A Mathematical Model of Malaria Transmission Incorporating Four Optimal Control Strategies.

Abstract – **Malaria is an endemic disease in many countries around the world and especially in Africa. In this paper, we studied SEIR-SEI mathematical model for malaria with vertical transmission, the basic properties of the model was investigated, the next generation matrix approach was used to obtained the reproduction number, $\mathscr{R}_0$, and lastly, we discussed the existence of the endemic equilibrium. Furthermore, we applied four optimal control strategies to control the number of exposed and infected humans. These control strategies are: the use of treated bed nets $u_1(t)$; intermittent prophylactic treatment in pregnancy,$u_2(t)$; the prompt and effective case management,$u_3(t)$, and the use of insecticides spray,$u_4(t)$. According to our numerical simulation results, the use of all control strategies together is a very effective approach in reducing the number of exposed and infected humans while the number of recovered humans also increases. Thus, through these strategies, we can be able to control the disease in a short period of time.**

Keywords – **Malaria transmission; Vertical transmission; Numerical Simulation; Optimal control.**

1. INTRODUCTION

Malaria is a life-threatening disease caused by *Plasmodium* parasites that are transmitted to humans through the bites of infected female Anopheles mosquitoes. In 2023, an estimated 263 million cases of malaria were reported globally, resulting in approximately 597,000 deaths. The African Region accounted for 94% of the total malaria cases and 95% of the associated deaths, underscoring its disproportionate burden of the disease. [1], [2]. There are five species of *Plasmodium* that can infect humans they are:*P.falciparum, P.vivax, P.ovale, P.malariae* and *P.knowlesi*. Among these species of human malaria parasites, *P.falciparum* is the most serious one [3], [4]. Globally, in 2023, malaria was estimated to have caused approximately 597,000 deaths, corresponding to a mortality rate of 13.7 per 100,000 population. This represents a steady decline from 2020, when the number of malaria deaths was estimated at 622,000 and the mortality rate stood at 14.9 per 100,000. The WHO African Region continues to bear the highest burden of malaria mortality, accounting for 95% of the estimated global malaria deaths. [1]. The estimated 228 million malaria cases was reported in the WHO African Region in 2020. Moreover, the case incidence reduced from 268 to 222 per 1000 population at risk between 2000-2019, but increased to 232 in 2020 mainly because of perturbations to services during the Covid-19 pandemic [5].

Mathematical models are essential tools that can be used to understand the behaviour and transmission of malaria. In addition to that, it can help in predicting the disease future and how we can prevent and control it. Several researchers have tried to control malaria disease through many optimal control strategies using mathematical approaches. Kazeem O.Okosun *et al* [7], studied and analyzed malaria transmission model using the treated bed nets,treatment of infective human and spray of insecticides as three control strategies to control and reduce the disease transmission with minimum cost. According to their simulation results the effective strategy for eliminating malaria is the combination between spray of insecticides and treatment of infective human. Gasper G.Mwanga *et al* [8], studied and discussed the optimal control practices and its strategies to control the malaria transmission using mathematical model with asymptomatic carriers of the disease and two age compartments of human population with four optimal control strategies. These control strategies are: Long -Lasting treated mosquito nets, Indoor residual spraying, screening and treatment of infected individuals. Their simulation result shows that using the four optimal control together will reduce the malaria. Romero-Leiton *et al* [9], developed their model with three optimal control strategies: bed nets, intermittent prophylactic treatment in pregnancy and prompt and effective case management to eliminate the malaria in colombia. Their result shows that the death due to malaria and congenital transmission have important effects in the disease infection outcome [9]. M.Osman *et al*, studied the effects of incidence function in occurrence of phenomena of backward bifurcation in malaria transmission mathematical model and they formulated an optimal problem with three control strategies to control and reduce the disease transmission in Democratic Republic of Congo. Their control strategies included: Long-Last insecticide treated net $u_1(t)$, treatment with drug of infected human $u_2(t)$ and insecticide spray $u_3(t)$. Their optimal control simulation result shows that using $u_1(t)$ and $u_2(t)$ together is an effective strategy to reduce and control the infected human in the country [10]. Many of modern research papers has discussed the optimal control strategies of malaria and how to control and eliminate the disease including the co-infection of malaria with other diseases. For more information see [11]–[18]. In this paper, we modified the SEIR-SI models in [22] by adding the exposed class of mosquito population with progression rate from exposed mosquitoes to infected mosquitoes. Furthermore, we included to model [9], the insecticides spray as control strategy. Our target is to minimize the number of infected human with minimum cost.

This paper is organized as follows: In Section 2, we introduce the model that describes the malaria transmission with vertical transmission and proved the positivity and boundedness of the solutions. In section 3, we obtained the basic reproduction number and discussed the existence of the disease free and endemic equilibria. Numerical simulation of the model is presented in section 4. In section 5, we introduced four control strategies to model(1) with their analysis to reduce and control the disease(number of infected humans). Section 6, is devoted to the numerical analysis of optimal control strategies. Finally, the conclusion is given in section 7.

2. Mathematical Model

A. Model formulation

We modified the ordinary differential equations for the malaria transmission mode with vertical transmission [22], [23]. This is a standard SEIRS model for human and SEI for mosquito. The total population of human,N_h, is sub-divided into four compartments at time t, represented by Susceptible $S_h(t)$, Exposed $E_h(t)$, Infected $I_h(t)$, and Recovery $R_h(t)$ that means $N_h(t) = S_h(t) + E_h(t) + I_h(t) + R_h(t)$. Also, the total mosquito population,$N_v(t)$ is sub-divided into three compartments at time t, they are Susceptible $S_v(t)$, Exposed $E_v(t)$, Infected $I_v(t)$ and $N_v(t) = S_v(t) + E_v(t) + I_v(t)$. We assumed that the susceptible humans and susceptible mosquitoes are recruited at constant rates, Λ_h and Λ_v respectively. The probability of human infected due to the bite of an infected mosquito,ϕ, is β_h. The progression rate from exposed human $E_h(t)$ to infected human $I_h(t)$ is α_1. Furthermore, $\frac{\psi}{2}$ represents the increase in susceptible humans due to birth of infected human and ψ represents the vertical transmission rate. Also, the progression rate from exposed mosquito $E_v(t)$ to infected mosquito $I_v(t)$ is α_2 and the probability of mosquito been infected is β_v. The rest of the model parameters are listed in Table (1) with their descriptions. From the above we have the following system

$$\begin{cases} \frac{dS_h}{dt} = \Lambda_h + wR_h - \frac{\beta_h \phi I_v}{N_h} S_h - \mu_h S_h, \\ \frac{dE_h}{dt} = \frac{\beta_h \phi I_v}{N_h} S_h - (\alpha_1 + \mu_h) E_h, \\ \frac{dI_h}{dt} = \frac{\psi}{2} I_h + \alpha_1 E_h - (\delta + \rho + \mu_h) I_h, \\ \frac{dR_h}{dt} = \delta I_h - (w + \mu_h) R_h, \\ \frac{dS_v}{dt} = \Lambda_v - \frac{\beta_v \phi I_h}{N_v} S_v - \mu_v S_v, \\ \frac{dE_v}{dt} = \frac{\beta_v \phi I_h}{N_v} S_v - (\alpha_2 + \mu_v) E_v, \\ \frac{dI_v}{dt} = \alpha_2 E_v - \mu_v I_v, \end{cases} \tag{1}$$

With the initial conditions: $S_h(0) > 0, E_h(0) \geq 0, I_h(0) \geq 0$, $R_h(0) \geq 0$, $S_v(0) > 0$, $E_v(0) \geq 0$, $I_v(0) \geq 0$.

B. Positivity and boundedness of the solutions

Lemma 1. *Let* $\rho^* = \frac{\frac{\psi}{2} - \rho}{\mu_h}$ *and* $n = \min(\frac{\Lambda_h}{\mu_h}, \frac{\Lambda_h}{\mu_h(1-\rho^*)})$ *then the set* Δ *defined by* $\Delta = \left\{ (S_h, E_h, I_h, R_h, S_v, E_v, I_v) \in \mathbb{R}^7_+ : 0 \leq N_h \leq n, 0 \leq N_v \leq \frac{\Lambda_v}{\mu_v} \right\}$ *is positively invariant with respect to the solution of the model* (1) *if* $\rho^* < 1$

TABLE. 1: Parameter values for model (1)

Parameter	Description	Value	Unit	Source
α_2	Exposed mosquitoes progression rate	0.055	Day^{-1}	[7]
Λ_h	Humans recruitment rate	0.01	Day^{-1}	[24]
Λ_v	Mosquitoes recruitment rate	0.018	Day^{-1}	[24]
ϕ	Mosquitoes biting rate	0.45	Day^{-1}	[24]
β_h	Humans transmission rate	0.70	Dimensionless	[24]
β_v	Mosquitoes transmission rate	0.20	Dimensionless	[24]
μ_h	Humans natural death rate	0.00102	Day^{-1}	[24]
μ_v	Mosquitoes natural death rate	0.0039	Day^{-1}	[24]
α_1	Exposed humans progression rate	0.10	Day^{-1}	[24]
ψ	Vertical transmission rate	0.0091	Day^{-1}	[24]
ρ	Disease induced death rate	0.0090	Day^{-1}	[24]
δ	Infection humans recovery rate	0.00297	Day^{-1}	[24]
w	Loss of immunity rate	0.012	Day^{-1}	[24]

Proof. By adding the first four equations of humans and the last three of the mosquitoes in (1) we obtain respectively.

$$\frac{dN_h}{dt} = \Lambda_h - \mu_h N_h + \mu_h \rho^* I_h, \tag{2}$$

$$\frac{dN_v}{dt} = \Lambda_v - \mu_v N_v, \tag{3}$$

from (2) clearly, if $\rho^* < 1$, then $\frac{dN_h}{dt} \leq \Lambda_h - \mu_h N_h$. Thus the solution is given by

$$N_h \leq \frac{\Lambda_h}{\mu_h} + (N_h(0) - \frac{\Lambda_h}{\mu_h})e^{-\mu_h t}, \tag{4}$$

we verify that from (4) $N_h \leq \frac{\Lambda_h}{\mu_h}$ when $N_h(0) \leq \frac{\Lambda_h}{\mu_h}$. By substituting,$0 < \rho^* < 1$ when $I_h \leq N_h$ in (2) we get

$$\frac{dN_h}{dt} = \Lambda_h - (1-\rho^*)\mu_h N_h, \tag{5}$$

by integrating equation (5) with respect to t, we obtain the following solution

$$N_h(t) \leq \frac{\Lambda_h}{\mu_h(1-\rho^*)} + (N_h(0) - \frac{\Lambda_h}{\mu_h(1-\rho^*)})e^{-\mu_h(1-\rho^*)t}, \tag{6}$$

we observe obviously, $N_h(t) \leq \frac{\Lambda_h}{\mu_h(1-\rho^*)}$ when $N_h(0) \leq \frac{\Lambda_h}{\mu_h(1-\rho^*)}$. Consequently, we have $N_h(0) \leq n$. In addition to that, integrating equation (3) we get $N_v \leq \frac{\Lambda_v}{\mu_v}$. Finally, we conclude that the set is positively invariant with respect to the solution of model (1) under the condition $\rho^* < 1$ ■

3. Existence of equilibria

When $E_h = I_h = R_h = E_v = I_v = 0$, that means the model (1) has a disease-free equilibrium (DFE) D_0 , which is obtained by setting all the right hand sides of the system (1) to zero and its defined by $D_0 = (\frac{\Lambda_h}{\mu_h}, 0, 0, 0, \frac{\Lambda_v}{\mu_v}, 0, 0)$. According to the next generation matrix technique in [25], [26], we calculate the basic reproduction number of system (1) and it is given by:

$$\mathscr{R}_0 = \sqrt{\frac{\alpha_1 \alpha_2 \beta_h \beta_v \phi^2}{\mu_v(\alpha_1 + \mu_h)(\alpha_2 + \mu_v)(\mu_h(1-\rho^*) + \delta)}}, \tag{7}$$

thus, we have the following lemma

Lemma 2. *The disease-free equilibrium* (DFE) D_0 *is locally asymptotically stable if* $\mathscr{R}_0 < 1$ *and unstable if* $\mathscr{R}_0 > 1$.

We assume that there exist an endemic equilibrium for model (1) denoted by:
$D^* = (S_h^*, E_h^*, I_h^*, R_h^*, S_v^*, E_v^*, I_v^*)$. Set all the right hand side of the model(1) equations to zero as:

$$
\begin{cases}
\Lambda_h + wR_h^* - \dfrac{\beta_h \phi I_v^*}{N_h^*} S_h^* - \mu_h S_h^* = 0, \\
\dfrac{\beta_h \phi I_v^*}{N_h^*} S_h^* - (\alpha_1 + \mu_h) E_h^* = 0, \\
\dfrac{\psi}{2} I_h^* + \alpha_1 E_h^* - (\delta + \rho + \mu_h) I_h^* = 0, \\
\delta I_h^* - (w + \mu_h) R_h^* = 0, \\
\Lambda_v - \dfrac{\beta_v \phi I_h^*}{N_v^*} S_v^* - \mu_v S_v^* = 0, \\
\dfrac{\beta_v \phi I_h^*}{N_v^*} S_v^* - (\alpha_2 + \mu_v) E_v^* = 0, \\
\alpha_2 E_v^* - \mu_v I_v^* = 0,
\end{cases}
\tag{8}
$$

Then solve for $S_h^*, E_h^*, I_h^*, R_h^*, S_v^*, E_v^*$ and I_v^*, we obtain the following

$$
\begin{aligned}
&S_h^* = \frac{\Lambda_h \Lambda_v \alpha_1 \alpha_2 \beta_v \phi k_3 + (\Lambda_v \mu_v k_4 (\alpha_1 \delta w - k_2 (k_1 - \frac{\psi}{2})) - \Lambda_h \alpha_1 \beta_v \phi k_3 k_4 \mu_v) I_v^*}{\alpha_1 \beta_v \phi \mu_v k_3 (\alpha_2 \Lambda_v - \mu_v k_4 I_v^*)},\ E_h^* = \frac{\Lambda_v \mu_v k_4 (k_1 - \frac{\psi}{2}) I_v^*}{\alpha_1 \beta_v \phi (\alpha_2 \Lambda_v - \mu_v k_4 I_v^*)}, \\
&I_h^* = \frac{\Lambda_v \mu_v k_4 I_v}{\beta_v \phi (\alpha_2 \Lambda_v - \mu_v k_4 I_v^*)},\ R_h^* = \frac{\Lambda_v \mu_v \delta k_4 I_v^*}{\beta_v \phi k_3 (\alpha_2 \Lambda_v - \mu_v k_4 I_v^*)},\ S_v^* = \frac{\Lambda_v \alpha_2 - \mu_v k_4 I_v^*}{\mu_v \alpha_2},\ E_v^* = \frac{\mu_v I_v^*}{\alpha_2},
\end{aligned}
\tag{9}
$$

by adding the first two equations and substituting the points in (9) we obtain the equation

$$
\beta_v \Lambda_h \Lambda_v \alpha_1 \alpha_2 \phi k_3 + (k_1 k_2 (1 - k_3) + \tfrac{\psi}{2} \Lambda_v \mu_v k_4 (k_1 k_3 - k_2) - \beta_v \Lambda_h \alpha_1 \phi \mu_v k_3 k_4) I_v^* = 0,
\tag{10}
$$

then we get $I_v^* = \frac{\beta_v \Lambda_h \Lambda_v \alpha_1 \alpha_2 \phi k_3}{\beta_v \Lambda_h \alpha_1 \phi \mu_v k_3 k_4 - \Lambda_v \mu_v k_4 (k_1 k_3 - k_2)}$, where $k_1 = (\delta + \rho + \mu_h), k_2 = (\alpha_1 + \mu_h)$, $k_3 = (w + \mu_h)$ and $k_4 = (\alpha_2 + \mu_v)$. Thus, there exists one endemic equilibrium.

4. Numerical Simulation

In this section, based on the parameter values that are presented in Table(1), we carry out our numerical simulation using Matlab software. Fig(1)(a) displays the solution of model(1) for exposed $E_h(t)$,infected $I_h(t)$ and recovery humans $R_h(t)$ and the disease will die out after day (1400). Fig(1) (b), shows the solution of model(1) for susceptible $S_v(t)$, exposed $E_v(t)$, infected $I_v(t)$ respectively. Clearly, mosquitoes population will be controlled after day(600). Also, Fig(2)(a) describes the influence of humans transmission rate β_h, when the value of β_h is decreased the number of infected humans $I_h(t)$ will also decreased accordingly. Clearly, from Fig(2)(a) decreasing β_h will decrease the number of infected malaria cases and vice versa. Similarly, from Fig(2)(b) increasing the value of the mosquitoes transmission rate,β_v, will increase the infected individuals.

Fig(3) (a) shows the effect of initial population of susceptible mosquitoes on the infected humans, that is the number of infected human increases according to increasing the initial conditions of mosquitoes. Furthermore, Fig 3 (b) shows the effect of $\beta_h, \beta_v, \phi,$ *and* ψ on the reproduction number $\mathscr{R}_0$.

To control and eliminate the malaria we have to reduce the transmission rates, to achieve this job we applied four control strategies to control the disease.

(a) (b)

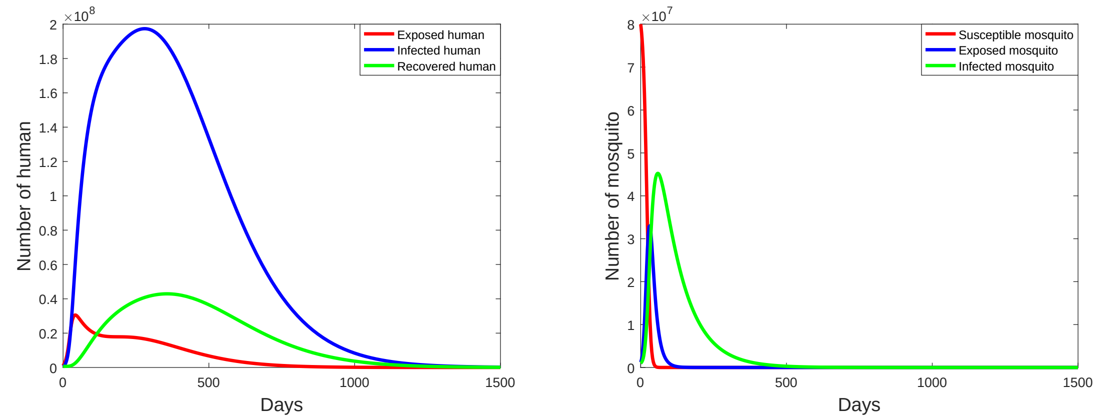


Fig. 1. (a) and (b) Represent the solution of model (1) with parameter values from Table (1).

(a) (b)

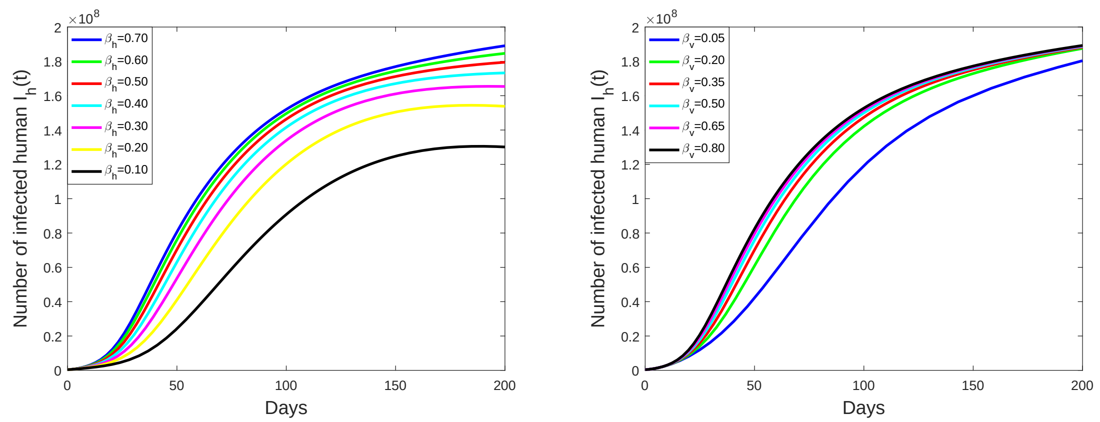


Fig. 2. (a) Represent the influence of β_h on the number of infected humans $I_h(t)$. (b) Represent the influence of β_v on the number of infected humans $I_h(t)$.

(a) (b)

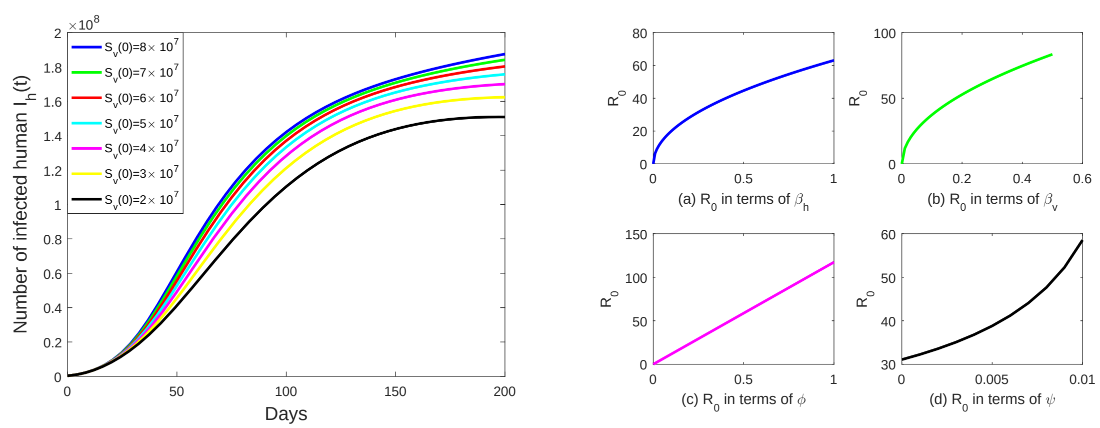


Fig. 3. (a) Represent the influence of S_v on the number of infected humans $I_h(t)$. (b) Represent the influence of $\beta_h, \beta_v, \phi, \psi$ on R_0.

5. Optimal Control

In this section, we included four optimal control strategies to model(1). These controls are: the use of treated bed nets $u_1(t)$, intermittent prophylactic treatment in pregnancy $u_2(t)$, fast and effective case management $u_3(t)$, the use of insecticides $u_4(t)$, where $c_1 \in [0,1]$ is the efficacy of the treatment and $c_2 \in [0,1]$ is the efficacy of insecticides spray. From the above controls and all the assumption in model(1). We have the following equations

$$\begin{cases} \frac{dS_h}{dt} = \Lambda_h + wR_h - (1-u_1(t))\frac{\beta_h \phi I_v}{N_h}S_h - \mu_h S_h, \\ \frac{dE_h}{dt} = (1-u_1(t))\frac{\beta_h \phi I_v}{N_h}S_h - (\alpha_1 + \mu_h)E_h, \\ \frac{dI_h}{dt} = (1-u_2(t))\frac{\psi}{2}I_h + \alpha_1 E_h - (\delta + \rho + c_1 u_3(t) + \mu_h)I_h, \\ \frac{dR_h}{dt} = (\delta + c_1 u_3(t))I_h - (w + \mu_h)R_h, \\ \frac{dS_v}{dt} = \Lambda_v - (1-u_1(t))\frac{\beta_v \phi I_h}{N_v}S_v - (c_2 u_4(t) + \mu_v)S_v, \\ \frac{dE_v}{dt} = (1-u_1(t))\frac{\beta_v \phi I_h}{N_v}S_v - (\alpha_2 + c_2 u_4(t) + \mu_v)E_v, \\ \frac{dI_v}{dt} = \alpha_2 E_v - (c_2 u_4(t) + \mu_v)I_v, \end{cases} \tag{11}$$

our objective function is defined as:

$$J(u_1(t),u_2(t),u_3(t),u_4(t)) = \int_0^{t_f}(A_1E_h + A_2I_h + \tfrac{1}{2}(a_1u_1^2(t) + a_2u_2^2(t) + a_3u_3^2(t) + a_4u_4^2(t)))dt, \tag{12}$$

where A_1, A_2 represent balancing cost coefficients and a_1, a_2, a_3 and a_4 are the weighting constants for $u_1(t), u_2(t), u_3(t)$ and $u_4(t)$ respectively. We required an optimal control $u_1^*(t), u_2^*(t), u_3^*(t)$ and $u_4^*(t)$ such that:

$$J(u_1^*, u_2^*, u_3^*, u_4^*) = \min\{J(u_1,u_2,u_3,u_4)|u_1,u_2,u_3,u_4 \in \Upsilon\}, \tag{13}$$

where the control set $\Upsilon = \{(u_1,u_2,u_3,u_4)|u_i : [0,t_f] \to [0,1],\ i = 1,2,3,4\}$ is a Lebegue measurable set. The optimal system is considered with the necessary conditions that an optimal control should satisfy using Pontryagin's Maximum Principle [27]. It changes the model(11) and (12) into a problem of Hamiltonian principle H, pointwise with respect to optimal controls $u_1(t), u_2(t), u_3(t)$ and $u_4(t)$.

$$\begin{aligned} H = {} & A_1E_h + A_2I_h + \tfrac{a_1}{2}u_1^2(t) + \tfrac{a_2}{2}u_2^2(t) + \tfrac{a_3}{2}u_3^2(t) + \tfrac{a_4}{2}u_4^2(t) \\ & + \lambda_1\left\{\Lambda_h + wR_h - (1-u_1(t))\tfrac{\beta_h\phi I_v}{N_h}S_h - \mu_h S_h\right\} \\ & + \lambda_2\left\{(1-u_1(t))\tfrac{\beta_h\phi I_v}{N_h}S_h - (\alpha_1+\mu_h)E_h\right\} \\ & + \lambda_3\left\{(1-u_2(t))\tfrac{\psi}{2}I_h + \alpha_1E_h - (\delta+\rho+c_1u_3(t)+\mu_h)I_h\right\} \\ & + \lambda_4\left\{(\delta+c_1u_3(t))I_h - (w+\mu_h)R_h\right\} \\ & + \lambda_5\left\{\Lambda_v - (1-u_1(t))\tfrac{\beta_v\phi I_h}{N_v}S_v - (c_2u_4(t)+\mu_v)S_v\right\} \\ & + \lambda_6\left\{(1-u_1(t))\tfrac{\beta_v\phi I_h}{N_v}S_v - (\alpha_2+c_2u_4(t)+\mu_v)E_v\right\} \\ & + \lambda_7\left\{\alpha_2E_v - (c_2u_4(t)+\mu_v)I_v\right\} \end{aligned} \tag{14}$$

Theorem 3. *Assume the optimal controls $u_1^*, u_2^*, u_3^*, u_4^*$ and the solutions $S_h, E_h, I_h, R_h, S_v, E_v$ and I_v of corresponding state system* (12),(13)*, that minimizes $J(u_1,u_2,u_3,u_4)$ over Υ. Then there exist adjoint variables $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7$ satisfying the following*

$$-\frac{d\lambda_i}{dt} = \frac{\partial H}{\partial i}, \tag{15}$$

where $i = 1,2,\cdots,7$ with transversality conditions

$$\lambda_1(t_f) = \lambda_2(t_f) = \lambda_3(t_f) = \lambda_4(t_f) = \lambda_5(t_f) = \lambda_6(t_f) = \lambda_7(t_f) = 0, \tag{16}$$

and

$$
\begin{aligned}
u_1^* &= \frac{1}{a_1}\Big(\frac{\beta_h \phi I_v S_h}{N_h}(\lambda_2-\lambda_1)+\frac{\beta_v \phi I_h S_v}{N_v}(\lambda_6-\lambda_5)\Big),\\
u_2^* &= \frac{1}{a_2}\frac{\psi}{2} I_h \lambda_3,\\
u_3^* &= \frac{c_1}{a_3} I_h(\lambda_3-\lambda_4),\\
u_4^* &= \frac{c_2}{a_4}(S_v\lambda_5+E_v\lambda_6+I_v\lambda_7),
\end{aligned}
$$

Proof. Combined with the convexity of the integrand of $J(u_1,u_2,u_3,u_4)$ with respect to u_1,u_2,u_3 and u_4, a priori bound of the state solutions, and the resulting Lipschitz characteristics of the state system (11), Theorem 4.1 and Corollary 4.1 of [27] guarantee the existence of an optimal control. Thus, the derivatives of H with respect to the adjoint variables are given by

$$
\begin{aligned}
\frac{d\lambda_1}{dt} &= \mu_h\lambda_1+(1-u_1(t))\frac{\beta_h\phi I_v}{N_h}(\lambda_1-\lambda_2)+(1-u_1(t))\frac{\beta_h\phi I_v}{N_h^2}S_h(\lambda_2-\lambda_1)\\
\frac{d\lambda_2}{dt} &= -A_1+(1-u_1(t))\frac{\beta_h\phi I_v}{N_h^2}S_h(\lambda_2-\lambda_1)+(\alpha_1+\mu_h)\lambda_2-\alpha_1\lambda_3,\\
\frac{d\lambda_3}{dt} &= -A_2+(1-u_1(t))\frac{\beta_h\phi I_v}{N_h^2}S_h(\lambda_2-\lambda_1)+(1-u_1(t))\frac{\beta_v\phi}{N_v}S_v(\lambda_5-\lambda_6)\\
&\quad -(1-u_2(t))\frac{\psi}{2}\lambda_3-(\delta+\rho+c_1u_3(t)+\mu_h)\lambda_3-(\delta+c_1u_3(t))\lambda_3,\\
\frac{d\lambda_4}{dt} &= (1-u_1(t))\frac{\beta_h\phi I_v}{N_h^2}(\lambda_2-\lambda_1)+(w+\mu_h)\lambda_1-w\lambda_1,\\
\frac{\lambda_5}{dt} &= (1-u_1(t))\frac{\beta_v\phi I_h}{N_v}(\lambda_5-\lambda_6)+(1-u_1(t))\frac{\beta_v\phi I_h}{N_v^2}S_v(\lambda_6-\lambda_5)+(c_2u_2(t)+\mu_v)\lambda_5,\\
\frac{\lambda_6}{dt} &= (1-u_1(t))\frac{\beta_v\phi I_h}{N_v^2}S_v(\lambda_6-\lambda_5)+(c_2u_4(t)+\alpha_2+\mu_v)\lambda_6-\alpha_2\lambda_7,\\
\frac{\lambda_7}{dt} &= (1-u_1(t))\frac{\beta_h\phi}{N_h}S_h(\lambda_1-\lambda_2)+(1-u_1(t))\frac{\beta_v\phi I_h}{N_v^2}S_v(\lambda_6-\lambda_5)+(c_2u_4(t)+\mu_v)\lambda_7,
\end{aligned}
$$

■

6. Numerical Simulation of Optimal Control

In this section, we discuss and analyze the numerical simulation of optimal control and the effects of our control strategies $u_1(t), u_2(t), u_3(t)$ and $u_4(t)$ on model(1) using the parameter values that are provided in Table(1). We used approximately (4-5 months)140 days as implementation time of our strategies with the initial conditions.

A. Combination of $u_2(t), u_3(t), u_4(t)$ together when $u_1(t)$=0

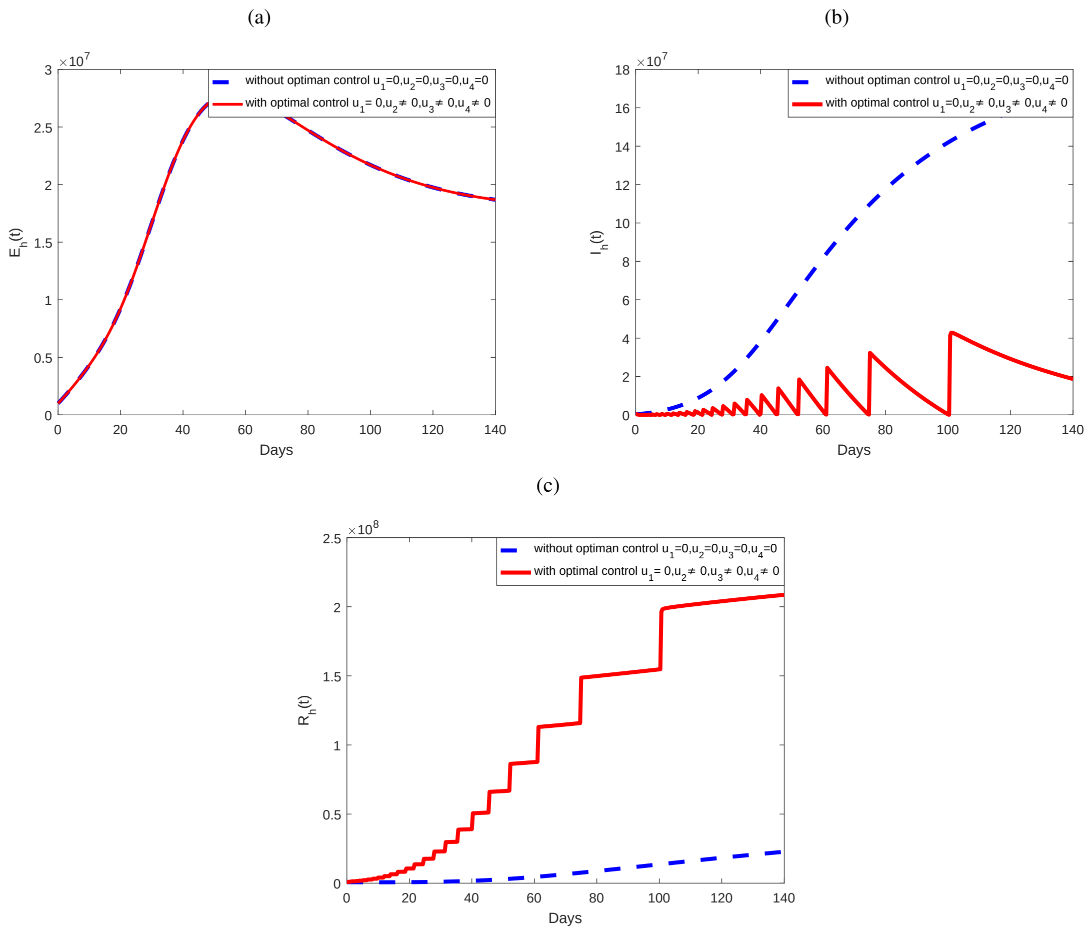


Fig. 4. (a), (b) and (c) represent the behaviors of exposed, infected and recovered humans respectively. Dashed line represents system without control $(u_1(t) = 0, u_2(t) = 0, u_3(t) = 0, u_4(t))$ and solid line shows the system with control $(u_1(t) = 0, u_2(t) \neq 0, u_3(t) \neq 0, u_4(t) \neq 0.)$

This is the strategy of applying intermittent prophylactic treatment in pregnancy $u_2(t)$, fast and effective case management $u_3(t)$ and the use of insecticides $u_4(t)$ together. From figure 4(a) it can be seen that this strategy has no effect on the exposed humans. On the other hand, this strategy greatly reduces the infected humans and increase the recovered humans.

B. Combination of $u_1(t), u_3(t), u_4(t)$ together when $u_2(t) = 0$.

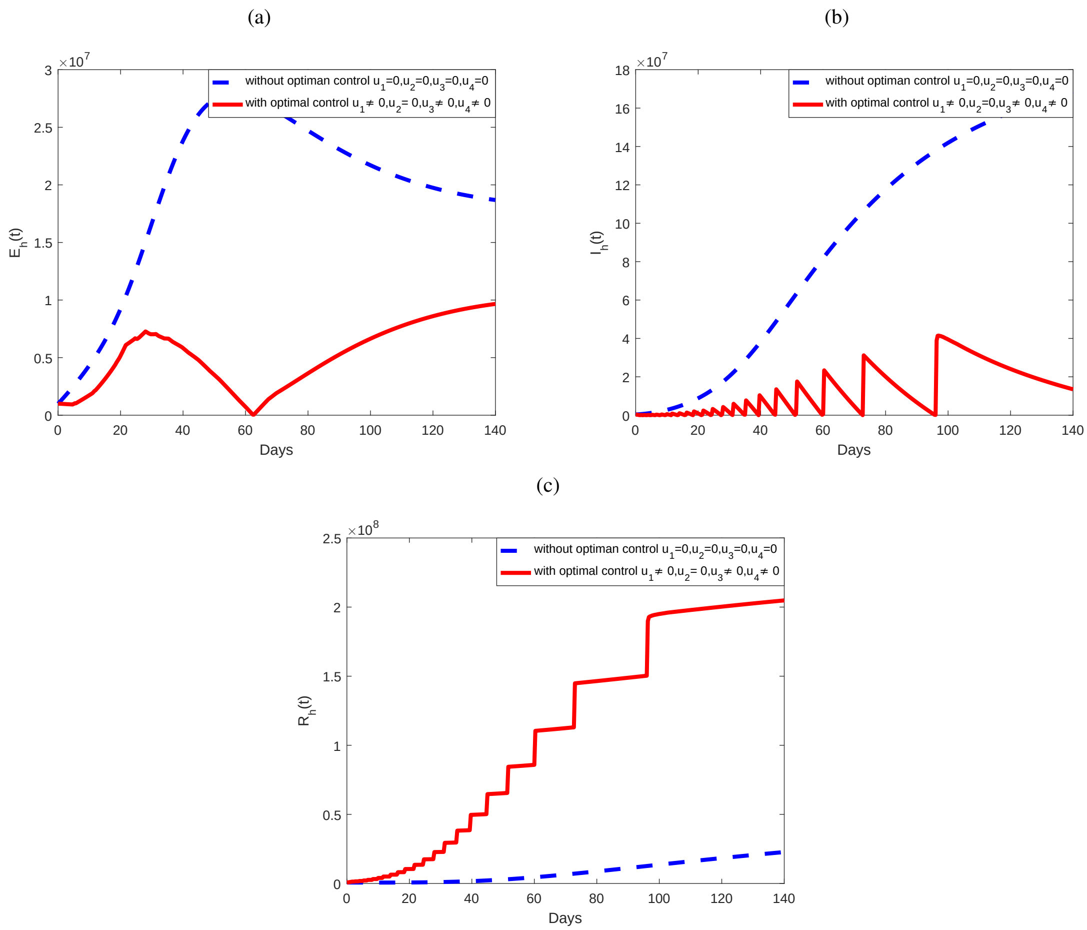


Fig. 5. (a), (b) and (c) represent the behaviors of exposed, infected and recovered humans respectively. Dashed line represents system without control $(u_1(t) = 0, u_2(t) = 0, u_3(t) = 0, u_4(t))$ and solid line shows the system with control $(u_1(t) \neq 0, u_2(t) = 0, u_3(t) \neq 0, u_4(t) \neq 0)$

This is the strategy of using treated bed net, fast and effective case management and the use of insecticides together. This strategy significantly reduces the number of exposed and infected humans evident from figures 5(a) and 5(b) respectively. From fig. 5(c), the graph indicates that this strategy greatly increases the number of recovered humans.

C. Combination of $u_1(t), u_2(t), u_4(t)$ together when $u_3(t) = 0$.

In Fig(6) (a),(b) the controls $u_1(t), u_2(t), u_4(t)$ are used together while $u_3(t) = 0$. Clearly, there is a significant effects between the states with control and those without the control. The number of exposed human,$E_h(t)$, decreases very fast after day 60 of implementation process. Similarly, the number of infected human, $I_h(t)$, begin to decrease rapidly after day 60. In addition to that Fig(6) (c) indicates that this strategy has no effect on the recovered humans.

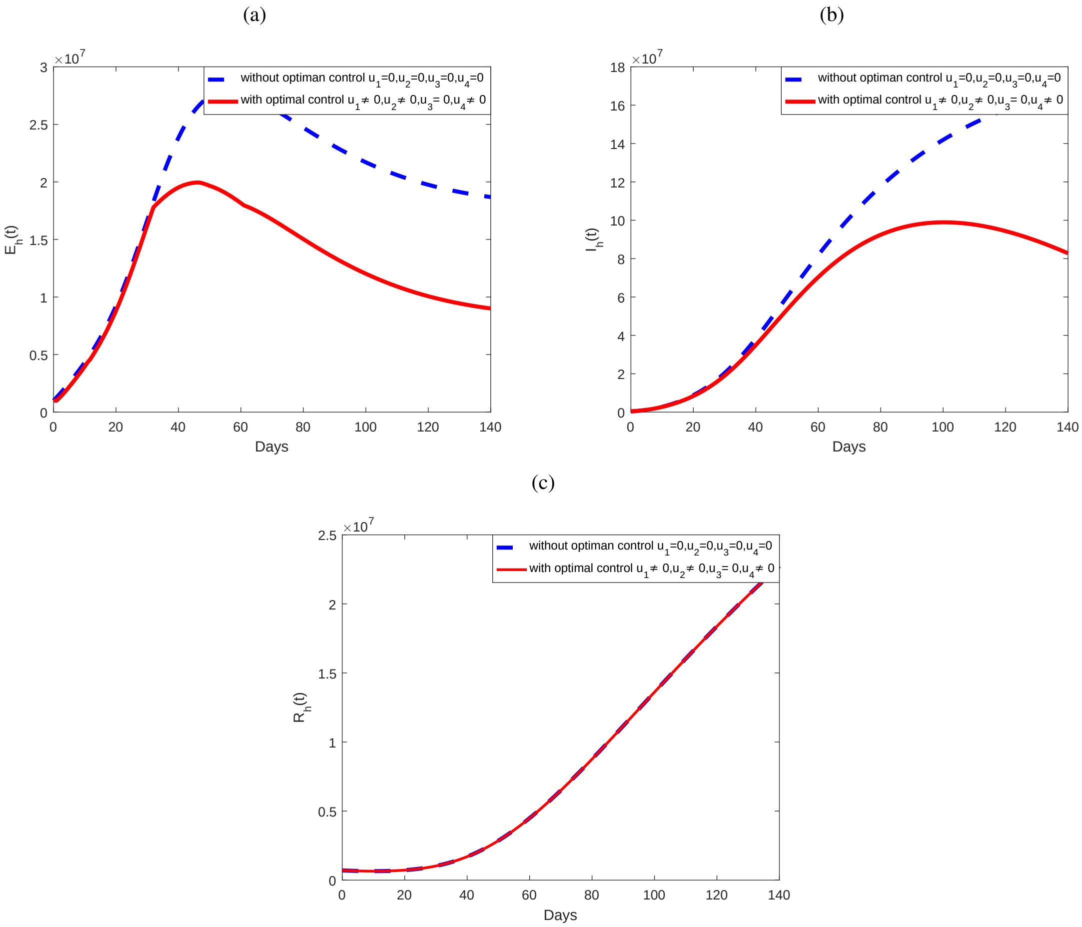


Fig. 6. (a), (b) and (c) represent the behaviors of exposed, infected and recovered humans respectively. Dashed line represents system without control ($u_1(t) = 0, u_2(t) = 0, u_3(t) = 0, u_4(t)$) and solid line shows the system with control ($u_1(t) \neq 0, u_2(t) \neq 0, u_3(t) = 0, u_4(t) \neq 0$.)

D. Combination of $u_1(t), u_2(t), u_3(t)$ together when $u_4(t) = 0$

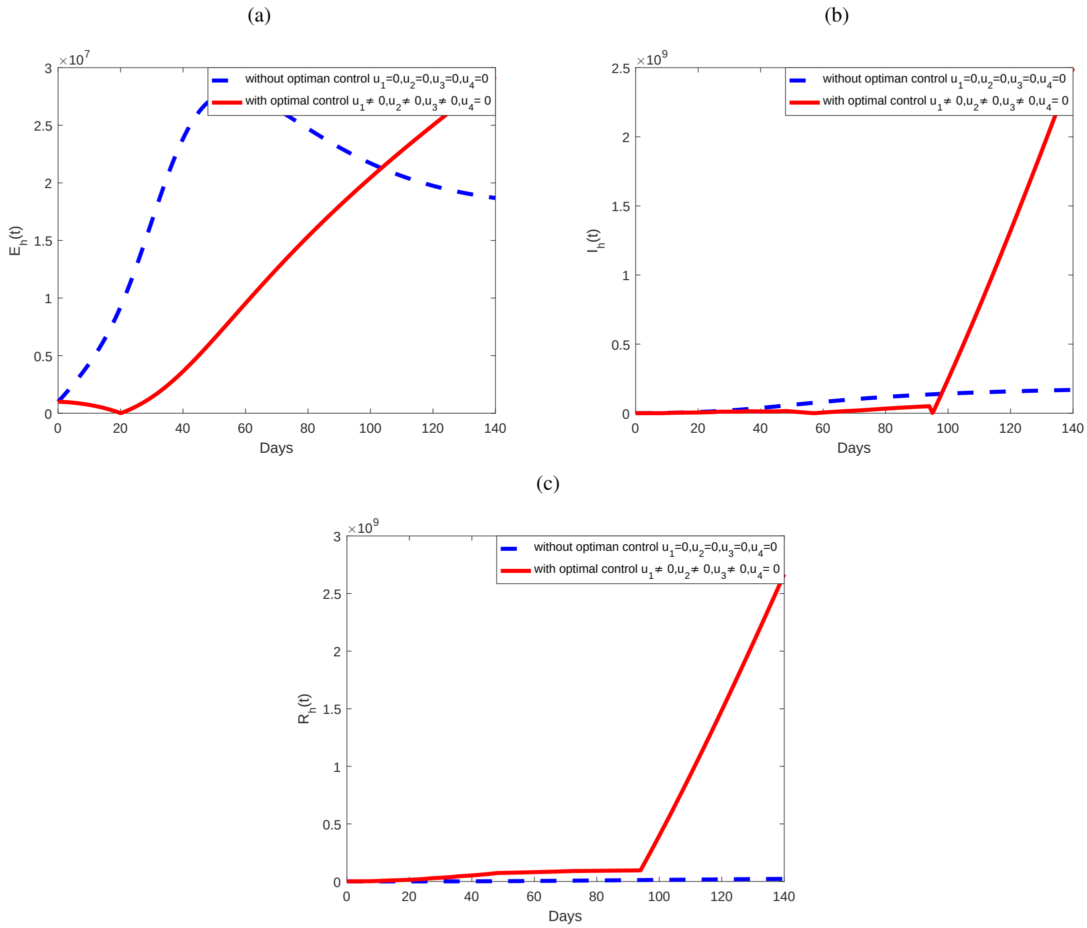
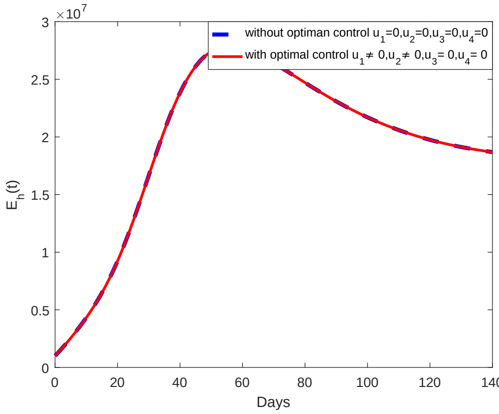


Fig. 7. (a), (b) and (c) represent the behaviors of exposed, infected and recovered humans respectively. Dashed line represents system without control $(u_1(t) = 0, u_2(t) = 0, u_3(t) = 0, u_4(t))$ and solid line shows the system with control $(u_1(t) \neq 0, u_2(t) \neq 0, u_3(t) \neq 0, u_4(t) = 0.)$

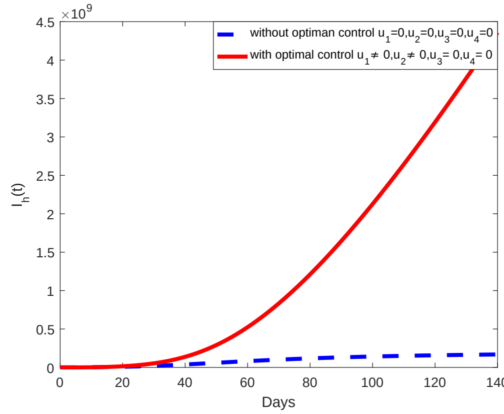
This strategy combines the use of treated bed net, intermittent prophylactic treatment in pregnancy and fast and effective case management. From the above graphs, Fig(7) (a) indicates that the strategy is effective in reducing the number of exposed humans until after day 100 where the effectiveness of this strategy is no more felt since the number without the control is lower than the number with the control strategy. From Fig(7) (b) the effectiveness of this strategy in seen after day 98 where the number with the control significantly increases. Additionally, Fig(7) (c) shows that this strategy shows no significant difference in improving the number of recovered humans until after day 98.

E. Combination of $u_1(t), u_2(t)$ together when $u_3(t) = 0, u_4(t) = 0$.

(a)



(b)



(c)

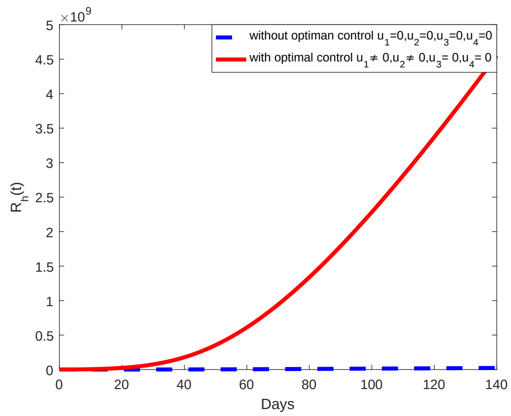


Fig. 8. (a), (b) and (c) represent the behaviors of exposed, infected and recovered humans respectively. Dashed line represents system without control $(u_1(t) = 0, u_2(t) = 0, u_3(t) = 0, u_4(t))$ and solid line shows the system with control $(u_1(t) \neq 0, u_2(t) \neq 0, u_3(t) = 0, u_4(t) = 0.)$

This strategy combines the use of treated bed net and intermittent prophylactic treatment in pregnancy together. From Fig(8) (a), we conclude that this strategy has no effect on the exposed humans since it doesn't show any difference between the number with control and the number without control. However,with the application of this strategy Fig(8) (a)and (b) indicate a sharp increase in the number of infected and recovered humans respectively, after day 40.

F. Combination of $u_1(t), u_3(t)$ together when $u_2(t) = 0, u_4(t) = 0$.

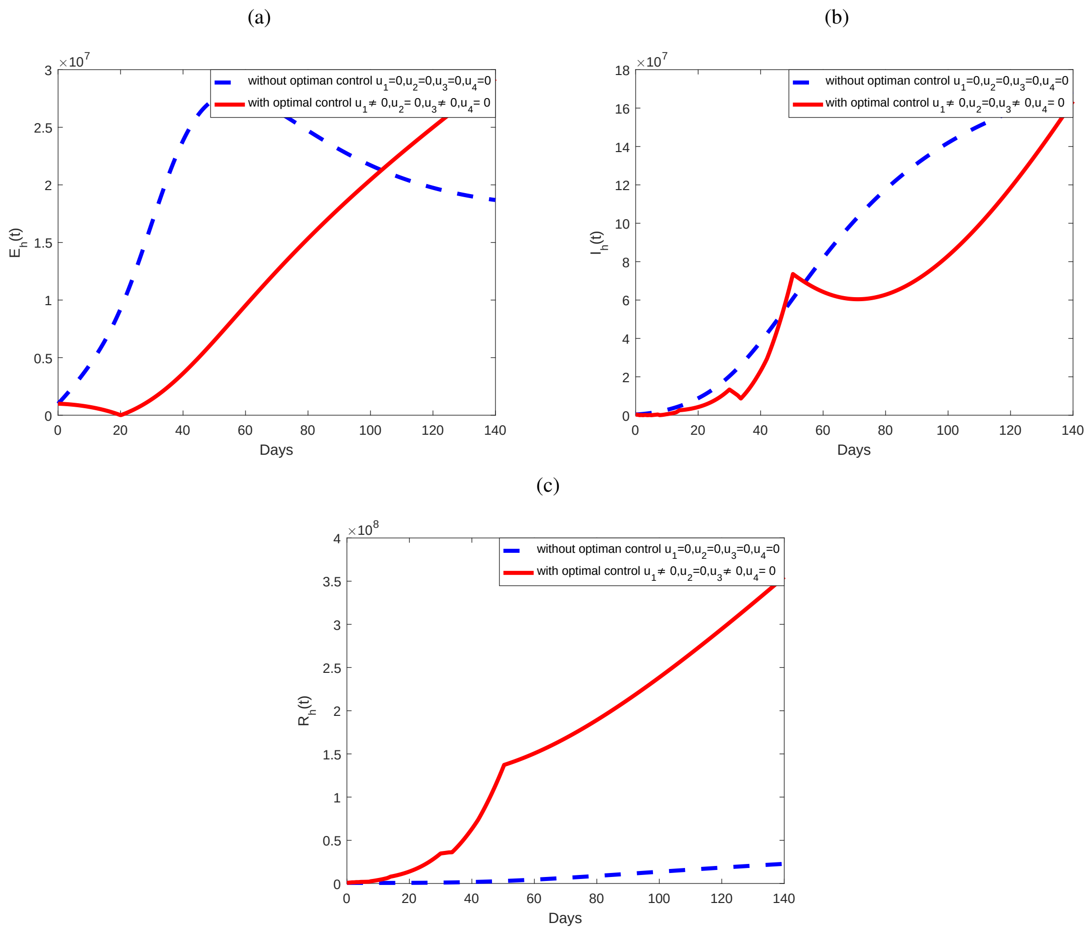


Fig. 9. (a), (b) and (c) represent the behaviors of exposed, infected and recovered humans respectively. Dashed line represents system without control $(u_1(t) = 0, u_2(t) = 0, u_3(t) = 0, u_4(t))$ and solid line shows the system with control $(u_1(t) \neq 0, u_2(t) = 0, u_3(t) \neq 0, u_4(t) = 0.)$

With this strategy, we combined the use of treated bed net and fast and effective case management. From the graph on Fig(9) (a), this strategy decreases the number of exposed humans until after day 100 where the number with control begins to increase more than the number without the control. This is not a good strategy to control the exposed humans. From Fig(9) (b), this strategy reduces the number of infected humans after day 60 until up to day 140. On the other side, this strategy sharply increases the number of recovered humans from day 15 until the end of the simulation period.

G. Combination of $u_3(t), u_4(t)$ *together when* $u_1(t) = 0, u_2(t) = 0$.

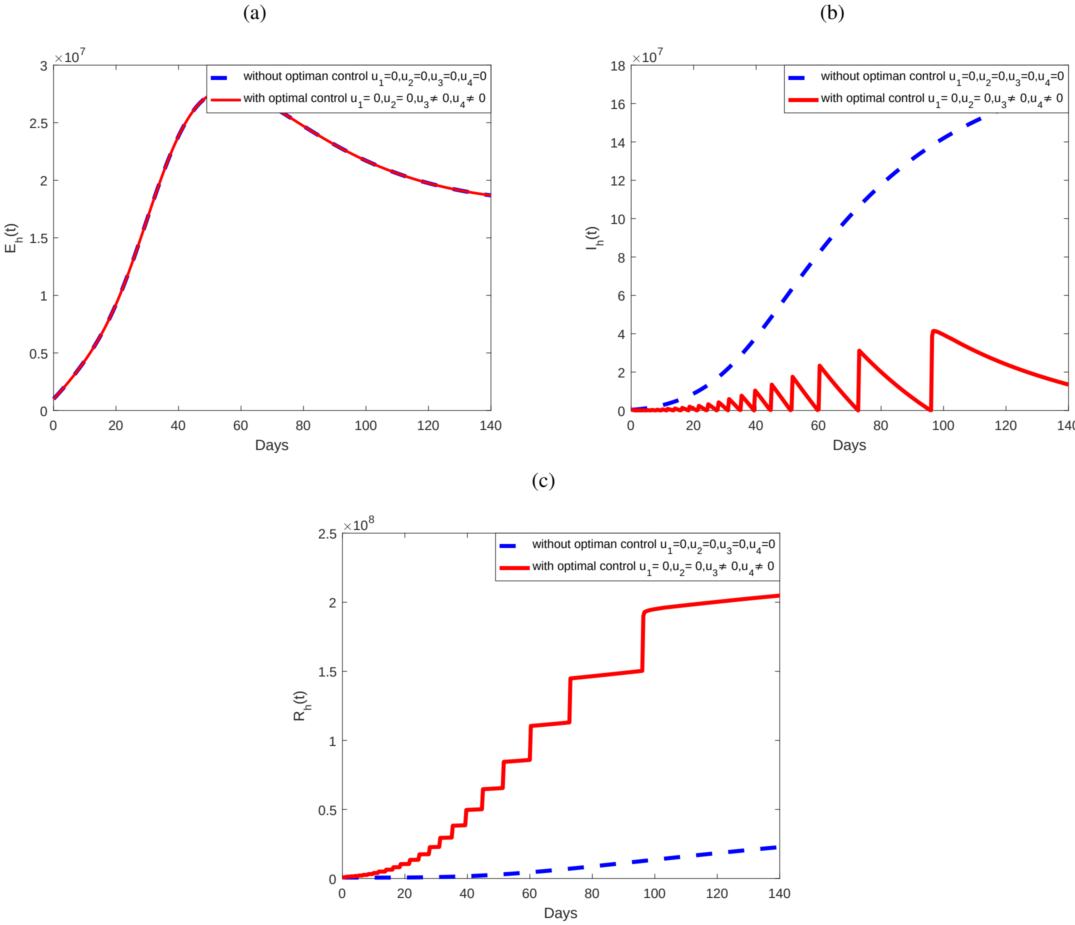


Fig. 10. (a), (b) and (c) represent the behaviors of exposed, infected and recovered humans respectively. Dashed line represents system without control $(u_1(t) = 0, u_2(t) = 0, u_3(t) = 0, u_4(t))$ and solid line shows the system with control $(u_1(t) = 0, u_2(t) = 0, u_3(t) \neq 0, u_4(t) \neq 0.)$

In this strategy, we combined the fast and effective case management and use of insecticides. This strategy shows no difference between the number with control and the number without control of the exposed humans, this can be seen from Fig(10) (a). However, from Fig(10) (b) and (c), this strategy significantly reduces the number of infected humans and sharply increases the number of recovered humans respectively, until the end of the simulation period.

H. Combination of $u_2(t), u_4(t)$ together when $u_1(t) = 0, u_3(t) = 0$.

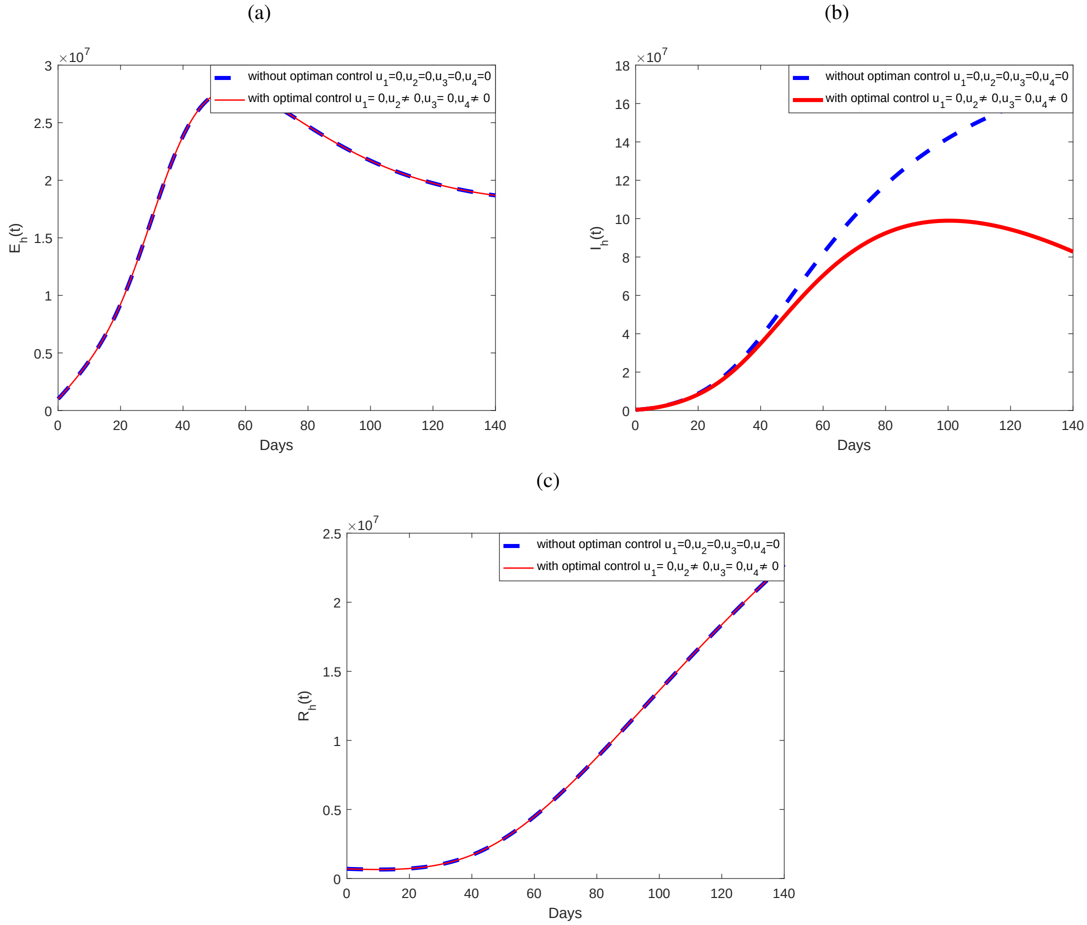


Fig. 11. (a), (b) and (c) represent the behaviors of exposed, infected and recovered humans respectively. Dashed line represents system without control $(u_1(t) = 0, u_2(t) = 0, u_3(t) = 0, u_4(t))$ and solid line shows the system with control $(u_1(t) = 0, u_2(t) \neq 0, u_3(t) = 0, u_4(t) \neq 0.)$

In this strategy, we combined intermittent prophylactic treatment in pregnancy and the use of insecticides. From Fig(11) (a) and (c), it can be seen that this strategy does not affect the number of exposed humans as well as the number of recovered humans respectively. However, after day 50, this strategy then begins to reduce the number of infected humans. This can be clearly seen from Fig(11) (b).

I. Combination of $u_1(t), u_2(t), u_3(t)$ and, $u_4(t)$ together.

This is the strategy of using all controls together. The graphs for this strategy are shown in Fig(12)(a),(b)and (c). Clearly there is a significant effects between the states with and without control. The number of exposed human,$E_h(t)$, starts to decrease very fast after day 50 see Fig(12)(a). In Fig(12)(b) the number of infected human, $I_h(t)$ decrease periodically up to day 100 then start to decrease rapidly. Thus, using all the control strategies is better than using them individually in reducing and controlling the disease. Furthermore, from Fig(12) (c) the number of recovered human ,$R_h(t)$ increased significantly.

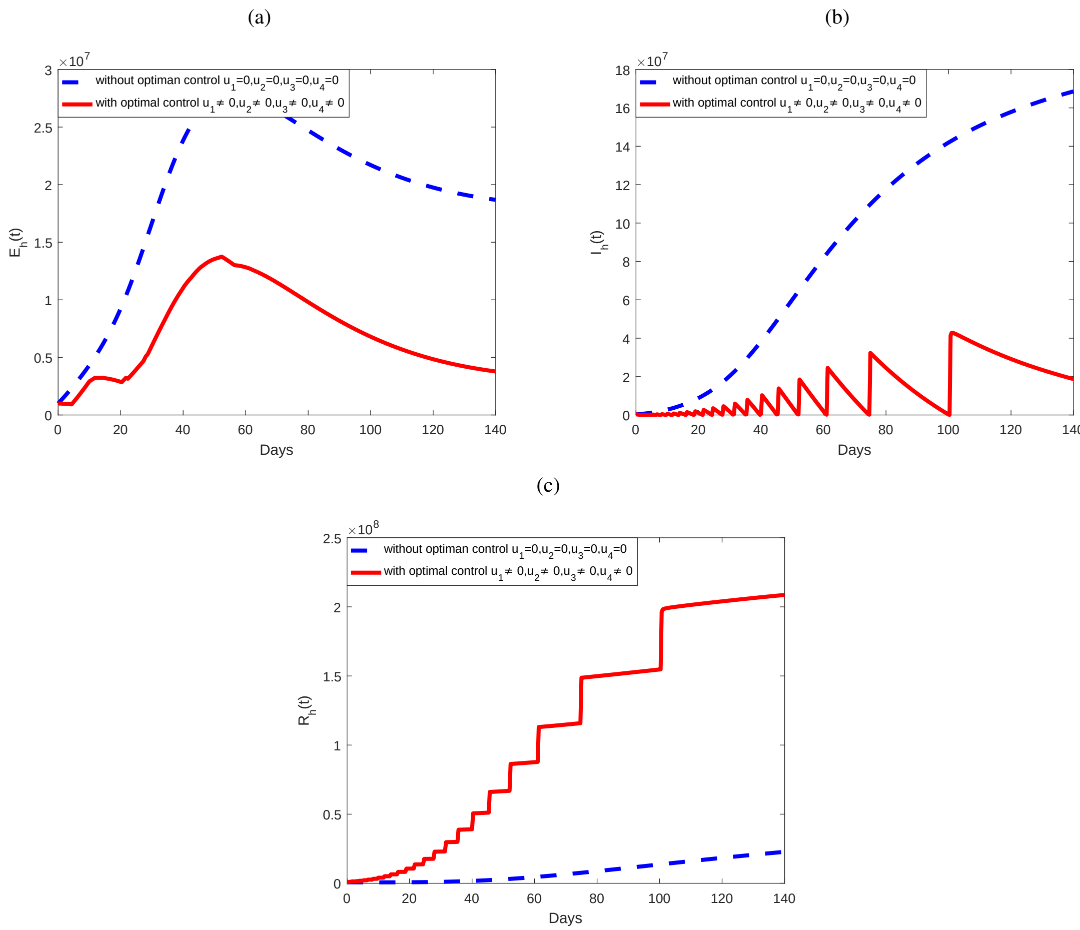


Fig. 12. (a), (b) and (c) represent the behaviors of exposed, infected and recovered humans respectively. Dashed line represents system without control ($u_1(t) = 0, u_2(t) = 0, u_3(t) = 0, u_4(t)$) and solid line shows the system with control ($u_1(t) \neq 0, u_2(t) \neq 0, u_3(t) \neq 0, u_4(t) \neq 0$.)

7. Conclusion

In this paper, we studied standard SEIR-SEI ordinary differential equation mathematical model for malaria transmission. The fundamental properties of the model was investigated, $\mathscr{R}_0$ is also obtained. Thus, when $\mathscr{R}_0 > 1$ the disease persist in the area and when $\mathscr{R}_0 < 1$ the disease will die out. Furthermore, four control strategies $u_1(t), u_2(t), u_3(t)$ and $u_4(t)$ were included to the model to reduce and control the disease in human population. According to our optimal control numerical simulation results the best strategies for reducing and controlling the number of exposed and infected humans is the use of all the four control strategies together.

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