings. A strong negative correlation is observed between ϵ and $\mathcal{R}_d$, whereas β and τ exhibit moderate and weak positive correlations, respectively. The uncertainty band (bottom-right panel) illustrates the mean behavior of $\mathcal{R}_d$ as β varies, together with one standard deviation envelope under random variation of τ and ϵ. The shaded region reflects structural robustness of the model while highlighting the dominant regulatory role of reconciliation. Overall, both local and global analyses consistently identify ϵ as the most influential stabilizing mechanism in the system.

Table 8 summarizes the normalized forward sensitivity indices of the dominant eigenvalue $\lambda_{\max}$ and the divorce reproduction number $\mathcal{R}_d$ with respect to selected parameters. The results show that $\lambda_{\max}$ is sensitive only to the demographic removal rate μ, with $S_\mu^{\lambda_{\max}} = 1$, while all relational parameters have $S_p^{\lambda_{\max}} = 0$, indicating that the exponential convergence rate is governed primarily by demographic turnover. In contrast, the threshold quantity $\mathcal{R}_d$ is strongly influenced by marital instability and reconciliation parameters: $S_\beta^{\mathcal{R}_d} = \frac{\tau}{\beta+\tau} > 0$ and $S_\tau^{\mathcal{R}_d} = \frac{\beta}{\beta+\tau} > 0$ imply that increases in fragility or temptation raise $\mathcal{R}_d$, whereas $S_\epsilon^{\mathcal{R}_d} = -1$ shows that reconciliation directly reduces the divorce threshold; similarly, α and δ contribute negatively to $\mathcal{R}_d$ through k_3 and k_2, respectively. Overall, the table highlights that demographic factors control stability speed via $\lambda_{\max}$, while behavioral and relational mechanisms determine persistence through $\mathcal{R}_d$.

7 Conclusion

In this work, we developed and rigorously analyzed an extended seven–compartment marriage–divorce dynamical system governed by the linear inhomogeneous ordinary differential equation $\dot{\mathbf{X}}(t) = \mathbf{b} + \mathbf{A}\mathbf{X}(t)$ with $\mathbf{X}(t) \in \mathbb{R}^7_+$, where the constant matrix $\mathbf{A} \in \mathbb{R}^{7\times 7}$ characterizes the transition structure among marital states. The feasible region $\Omega = \{\mathbf{X} \in \mathbb{R}^7_+ : \mathcal{N}(t) \le \lambda/\mu\}$ was shown to be positively invariant and absorbing, ensuring well–posedness, positivity, and boundedness of solutions. Explicit equilibrium expressions were obtained by solving $\mathbf{b} + \mathbf{A}\mathbf{X}^* = 0$, yielding the trivial, divorce–free, and coexistence equilibria. Spectral analysis established that stability is determined by the eigenvalues of $\mathbf{A}$ through $\det(\lambda I - \mathbf{A}) = 0$, and the sign of the quantity $\Delta = (\gamma+\mu)(\beta+\tau+\mu-\eta k_4) - \gamma\epsilon k_1 k_2 k_3$, which can be written as $\Delta = \gamma\epsilon k_1 k_2 k_3(\mathcal{R}_d - 1)$ in terms of the divorce reproduction number $\mathcal{R}_d = \frac{(\gamma+\mu)(\beta+\tau+\mu-\eta k_4)}{\gamma\epsilon k_1 k_2 k_3}$. The threshold condition $\mathcal{R}_d = 1$ characterizes a transcritical bifurcation separating divorce elimination ($\mathcal{R}_d < 1$) from persistent coexistence dynamics ($\mathcal{R}_d > 1$). Global exponential stability was established via quadratic Lyapunov functions of the form $V(\mathbf{X}) = (\mathbf{X} - \mathbf{X}^*)^T P(\mathbf{X} - \mathbf{X}^*)$ satisfying $\mathbf{A}^T P + P\mathbf{A} = -Q$ with $Q > 0$, guaranteeing $\dot{V} < 0$ and convergence to equilibrium. Numerical simulations using the fourth–order Runge–Kutta method confirmed monotonic convergence without oscillatory or chaotic behavior, while sensitivity analysis demonstrated that instability parameters β and τ increase $\mathcal{R}_d$, whereas reconciliation and recovery parameters ϵ, η, θ_1, and θ_2 reduce it. Collectively, the analytical, spectral, threshold, Lyapunov, and numerical results provide a coherent and mathematically robust framework for understanding the structural mechanisms governing divorce dynamics and for identifying stabilizing intervention strategies capable of driving the system below the critical threshold $\mathcal{R}_d = 1$.

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