Edge Induced V_4-Magic Labeling of Line graphs

Abstract

Let $V_4 = \{0, a, b, c\}$ be the Klein-4-group with identity element 0 and $G = (V(G), E(G))$ be a graph. Let $f : E(G) \to V_4 \setminus \{0\}$ be an edge labeling and $f^+ : V(G) \to V_4$ denotes the induced vertex labeling of f defined by $f^+(u) = \sum_{uv \in E(G)} f(uv)$ for all $u \in V(G)$. Then f^+ again induces an edge labeling $f^{++} : E(G) \to V_4$ defined by $f^{++}(uv) = f^+(u) + f^+(v)$, for all $uv \in E(G)$. A graph $G = (V(G), E(G))$ is said to be an edge induced V_4-magic graph, if there exists an edge labeling f for which the function f^{++} is a constant function. The function f, so obtained is called an Edge Induced V_4-Magic Labeling (EIML) of G. The present paper deals with basic results regarding EIML of line graphs and the characterization of EIML of line graph of certain named graphs.

AMS classification: 05C78, 05C25.
Keywords: Klein-4-group, Edge induced V_4-magic graphs, Line graph of a graph.

1 Introduction

The present paper intends to deal exclusively with simple, connected, finite and undirected graphs. Also note that, the Klein 4-group is denoted by $V_4 = \{0, a, b, c\}$ which is a non cyclic abelian group of order 4 with every non identity element has order 2. Let $G = (V(G), E(G))$ be the graph with vertex set $V(G)$ and edge set

$E(G)$. The reader may check [5] and [1] for the standard terminology and notations related to Graph theory.

Let $f : E(G) \to V_4 \setminus \{0\}$ be an edge labeling and $f^+ : V(G) \to V_4$ denotes the induced vertex labeling of f defined by $f^+(u) = \sum_{uv \in E(G)} f(uv)$ for all $u \in V(G)$. Then f^+ again induces an edge labeling, say, $f^{++} : E(G) \to V_4$ defined by $f^{++}(uv) = f^+(u) + f^+(v)$, for all $uv \in E(G)$. A graph $G = (V(G), E(G))$ is said to be an edge induced V_4-magic graph, if there exists an edge labeling f for which $f^{++}(e)$ is a constant for all $e \in E(G)$. If this constant is x, then x is said to be the induced edge sum of the graph G and the function f, so obtained is called an edge induced V_4-magic labeling (EIML) of G. This paper aims to discuss edge induced V_4-magic labeling of line graph of some graphs which belong to the following categories:

(i) $\sigma_a(V_4) :=$ Set of all edge induced V_4-magic graphs with edge induced magic labeling f satisfying $f^{++}(u) = a$ for all $u \in V$.

(ii) $\sigma_0(V_4) :=$ Set of all edge induced V_4-magic graphs with edge induced magic labeling f satisfying $f^{++}(u) = 0$ for all $u \in V$.

(iii) $\sigma(V_4) := \sigma_a(V_4) \bigcap \sigma_0(V_4)$.

2 PRELIMINARIES

Definition 2.1. [2] The corona $P_n \circ K_1$ is called the comb graph CB_n.

Definition 2.2. [3] A flag graph is denoted by Fl_n and it is obtained by joining one vertex of C_n to an extra vertex called the root.

Definition 2.3. [3] The sun graph on $m = 2n$ vertices, denoted by Sun_n, is the graph obtained by attaching a pendant vertex to each vertex of a $n-$ cycle.

Definition 2.4. *The bistar $B_{m,n}$ is the graph obtained by joining the central or apex vertex of $K_{1,m}$ and $K_{1,n}$ by an edge.*

Definition 2.5. [3] A triangular snake graph TS_n is obtained from a path $v_1, v_2, v_3 \cdots, v_n$ by joining v_i and v_{i+1} to a new vertex w_i for $i = 1, 2, 3, \cdots, n-1$.

Theorem 2.6. [4]Let $G = (V, E)$ be a graph with either each vertex is of odd degree or even degree then $G \in \sigma_0(V_4)$.

Theorem 2.7. [4] ***(Induced edge sum theorem)***.
For any graph G, f is an edge induced V_4-Magic labeling of G if and only if the induced edge sum

$$x = f^{++}(uv) = \sum_{u\alpha\in E,\ \alpha\neq v} f(u\alpha) + \sum_{\beta v\in E,\ \beta\neq u} f(\beta v),\ \textit{for all } (u,v)\in E \qquad (1)$$

The Equation (1) corresponding to an edge uv in G, is called induced edge sum equation of the edge uv.

Theorem 2.8. [4] For the path graph P_n, we have the following:

(i) $P_2 \in \sigma_0(V_4)$ and $P_2 \notin \sigma_a(V_4)$.

(ii) $P_3 \in \sigma_a(V_4)$ and $P_3 \notin \sigma_0(V_4)$.

(iii) $P_4 \in \sigma_a(V_4)$ and $P_4 \notin \sigma_0(V_4)$.

(iv) P_n is not an edge induced magic graph for any $n \geq 5$.

Theorem 2.9. [4] For the cycle graph C_n, we have the following:

(i) $C_n \in \sigma_0(V_4)$ for all n.

(ii) $C_n \in \sigma_a(V_4)$ if and only if n is a multiple of 4.

Theorem 2.10. [4] Let K_n be the complete graph with n vertices, then $K_n \in \sigma_0(V_4)$ for all n.

Definition 2.11. [5] Let G be a graph, then the line Line graph graph of G is denoted by $L(G)$ and it is a graph whose vertex set is in $1-1$ correspondence with the edge set of G and two vertices of $L(G)$ are joined by an edge if and only if the corresponding edges of G are adjacent in G.

3 MAIN RESULTS

Theorem 3.1. *For the line graph of a path graph P_n, we have the following:*

(i) $L(P_3) \in \sigma_0(V_4)$ *and* $L(P_3) \notin \sigma_a(V_4)$.

(ii) $L(P_4) \in \sigma_a(V_4)$ *and* $L(P_4) \notin \sigma_0(V_4)$.

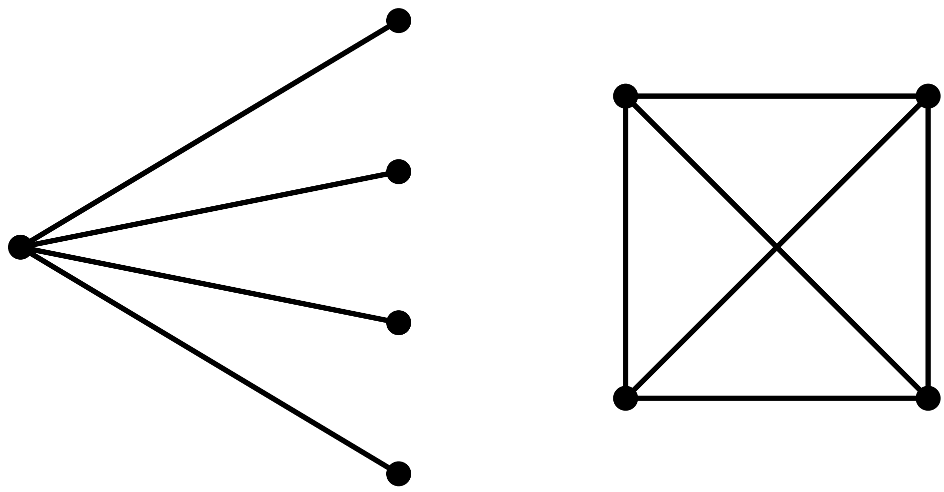

Fig. 1. $K_{1,4}$ **Graph and its line graph**

(iii) $L(P_5) \in \sigma_a(V_4)$ *and* $L(P_5) \notin \sigma_0(V_4)$.

(iv) $L(P_n)$ *is not an edge induced magic graph for any* $n \geq 6$.

Proof. Consider the path graphs,

(i) Since $L(P_3) = P_2$, proof of (i) follows directly from Theorem 2.8 (i).

(ii) Since $L(P_4) = P_3$ proof of (ii) follows directly from Theorem 2.8 (ii).

(iii) Since $L(P_5) = P_4$, proof of (iii) follows directly from Theorem 2.8 (iii).

(iv) Note that $L(P_n) = P_{n-1}$. Thus if $n \geq 6$, then $L(P_n) = P_{n-1}$ and $n - 1 \geq 5$, therefore the proof of (iv) follows from Theorem 2.8 (iv).

□

Corollary 3.2. $L(P_n) \notin \sigma(V_4)$ *for any* $n \geq 3$.

Proof. Proof of the corollary follows from Theorem 3.1. □

Theorem 3.3. $L(C_n) \in \sigma_0(V_4)$ *for all* n.

Proof. Since $L(C_n) = C_n$, the proof follows from Theorem 2.9 (i). □

Theorem 3.4. $L(C_n) \in \sigma_a(V_4)$ *if and only if* n *is a multiple of* 4.

Proof. Since $L(C_n) = C_n$, the proof follows from Theorem 2.9 (ii). □

Corollary 3.5. *$L(C_n) \in \sigma(V_4)$ if and only if n is a multiple of 4.*

Proof. The proof follows from Theorem 3.3 and Theorem 3.4. □

Theorem 3.6. *Let $K_{1,n}$ be the star graph, then $L(K_{1,n}) \in \sigma_0(V_4)$ for all n.*

Proof. Since $L(K_{1,n}) = K_n$, the proof follows from Theorem 2.10. □

Theorem 3.7. *Let CB_n be the comb graph, then we have the following.*

(i) *$L(CB_n) \notin \sigma_0(V_4)$ for any n.*

(ii) *$L(CB_n) \notin \sigma_a(V_4)$ for any n.*

Proof. Let $\{u_i, v_i : i = 1, 2, 3, \dots, n\}$ be the vertex set of CB_n, where u_i is the pendant vertex adjacent to v_i. Also let $w_i = u_i v_i$, $i = 1, 2, 3, \dots, n$ and $t_k = u_k u_{k+1}$, $k = 1, 2, 3, \dots, n-1$ be the edges in CB_n. Then $\{w_i,\ t_k : i = 1, 2, 3, \dots, n,\ k = 1, 2, 3, \dots, n-1\}$ are the vertices of $L(CB_n)$.

(i) : If possible, suppose $L(CB_n) \in \sigma_0(V_4)$ for some n. Then there exists an EIML say $f : E(L(CB_n)) \to V_4 \setminus \{0\}$ with $f^{++}(e) = 0$ for all $e \in E(L(CB_n))$. Let $f(t_i w_i) = \alpha_i$, $f(t_i w_{i+1}) = \beta_i$, for $i = 1, 2, 3, \dots, n-1$ and $f(t_j t_{j+1}) = \gamma_j$, for $j = 1, 2, 3, \dots, n-2$. Then the induced edge sum equation of the edges $t_i w_i$ gives the equation,

$$\begin{aligned}
\gamma_1 + \beta_1 &= \gamma_1 + \beta_1 + \gamma_2 + \beta_2 \\
&= \gamma_2 + \beta_2 + \gamma_3 + \beta_3 \\
&\cdots \\
&= \gamma_{n-3} + \beta_{n-3} + \gamma_{n-2} + \beta_{n-2} \\
&= \gamma_{n-2} + \beta_{n-2} + \beta_{n-1}.
\end{aligned}$$

But since $f^{++} \equiv 0$, we get $\gamma_1 + \beta_1 = 0$, using this fact in the above system of equations, we get

$$\gamma_1 + \beta_1 = \gamma_2 + \beta_2 = \gamma_3 + \beta_3 = \cdots = \gamma_{n-2} + \beta_{n-2} = 0.$$

That is $\gamma_i = \beta_i$ for $i = 1, 2, 3, \dots, n-2$. Thus using $\gamma_{n-2} = \beta_{n-2}$ in the equation $\gamma_{n-2} + \beta_{n-2} + \beta_{n-1} = 0$, we get $\beta_{n-1} = 0$. That is $f(t_{n-1} w_n) = 0$, which is a contradiction. Hence our assumption is wrong, that is $L(CB_n) \notin \sigma_0(V_4)$ for all n.

(ii) : If possible, suppose $L(CB_n) \in \sigma_a(V_4)$ for some n. Then there exists an EIML say $g : E(L(CB_n)) \to V_4 \setminus \{0\}$ with $g^{++}(e) = a$ for all $e \in E(L(CB_n))$.
Let $g(t_i w_i) = \alpha_i$, $g(t_i w_{i+1}) = \beta_i$, for $i = 1, 2, 3, \ldots, n-1$ and $g(t_j t_{j+1}) = \gamma_j$, for $j = 1, 2, 3, \ldots, n-2$.
Then the induced edge sum of the edge $t_1 w_1$ and $t_2 w_2$ gives $\gamma_1 + \beta_1 = \gamma_1 + \beta_1 + \gamma_2 + \beta_2 = a$. Thus we get $\gamma_1 + \beta_1 = a$ and $\gamma_2 + \beta_2 = 0$.
Similarly the induced edge sum of the edges $t_1 w_2$ and $t_1 t_2$ gives $\alpha_1 + \alpha_2 + \gamma_1 = \alpha_1 + \alpha_2 + \beta_1 + \beta_2 + \gamma_2$. Since $\gamma_2 + \beta_2 = 0$ the above equation reduces to $\alpha_1 + \alpha_2 + \gamma_1 = \alpha_1 + \alpha_2 + \beta_1$ and which implies that $\gamma_1 = \beta_1$. That is $\gamma_1 + \beta_1 = 0$, which is contradiction. Hence our assumption is wrong, that is $L(CB_n) \notin \sigma_a(V_4)$ for any n.

Hence the Proof. □

Theorem 3.8. *For the flag graph Fl_n, we have $L(Fl_n) \notin \sigma_a(V_4)$ for any n.*

Proof. Let $V(Fl_n) = \{w, v_1, v_2, v_3, \ldots, v_n\}$, where $v_1, v_2, v_3, \ldots, v_n$ are the vertices of corresponding cycle graph C_n and w is the root vertex adjacent to the vertex v_1. Also suppose $e_i = v_i v_{i+1}$ and $e = v_1 w$ are the edges in Fl_n. Therefore we can take the vertex set of $L(Fl_n)$ equal to $\{e, e_1, e_2, e_3, \ldots, e_n\}$.
If possible, suppose $L(Fl_n) \in \sigma_a(V_4)$ for some n. Then there exists an EIML say $f : E(L(Fl_n)) \to V_4 \setminus \{0\}$ with $f^{++}(e) = a$ for all $e \in E(L(Fl_n))$.
Let $f(e_i e_{i+1}) = \alpha_i$, for $i = 1, 2, 3, \ldots, n$ with $i+1$ is taken modulo n, $f(e_1 e) = \alpha$ and $f(e_n e) = \beta$. Then the induced edge sum equation of the edges $e_n e_1$, $e_1 e$, and $e_n e$ gives the equation:

$$\alpha_{n-1} + \alpha_1 + \alpha + \beta = \alpha_n + \alpha_1 + \beta = \alpha_{n-1} + \alpha_n + \alpha = a.$$

but $\alpha_n + \alpha_1 + \beta = \alpha_{n-1} + \alpha_n + \alpha$ implies $\alpha + \beta = \alpha_1 + \alpha_{n-1}$, thus $\alpha_{n-1} + \alpha_1 + \alpha + \beta = 0$, which is a contradiction. Hence our assumption is wrong, that is $L(Fl_n) \notin \sigma_a(V_4)$ for any n.
Hence the Proof. □

Theorem 3.9. *For the sun graph Sun_n, we have $L(Sun_n) \in \sigma_0(V_4)$ for all n.*

Proof. Note that in $L(Sun_n)$ every vertex is of even degree. Therefore by Theorem 2.6, we have $L(Sun_n) \in \sigma_0(V_4)$ for all n. □

Theorem 3.10. *Consider bistar graph $B_{m,n}$, then $L(B_{m,n}) \in \sigma_0(V_4)$ for m and n are even.*

Proof. Note that for m and n are even, every vertex in $L(B_{m,n})$ is of even degree. Therefore by Theorem 2.6, we have $L(B_{m,n}) \in \sigma_0(V_4)$ for all n. □

Theorem 3.11. *For the triangular snake graph TS_n, we have $L(TS_n) \in \sigma_0(V_4)$ for all n.*

Proof. Since every vertex in $L(TS_n)$ is of even degree, by Theorem 2.6 the proof follows. □

4 Conclusion

This paper has attempted to investigate the key results pertaining to edge-induced V4- magic labeling of line graphs of for a certain class of graphs, such as P_n, C_n, $K_{1,n}$, CB_n, Fl_n, Sun_n, $B_{m,n}$, and TS_n.

References

[1] Bondy, J. A., & Murty, U. S. R. *Graph Theory.* Springer, Berlin.(2008).

[2] Frank Harary *Graph theory*, Narosa Publishing House.

[3] Joseph A. Gallian, *A dynamic survey of labeling.* The Electronic Journal of Combinatorics (2018).

[4] K. B. Libeeshkumar and V. Anil Kumar, *Edge Induced V_4- Magic Labeling of Graphs,* Malaya Journal of Matematik, Vol. 8, No. 4, 1916-1921, (2020).

[5] R Balakrishnan and K Ranganathan, *A text book of graph theory.* Springer-Verlag, NewYork (2012).