Original Research Article

Unit Regular Elements on the Space of Transformations

Abstract. This paper presents a characterization of unit regular elements of the set of all transformations defined on a set. Also, we analyze unit regular elements of the space of continuous transformations on a topological space.

KEY WORDS: Semigroup, Regular element, Monoid, Unit regular transformation.

1 Introduction

The concept of regularity in ring theory was introduced by von Neumann [1] to clarify certain aspects of operator algebras. A stronger form of regularity, namely unit regularity, was introduced by Ehlrich [2] as a finiteness condition on rings. In fact, these concepts are essentially of the domain of semigroup theory, and regularity, especially, is extensively studied as a semigroup theoretic notion [3,4]. In recent years, transformation semigroups have been used to model dynamic systems and non-invertible processes due to the rapid development in computational algebra [5,6]. They are semigroup analogues of permutation groups, representing irreversible state changes in automata, mapping structure in algebras. In this paper, we consider unit regular elements in the space of transformations defined on an arbitrary set. Also, we give a characterization of unit regular elements among the continuous transformations defined on topological spaces.

When we compare group structure with the structure present in general semigroups they lack smooth and neat properties like existence of inverse and identity. It is in this context that we search for regions in the semigroup space where elements behave nearly like invertible elements, helping us to use group-theoretic tools in semigroups. With this background, we define regular elements in monoids (semigroups with identity).

2 Preliminary

An element t in a monoid S is said to be a *regular element* if $t = tt't$ for some $t' \in S$. An element t in S is said to be *unit regular* if $t = tut$ for some invertible element $u \in S$. A monoid in which each element is a regular element (unit regular element) is called *regular semigroup (unit regular semigroup)*. We use the notation $T(X)$ to represent the set of all transformations from the set X to itself and the the notation $R(t)$ to represent the range of $t \in T(X)$ given by $R(t) = \{y \in X : y = t(x), x \in X\}$ and kernel of transformation $t \in T(X)$ is defined by $\ker t = \{(x, x') : t(x) = t(x')\}$. The relation ρ_t defined on X by $x \rho_t x'$ if and only if $t(x) = t(x')$ is an equivalence relation on X and it partitions X into disjoint equivalent classes $\rho_t(x) = \{x' \in X : t(x) = t(x')\}$. The family of equivalence classes of the equivalence relation ρ is denoted by X/ρ. By a *cross section* of an equivalence relation ρ we mean a subset X_0 of X which contains exactly one element from each equivalence class of the equivalence relation. That is, $X_0 \cap \rho(x)$ is a singleton set. Note that the cardinality $|X_0|$ of the cross section X_0 is same as the cardinality of the family of equivalence classes X/ρ. We will be making use of the following results as and when needed.

Theorem 2.1. *Let $t : X \to Y$ be a mapping, where X and Y are non-empty sets and $\rho = ker\ t$. Let $\pi : X \to X/\rho$ be the natural projection defined by $\pi(x) = \rho(x)$. Then $\phi : X/\rho \to Y$ defined by $\phi(\rho(x)) = t(x)$ is an injection with $\phi\pi = t$.*

Theorem 2.2. *Let ρ be an equivalence relation on X and X_0 be a cross section of ρ. Let $\phi : X/\rho \to X_o$ be the associated choice function and $\pi : X \to X/\rho$ be the projection defined by $\pi(x) = \rho(x)$. Then $\pi\phi$ is the identity function on X/ρ and $e = \phi\pi : X \to X$ is an idempotent mapping.*

Theorem 2.3. *If $e : X \to X$ is an idempotent mapping, then range of e, $R(e)$, is a cross section of $\ker e$. Moreover, if $\phi : X/\rho \to R(e)$ is the associated choice function and $\pi : X \to X/\rho$ is natural projection, then $\phi\pi = e$.*

Let X be a topological space, Y be a set and $t : X \to Y$ be a surjective mapping. Then the collection $\{H \subseteq Y : t^{-1}(H) \text{ is open in X}\}$ is a topology on Y called the *quotient topology induced by t*. The space Y with this topology is called *quotient space* and the mapping t is called *quotient mapping* [7]. Let ρ be an equivalence relation on X. By a quotient topology on X/ρ, we mean the quotient topology induced by the natural projection $\pi : X \to X/\rho$ defined by $\pi(x) = \rho(x)$. A cross section X_0 of an equivalence relation ρ is called a continuous cross section of ρ if the choice function $\phi : X/\rho \to X_0$ is continuous with respect to the quotient topology on X/ρ. The following theorems are continuous analogues of theorems 2.2 and 2.3.

Theorem 2.4. *Let ρ be an equivalence relation on a topological space X and X_0 be a cross section of ρ. Let $\phi : X/\rho \to X_0$ be the associated choice function and*

$\pi : X \to X/\rho$ be the projection defined by $\pi(x) = \rho(x)$. Then $e = \phi\pi : X \to X$ is a continuous idempotent mapping.

Theorem 2.5. *If $e : X \to X$ be a continuous idempotent mapping, then range of e, $R(e)$, is a continuous cross section of ker e.*

A subspace Y of a topological space X is called a *retract* of X if there exists a continuous function $t : X \to Y$ whose restriction to Y is the identity function on Y. That is, if a topological space X has a subspace Y, which is a retract of X, then the unique identity mapping $i_Y : Y \to Y$ can be extended continuously to the function $t : X \to Y$. In this situation, the extended continuous mapping t is called a retraction on X.

3 Unit Regular Transformations

The set T(X) of all transformations from X to itself is a regular monoid [3], where X is non-empty set. But T(X) is not, in general, a unit regular semigroup. We prove the following characterization theorem of unit regular elements of T(X).

Theorem 3.1. *An element $t \in T(X)$ is unit regular if and only if there exits a cross section X_0 of kernel of t (ker t) in X with $|X - X_0| = |X - R(t)|$.*

Proof:
Assume that $t \in T(X)$ is a unit regular element. Then we can find a bijective mapping u:$X \to X$ with tut = t. Here e = ut is an idempotent mapping from X to itself. We claim that $X_0 = R(e)$ is a cross section of $\rho = ker\, t$. That is, we will prove that $X \cap \rho(x)$ is a singleton set for each $x \in X$. Let x_0 and y_0 be elements of $X_0 \cap \rho(x)$. Since $\rho(x)$ is an equivalence class from the equivalence relation ρ defined by $t(x) = t(y)$, we get that $t(x_0) = t(x) = t(y_0)$. Since $e = ut$ is an idempotent mapping, e restricted to $X_0 = R(e)$ is the identity mapping and so

$$x_0 = e(x_0) = ut(x_0) = ut(y_0) = e(y_0) = y_0$$

To prove that $|X - X_0| = |X - R(t)|$, first observe that the restriction u_0 of u to $R(t)$ is a bijection of R(t) onto X_0.

Let $y \in R(t)$. Then $y = t(x)$ for some $x \in X$. Note that

$$u_0(y) = u_0(t(x)) = u_0t(x) = ut(x) = e(x)$$

and so $u_0 : R(t) \to X_0$.

To prove that u_0 is a surjective mapping, let $x_0 \in X_0$ and $y_0 = t(x_0)$ and so $y_0 \in R(t)$. Since e is identity function on X_0, we have $u_0(y_0) = u(y_0) = u(t(x_0)) = ut(x_0) = e(x_0) = x_0$. Again, since the function u is an injective function, its restriction u_0 is also an injective function. Hence $u_0 : R(t) \to X_0$ is a bijective function.

Since u is a bijective function from X onto itself and u_0 is a bijection from $R(t)$ onto X_0, we see that the function u restricted to $X - R(t)$ is a bijection of $X - R(t)$ onto $X - X_0$. Hence $|X - X_0| = |X - R(t)|$.

Conversely, assume that there exits a cross section X_o of the equivalence relation $\rho = ker\, t$ defined by $x \rho y$ if and only if $t(x) = t(y)$ with $|X - X_0| = |X - R(t)|$ for $t \in T(X)$. We shall prove that t is a unit regular mapping.

Since $|X - X_0| = |X - R(t)|$, we can define a bijective mapping u_0 from $X - X_0$ onto $X - R(t)$. Next, we shall prove that $t_0 = t|X_0$, the restricted map of t to X_0 is a bijective mapping. To prove injection, let x_0 and y_0 be the elements of X_0 with $t(x_0) = t(y_0)$. That is,

$$x_0, y_0 \in X_0 \cap \rho(x_0) = X_0 \cap \rho(y_0)$$

Since X_0 is a cross section of the equivalence relation ρ, we get that $x_0 = y_0$. Hence, t_0 is an injective mapping from X_0 to $R(t)$. To prove that t_0 is a surjective mapping onto $R(t)$, let $y_0 \in R(t)$. Then $y_0 = t(x)$ for some $x \in X$. Since X_0 is a cross section of the equivalence relation ρ, $X_0 \cap \rho(x) = \{x_0\}$, a singleton set. Note that $t(x_0) = t(x)$ and $y_0 = t(x) = t(x_0) = t_0(x_0)$. Thus t_0 is a surjective mapping and hence it is a bijective mapping from X_0 onto $R(t)$.

Now, we define a mapping $u : X \to X$ by

$$u(x) = \begin{cases} t_0^{-1}(x) & \text{if } x \in R(t) \\ u_0(x) & \text{if } x \notin R(t) \end{cases}$$

By construction, the mapping u is a bijection. Let $x \in X$ and y = t(x). Since $y \in R(t)$, by definition of u, $u(y) = t_0^{-1}(y)$ and $t_0^{-1}(y) \in X_0$ so that

$$tut(x) = tu(t(x)) = tu(y) = t(u(y)) = t(t_0^{-1}(y)) = t_0(t_0^{-1}(y)) = t_0 t_0^{-1}(y) = y = t(x)$$

That is, $tut(x) = t(x)$. Hence t is a unit regular mapping. □

As an immediate consequence, we get the following result.

Corolary 3.1. *An injective (surjective) map which is not surjective (injective) is not unit regular.*

Proof:
In the first case, let $t \in T(X)$ be an injective mapping but not surjection. Since t is not surjective, $R(t) \neq X$ and so $|X - R(t)| \neq 0$.

But, since t is an injective mapping, each equivalence class corresponding to $ker\, t$ contains a single element and so the only cross section X_0 of $ker\, t$ is the whole set X. Then $|X - X_0| = 0$.

That is, there exists no cross section X_0 of $ker\,t$ with $|X - X_0| = |X - R(t)|$. So by theorem 3.1, t is not a unit regular element of $T(X)$.

In the second case, let $t \in T(X)$ be a surjective mapping which is not an injective mapping. Then, $|X - R(t)| = 0$.

Since t is not injective mapping, there will be at least one equivalence class of $\rho = ket\,t$ which contains more than one element of X. Note that any cross section X_0 of $ker\,t$ contains exactly one element in $X_o \cap \rho(x)$ for each $x \in X$, it follows that any cross section X_0 of $ker\,t$ is a proper subset of X and so $|X - X_0| \neq 0$. Since t is surjective mapping $R(t) = X$ and so $|X - R(t)| = 0$. That is, there exists no cross section X_0 of $ker\,t$ with $|X - X_0| = |X - R(t)|$. So by theorem 3.1, t is not a unit regular element of $T(X)$.

Now we describe a sub-family of elements of $T(X)$ in which each element is a unit regular element but the sub-family is not a unit regular semigroup.

Theorem 3.2. *Let X be a non-empty set and* $F = \{t \in T(X) : |R(t)| < \infty\}$. *Then each element of F is a unit regular element of* $T(X)$.

Proof:
Let $t \in F$. Then $t \in T(X)$ and $|R(t)| = m < \infty$. The equivalence relation ρ defined by $x\rho y$ if and only if $t(x) = t(y)$ has only a finite number m of equivalence classes. This implies that any cross section X_0 of the equivalence classes contains m number of elements. That is, $|X_0| = m = |R(t)|$. Now we prove the result in two cases.

In the first case, consider X as a finite set, $|X| = n$. Since $|X_0| = m = |R(t)|$, $|X - X_0| = n - m = |X - R(t)$. Hence by theorem 3.1, t is a unit regular element.

In the second case, consider X as an infinite set. Since X_0 and $R(t)$ are finite sets and X is an infinite set, $|X - X_0| = |X| = |X - R(t)|$. Hence by theorem 3.1, t is a unit regular element.

Using the above theorem, we give an alternate proof [8] for the following

Theorem 3.3. *Let X be non-empty set. The set* $T(X)$ *of all transformations on X is a unit regular semigroup if and only if X is a finite set.*

Proof:
Assume that X is a finite set and $t \in T(X)$. Then $|R(t)| < \infty$ and so by theorem 3.1, t is unit regular element. Since $T(X)$ is semigroup under composition of functions and also every element of $T(X)$ is unit regular, it is a unit regular semigroup.

Conversely, assume that $T(X)$ is a unit regular semigroup, where X is a non-empty set. If possible, let X be an infinite set and A be a finite non-empty subset of A. Since X is infinite, $|X - A| = |X|$. So we can define a bijective mapping $t : X \to X - A$. Then $t : X \to X$ is injective function but not a surjection and so

t can not be a unit regular mapping, by corollary 3.1. Hence X must be a finite set.

4 Continuous Unit Regular Transformations

In this section, we provide a characterization of unit regular elements in the semigroup of all continuous transformations from a topological space X to itself. We use the notation $CT(X)$ to denote the semigroup of all continuous transformations from a topological space X to itself. In general, $CT(X)$ is not a regular semigroup. We need the following results to characterize unit regular elements of $CT(X)$.

Lemma 4.1. *Every continuous cross section of an equivalence relation on a Hausdorff space is a closed set.*

Proof:
Let X_0 be a continuous cross section of an equivalence relation ρ on Hausdorff space X. Since X_0 is a continuous cross section of ρ, the choice function $\phi : X/\rho \to X_0$ defined by $\phi(x) = \rho(x)$ is continuous under the quotient topology on X/ρ, $\rho(x)$ is the equivalence class containing X. Let $\pi : X \to X/\rho$ defined by $\pi(x) = \rho(x)$ be natural projection. Then $e = \phi\pi$ is an idempotent mapping on X by theorem 2.2. Since ϕ and π are continuous $e = \phi\pi$ is also continuous. Observe that

$$e(X) = \phi(\pi(X)) = \phi(X/\rho) = X_0$$

We claim that X_0 is a closed set in X. Let $(x_\lambda : \lambda \in \Lambda)$ be a net in X_0 which converges to $x \in X$, where Λ is a directed set. Since $e : X \to X_0$ is continuous, the net $(e[x_\lambda] : \lambda \in \Lambda)$ in X_0 converges to $e(x)$ in X. Note that $e(x_\lambda) = x_\lambda$ as e restricted to X_0 is identity mapping on X_0. Also, note that $e(x) \in X_0$. That is, the net $(x_\lambda : \lambda \in \Lambda) = (e[x_\lambda] : \lambda \in \Lambda)$ converges to $x \in X$ and $e(x) \in X_0 \subset X$. Since X is a Hausdorff space, $e(x) = x$ and so $x \in X_0$. Hence X_0 is closed subset of X as every convergent net in X_0 converges in X_0 itself.

The following theorem is a characterization of regular elements in $CT(X)$ given in [9].

Theorem 4.1. *Let X be a topological space and $t \in CT(X)$. Then t is regular if and only if the following conditions are satisfied:*

1. *t is a quotient map onto $R(t)$*
2. *$R(t)$ is a retract of X*
3. *$\ker t$ has a continuous cross section*

Now we shall give a characterization of unit regular elements of $CT(X)$

Theorem 4.2. *Let X be a Hausdorff space and $t \in CT(X)$. Then t is unit regular if and only if the following conditions are satisfied:*

1. *$R(t)$ is closed*
2. *t is a quotient map onto $R(t)$*
3. *there exists a continuous cross section X_0 of $ker\, t$ and a homeomorphism from $\overline{X - X_0}$ to $\overline{X - R(t)}$ which coincides with t on the boundary of X_0.*

Proof:
Assume that $t \in CT(X)$ is a unit regular element. Then t is a regular element and so $R(t)$ is a retract of X and t is a quotient map onto $R(t)$ by theorem 4.1. Since X is a Hausdorff space, $R(t)$ is closed. By definition of unit regularity there exists a homeomorphism $u : X \to X$ such that $tut = t$. Then $e = ut$ is a continuous idempotent map from X to X and $X_0 = R(e)$ is a continuous cross section of $ker\, e$ by theorem 2.5. Also,

$$\begin{aligned}(x, y) \in ker\, e &\iff e(x) = e(y)\\ &\iff ut(x) = ut(y)\\ &\iff t(x) = t(y)\\ &\iff (x,\ y) \in ker\, t\end{aligned}$$

This shows that $ker\, e = ker\, t$ and so X_0 is a continuous cross section of $ker\, t$. Since u is a homeomorphism,

$$\begin{aligned}u\left(\overline{X - R(t)}\right) &= \overline{u(X - R(t))}\\ &= \overline{(u(X) - u(R(t))}\\ &= \overline{(X - R(ut))}\\ &= \overline{X - R(e)}\\ &= \overline{X - X_0}\end{aligned}$$

Therefore, u maps $\overline{X - R(t)}$ homeomorphically onto $\overline{X - X_0}$. Hence u^{-1} maps $\overline{X - X_0}$ homeomorphically onto $\overline{X - R(t)}$. If $bd(X)$ is the boundary of the set X, then:

$$bd(X_0) = \overline{X_0} \cap \overline{X - X_0} = X_0 \cap \overline{X - X_0}$$

as X_0 is closed, by lemma 4.1. Since $bd(X_0) \subset X_0$, and if $x \in bd(X_0)$, then:

$$ut(x) = e(x) = x$$

So $t(x) = u^{-1}(x)$. That is, the mapping t and the homeomorphism u^{-1} coincides on the boundary of X_0.

Conversely, assume that t satisfies the conditions (1), (2), and (3) of the theorem 4.2. Let $u_0 : \overline{X - X_0} \to \overline{X - R(t)}$ which coincides with t on $bd(X_0)$ and $t_0 = t|X_0$. Then t_0 is a bijection from X_0 onto $R(t)$, by theorem 3.1.

Let $\rho = ker\, t$ and $\phi : X/\rho \to X_0$ be the choice function and $\pi : X \to X/\rho$ be the projection mapping. Then $e = \phi\pi$ is a continuous idempotent map from X to itself. So for each $x \in X$, $e(x) = x_0$, where x_0 is the unique element in $\rho(x) \cap X_0$. Also, we have

$$t_0 e(x) = t_0(x_0) = t(x_0) = t(x)$$

as $x_0 \in \rho(x)$. Thus $e(x) = t_0^{-1}t(x)$ for each $x \in X$ and so $e = t_0^{-1}t$. Since e is continuous and t is a quotient map, it follows that t_0^{-1} is continuous. Hence t_0 is a homeomorphism of X_0 onto $R(t)$.

Define $v : X \to X$ by

$$v(x) = \begin{cases} u_0(x) & \text{if } x \in \overline{X - X_0} \\ t_0(x) & \text{if } x \in \overline{X_0} = X_0 \end{cases}$$

Since u_0 and t coincide on the $bd(X_0)$, the function v is a bijection and it is continuous by the pasting lemma.

Now we claim that u_0^{-1} coincides with t_0^{-1} on the boundary of $R(t)$.

Let $y \in bd(R(t)) = \overline{R(t)} \cap \overline{X - R(t)}$. Since $R(t)$ is closed, $\overline{R(t)} = R(t)$ and so $y = t_0(x)$ for some $x \in X_0$. Also, as $y \in \overline{X - R(t)}$, $y = u_0(t)$ for some $t \in \overline{X - X_0}$. Now $x \in X_0$ and $t \in \overline{X - X_0}$ implies $v(x) = v(t)$. Since v is an injective function, we get $x = t$. That is, $t_0^{-1}(y) = x = t = u_0^{-1}(y)$. Thus, the claim is proved.

So, the mapping $u : X \to X$ defined by

$$u(x) = \begin{cases} u_0^{-1}(x) & \text{if } x \in \overline{X - R(t)} \\ t_0^{-1}(x) & \text{if } x \in \overline{R(t)} = R(t) \end{cases}$$

is a bijection and is continuous by the pasting lemma. Hence u is a homeomorphism. Finally we prove that $tut = t$. Let $x \in X$.

Then, as $y = t(x) \in R(t)$, $\quad tut(x) = tu(y) = tt_0^{-1}(y)$.

Now $tt_0^{-1}(y) = t(t_0^{-1}(y)) = t_0(t_0^{-1}(y)) = y$ as $t_0^{-1}(y) \in X_0$.

Thus $tut(x) = y = t(x)$ for all $x \in X$. That is, t is a unit regular mapping.

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