On Reverse Sombor-Based Indices and Polynomial Analysis of Quadrilateral Snake Graphs

Abstract

Topological indices are widely used to describe the structural characteristics of graphs through numerical values. In this work, several variants of quadrilateral snake graphs are considered. For these graph classes, reverse degree-based indices including the reverse Sombor index, reverse Elliptic Sombor index, reverse Euler Sombor index, and reverse Harmonic index are computed using edge partition techniques. In addition, the reverse Sombor polynomial is obtained to give a more detailed representation of the graph structure. Exact results are derived for all cases and supported with examples. These findings add to the study of topological indices and help in understanding the structure of such graphs.
Keywords: Reverse Sombor index; Reverse Elliptic Sombor index; Reverse Euler Sombor index; Reverse Harmonic index; Reverse Sombor polynomial; Quadrilateral snake graph; Degree-based descriptors.

1 Introduction

Topological indices are numerical quantities that capture structural features of a graph by considering how its vertices are connected [1]-[3]. Over the years, researches worldwide have proposed numerous indices, each designed with particular goals and computational approaches in mind. The measures play a significant role in Quantity Structure-Property Relationship (QSPR) and Quantity Structure-Activity Relationship (QSAR) studies and have been widely applied in areas such as environmental analysis, pharmaceutical sciences, drug development, and materials engineering. In addition to these established applications, current research continues to identify new domains where such indices can offer valuable interpretations. Representing chemical compounds as graphs allows researchers to estimate molecular properties without relying on complex experimental procedures, thereby saving time, cost and effort [4]-[5].

A foundational advancement in this field was achieved by Harold Wiener, who introduced the wiener index, a concept that became central to chemical graph theory [8, 9]. This distance-based index effectively relates molecular structures to physicochemical properties, including boiling and melting points of alkanes. Although many new indices have since been introduced, the wiener index continues to serve as an important benchmark for evaluating newly developed descriptors. Building on this early contribution, a broad range of topological indices has been formulated and applied across disciplines such as materials science, medicinal chemistry, and pharmaceutical research [10]-[15].

In more recent developments, Ivan Gutman proposed the Sombor index in 2020 [16], which quickly attached attention and has been extensively investigated in subsequent studies [17]-[22]. This work led to the introduction of several related variants, including the Sombor Polynomial. the Elliptic Sombor index [23], and the Euler Sombor index [24]. Inspired by these advancements, researchers have further explored reverse degree-based adaptations of Sombor-type indices, such as the reverse Euler sombor index, and examined their structural properties across different classes of graphs[25]. Snake graphs have attracted attention for their structural importance in graph theory. P.Mahalank et al. (2021) [26] studied Zagreb indices, while Bhairaba Kumar Majhi et al. (2024)[27] introduces Revan indices and their polynomials. Chitra Ramaprakash (2024)[28] examined coloring aspects, whereas Gerand Rozario Joseph et al. (2025)[29] focused on Fibonacci prime labelling. More recently, Saranya K.M. and Manimekalai S (2025)[30] proposed Revan indices with polynomial formulations and computational implementation. Motivated by existing studies, this work investigates several variants of snake graphs, including quadrilateral, alternate quadrilateral, double quadrilateral, alternate double quadrilateral and cycle quadrilateral snake graphs. The novelty lies in systematically deriving reverse degree-based descriptors reverse Sombor, reverse Euler Sombor, reverse Elliptic Sombor, and reverse Harmonic indices for these graph classes. In addition, the reverse Sombor polynomial is formulated to provide a more detailed structural representation. The results enrich the theoretical development of reverse topological indices and offer insights into the structural properties of these graphs.

2 Basic Notions and Reverse Sombor Indices

Let $G = (V(G), E(G))$ be a finite, simple, and connected graph with vertex set $V(G)$ and edge set $E(G)$. The degree of a vertex $x \in V(G)$ is denoted by $d_G(x)$, and the maximum degree of G is written as $\Delta(G)$.

For each vertex $x \in V(G)$, the reverse degree is defined by

$$c_G(x) = \Delta(G) - d_G(x) + 1.$$

The reverse Sombor index of G is expressed as

$$RSO(G) = \sum_{xy \in E(G)} \sqrt{c_G(x)^2 + c_G(y)^2}.$$

The reverse Euler Sombor index [24] is given by

$$REUSO(G) = \sum_{xy \in E(G)} \sqrt{c_G(x)^2 + c_G(y)^2 + c_G(x)c_G(y)}.$$

The reverse Elliptic Sombor index [23] is defined as

$$RESO(G) = \sum_{xy \in E(G)} (c_G(x) + c_G(y))(\sqrt{c_G(x)^2 + c_G(y)^2}).$$

The reverse Harmonic index is written as

$$RH(G) = \sum_{xy \in E(G)} \frac{2}{c_G(x) + c_G(y)}.$$

The reverse Sombor polynomial [31] of G is expressed as

$$RSO(G) = \sum_{xy \in E(G)} \frac{1}{c_G(x)^2 + c_G(y)^2} X^{\left[c_G(x)^2 + c_G(y)^2\right]}.$$

Here, x and y denote the vertices of graph G, and X is a variable used to define the polynomial.

3 Results and Discussion

3.1 Quadrilateral Snake Graph

A Quadrilateral snake graph (QS_n) is derived from a path $v_1, v_2, v_3, v_4, ..., v_n$ by connecting new vertices u_i and w_i to v_i and v_{i+1} respectively, and then connecting u_i and w_i see [30]. let n denote the number of quadrilateral units in the graph. Hence, the Quadrilateral Snake Graph (QS_n) has $|V| = 3n + 1$ vertices and $|E| = 4n$ edges.

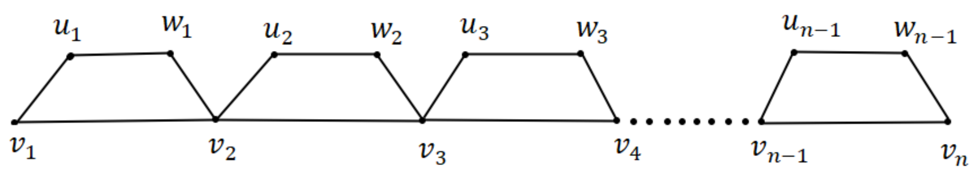

Figure 1: Quadrilateral Snake Graph. Adapted from[30]

Theorem 3.1. *Let G be the Quadrilateral snake graph $Q(S_n)$. Then*

$$\begin{aligned}
RSO(G) &= 3\sqrt{2}(n+2) + 2\sqrt{10}n + \sqrt{2}(n-2)\\
REUSO(G) &= 3\sqrt{3}(n+2) + 2\sqrt{13}n + \sqrt{3}(n-2)\\
RESO(G) &= 18\sqrt{2}(n+2) + 8\sqrt{10}n + 2\sqrt{2}(n-2)\\
RH(G) &= \frac{n+2}{3} + 2(n-1)\\
RSO(G,X) &= \frac{1}{8(5n+4)}X^{8(5n+4)}
\end{aligned}$$

Proof. Let G be the quadrilateral snake graph $Q(S_n)$ with $|V(G)| = 3n + 1$ and $|E(G)| = 4n$. The edge set can be partitioned as,

$$\begin{aligned}
E_1 &= \{xy \in E(G) | d_G(x) = d_G(y) = 2\}, |E_1| = n+2\\
E_2 &= \{xy \in E(G) | d_G(x) = 2, d_G(y) = 4\}, |E_2| = 2n\\
E_3 &= \{xy \in E(G) | d_G(x) = d_G(y) = 4\}, |E_3| = n-2
\end{aligned}$$

Here $\Delta(G) = 4$ and then, $c_G(x) = 5 - d_G(x)$

$$\begin{aligned}
CE_1 &= \{xy \in E(G) | c_G(x) = c_G(y) = 3\}, |CE_1| = n+2\\
CE_2 &= \{xy \in E(G) | c_G(x) = 3, c_G(y) = 1\}, |CE_2| = 2n\\
CE_3 &= \{xy \in E(G) | c_G(x) = c_G(y) = 1\}, |CE_3| = n-2
\end{aligned}$$

Using the reverse edge partition of graph $Q(S_n)$ and indices formula, we get

$$\begin{aligned}
RSO(G) &= (n+2)\sqrt{3^2+3^2} + 2n\sqrt{3^2+1^2} + (n-2)\sqrt{1+1}\\
&= 3\sqrt{2}(n+2) + 2\sqrt{10}n + \sqrt{2}(n-2)\\
REUSO(G) &= (n+2)\sqrt{3^2+3^2+3.3} + 2n\sqrt{3^2+1^2+3} + (n-2)\sqrt{1+1+1}\\
&= 3\sqrt{3}(n+2) + 2\sqrt{13}n + \sqrt{3}(n-2)\\
RESO(G) &= (n+2)(3+3)\sqrt{3^2+3^2} + 2n(3+1)\sqrt{3^2+1^2} + (n-2)(1+1)\sqrt{1+1}\\
&= 18\sqrt{2}(n+2) + 8\sqrt{10}n + 2\sqrt{2}(n-2)
\end{aligned}$$

$$\begin{aligned} RH(G) &= \frac{2(n+2)}{3+3} + \frac{2(2n)}{3+1} + \frac{2(n-2)}{1+1} \\ &= \frac{n+2}{3} + 2(n-1) \\ RSO(G,X) &= \frac{1}{(n+2)(18) + 2n(10) + (n-2)(2)} X^{[(n+2)(18)+2n(10)+(n-2)(2)]} \\ &= \frac{1}{8(5n+4)} X^{8(5n+4)} \end{aligned}$$

□

Example: For $n = 2$ in the Quadrilateral Snake Graph (QS_n), we have $|V| = 7$ vertices and $|E| = 8$ edges. Then, the indices are given by $RSO(G) = 12\sqrt{2} + 4\sqrt{10}$, $REUSO(G) = 12\sqrt{3} + 4\sqrt{13}$, $RESO(G) = 72\sqrt{2} + 16\sqrt{10}$, $RH(G) = \frac{10}{3}$, and $RSO(G,X) = \frac{1}{112} X^{112}$.

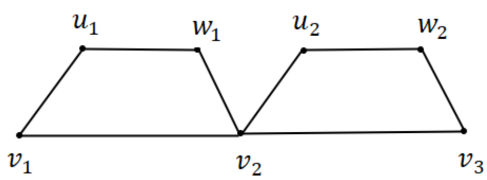

Figure 2: Quadrilateral snake graph (QS_2). Adapted from[30]

3.2 Alternating Quadrilateral Snake Graph

An Alternate Quadrilateral snake graph is derived from a path $v_1, v_2, v_3, v_4, ..., v_n$ by connecting new vertices u_i and w_i to v_i and v_{i+1}(alternatively) respectively, and then connecting u_i and w_i. It is denoted by AQS_n see [30]. let n denote the number of alternating quadrilateral units in the graph. Hence, the Alternating Quadrilateral Snake Graph (AQS_n) has $|V| = 4(n + 1)$ vertices and $|E| = 5n + 4$ edges.

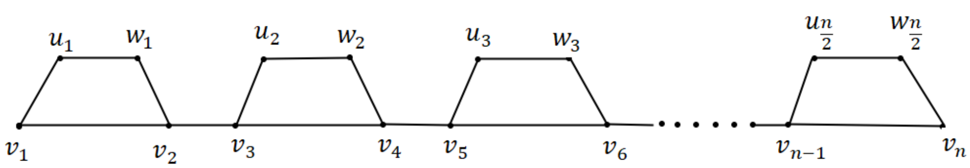

Figure 3: Alternating Quadrilateral Snake Graph. Adapted from[30]

Theorem 3.2. *Let G be the alternating quadrilateral snake graph (AQS_n). Then*

$$\begin{aligned} RSO(G) &= 2\sqrt{2}(n+3) + 2\sqrt{5}(n+1) + \sqrt{2}(2n-1) \\ REUSO(G) &= 2\sqrt{3}(n+3) + 2\sqrt{7}(n+1) + \sqrt{3}(2n-1) \\ RESO(G) &= 8\sqrt{2}(n+3) + 6\sqrt{5}(n+1) + 2\sqrt{2}(2n-1) \\ RH(G) &= \frac{n+3}{2} + \frac{4(n+1)}{3} + 2n - 1 \\ RSO(G,X) &= \frac{1}{2(11n+16)} X^{2(11n+16)} \end{aligned}$$

Proof. Let G be the quadrilateral snake graph $AQ(S_n)$ with $|V| = 4(n+1)$ vertices and $|E| = 5n+4$ edges. The edge set can be partitioned as,

$$\begin{aligned} E_1 &= \{xy \in E(G)|d_G(x) = d_G(y) = 2\}, |E_1| = n+3 \\ E_2 &= \{xy \in E(G)|d_G(x) = 2, d_G(y) = 3\}, |E_2| = 2(n+1) \\ E_3 &= \{xy \in E(G)|d_G(x) = d_G(y) = 3\}, |E_3| = 2n-1 \end{aligned}$$

Here $\Delta(G) = 3$ and then, $c_G(x) = 4 - d_G(x)$

$$\begin{aligned} CE_1 &= \{xy \in E(G)|c_G(x) = c_G(y) = 2\}, |CE_1| = n+3 \\ CE_2 &= \{xy \in E(G)|c_G(x) = 2, c_G(y) = 1\}, |CE_2| = 2(n+1) \\ CE_3 &= \{xy \in E(G)|c_G(x) = c_G(y) = 1\}, |CE_3| = 2n-1 \end{aligned}$$

Using the reverse edge partition of graph $AQ(S_n)$ and indices formula, we get

$$\begin{aligned} RSO(G) &= (n+3)\sqrt{2^2+2^2} + 2(n+1)\sqrt{2^2+1} + (2n-1)\sqrt{1+1} \\ &= 2\sqrt{2}(n+3) + 2\sqrt{5}(n+1) + \sqrt{2}(2n-1) \\ REUSO(G) &= (n+3)\sqrt{2^2+2^2+2.2} + 2(n+1)\sqrt{2^2+1+2.1} + (2n-1)\sqrt{1+1+1} \\ &= 2\sqrt{3}(n+3) + 2\sqrt{7}(n+1) + \sqrt{3}(2n-1) \\ RESO(G) &= (n+3)(2+2)\sqrt{2^2+2^2} + 2(n+1)(2+1)\sqrt{2^2+1} + (2n-1)(1+1)\sqrt{1+1} \\ &= 8\sqrt{2}(n+3) + 6\sqrt{5}(n+1) + 2\sqrt{2}(2n-1) \\ RH(G) &= \frac{2(n+3)}{4} + \frac{4(n+1)}{3} + \frac{2(2n-1)}{2} \\ &= \frac{n+3}{2} + \frac{4(n+1)}{3} + 2n - 1 \\ RSO(G,X) &= \frac{1}{(n+3)(8)+2(n+1)(5)+(2n-1)(2)} X^{[(n+3)(8)+2(n+1)(5)+(2n-1)(2)]} \\ &= \frac{1}{2(11n+16)} X^{2(11n+16)} \end{aligned}$$

□

Example: For $n = 1$ in the Alternating Quadrilateral Snake Graph (AQS_n), we have $|V| = 8$ vertices and $|E| = 9$ edges. Then, the indices are given by $RSO(G) = 9\sqrt{2} + 4\sqrt{5}$, $REUSO(G) = 9\sqrt{3} + 4\sqrt{7}$, $RESO(G) = 34\sqrt{2} + 12\sqrt{5}$, $RH(G) = \frac{17}{3}$, and $RSO(G,X) = \frac{1}{54}X^{54}$.

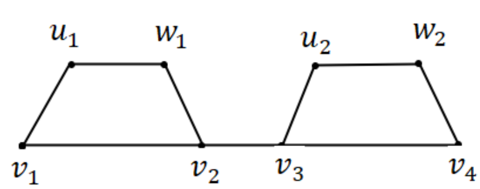

Figure 4: Alternate Quadrilateral snake graph (AQS_1). Adapted from[30]

3.3 Double Quadrilateral Snake Graph

A Double Quadrilateral snake graph consists of two quadrilateral snakes that have a common path and is denoted by DQS_n see [30]. let n denote the number of double quadrilateral units in the graph. Hence, the Double Quadrilateral Snake Graph (DQS_n) has $|V| = 5n+1$ vertices and $|E| = 7n$ edges.

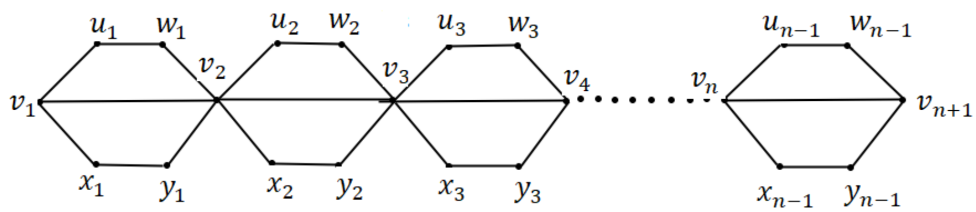

Figure 5: Double Quadrilateral Snake Graph. Adapted from[30]

Theorem 3.3. *Let G be the double quadrilateral snake graph. Then*

$$\begin{aligned}
RSO(G) &= 10\sqrt{2}n + 4\sqrt{41} + 4\sqrt{26}(n-1) + 2\sqrt{17} + (n-2)\sqrt{2} \\
REUSO(G) &= 10\sqrt{3}n + 4\sqrt{61} + 4\sqrt{31}(n-1) + 2\sqrt{21} + (n-2)\sqrt{3} \\
RESO(G) &= 100\sqrt{2}n + 36\sqrt{41} + 24\sqrt{26}(n-1) + 10\sqrt{17} + 2\sqrt{2}(n-2) \\
RH(G) &= \frac{2n}{5} + \frac{8}{9} + \frac{4(n-1)}{3} + \frac{4}{5} + n - 2 \\
RSO(G,X) &= \frac{1}{2(103n+45)}X^{2(103n+45)}
\end{aligned}$$

Proof. Let G be the double quadrilateral snake graph (DQS_n) with $|V| = 5n+1$ vertices and $|E| = 7n$ edges. The edge set can be partitioned as,

$$\begin{aligned}
E_1 &= \{xy \in E(G) | d_G(x) = d_G(y) = 2\}, |E_1| = 2n \\
E_2 &= \{xy \in E(G) | d_G(x) = 2, d_G(y) = 3\}, |E_2| = 4 \\
E_3 &= \{xy \in E(G) | d_G(x) = 2, d_G(y) = 6\}, |E_3| = 4(n-1) \\
E_4 &= \{xy \in E(G) | d_G(x) = 3, d_G(y) = 6\}, |E_4| = 2 \\
E_5 &= \{xy \in E(G) | d_G(x) = d_G(y) = 6\}, |E_5| = n-2
\end{aligned}$$

Here $\Delta(G) = 6$ and then, $c_G(x) = 7 - d_G(x)$

$$\begin{aligned}
CE_1 &= \{xy \in E(G) | c_G(x) = c_G(y) = 5\}, |CE_1| = 2n \\
CE_2 &= \{xy \in E(G) | c_G(x) = 5, c_G(y) = 4\}, |CE_2| = 4 \\
CE_3 &= \{xy \in E(G) | c_G(x) = 5, c_G(y) = 1\}, |CE_3| = 4(n-1) \\
CE_4 &= \{xy \in E(G) | c_G(x) = 4, c_G(y) = 1\}, |CE_4| = 2 \\
CE_5 &= \{xy \in E(G) | c_G(x) = c_G(y) = 1\}, |CE_5| = n-2
\end{aligned}$$

Using the reverse edge partition of graph $DQ(S_n)$ and indices formula, we get

$$\begin{aligned}
RSO(G) &= 2n\sqrt{5^2+5^2} + 4\sqrt{5^2+4^2} + 4(n-1)\sqrt{5^2+1} + 2\sqrt{4^2+1} + (n-2)\sqrt{1+1} \\
&= 10\sqrt{2}n + 4\sqrt{41} + 4\sqrt{26}(n-1) + 2\sqrt{17} + (n-2)\sqrt{2} \\
REUSO(G) &= 2n\sqrt{5^2+5^2+5.5} + 4\sqrt{5^2+4^2+5.4} + 4(n-1)\sqrt{5^2+1+5} + \\
&\quad 2\sqrt{4^2+1+4} + (n-2)\sqrt{1+1+1} \\
&= 10\sqrt{3}n + 4\sqrt{61} + 4\sqrt{31}(n-1) + 2\sqrt{21} + (n-2)\sqrt{3} \\
RESO(G) &= 2n(5+5)\sqrt{5^2+5^2} + 4(5+4)\sqrt{5^2+4^2} + 4(n-1)(5+1)\sqrt{5^2+1} + \\
&\quad 2(4+1)\sqrt{4^2+1} + (n-2)(1+1)\sqrt{1+1} \\
&= 100\sqrt{2}n + 36\sqrt{41} + 24\sqrt{26}(n-1) + 10\sqrt{17} + 2\sqrt{2}(n-2)
\end{aligned}$$

$$RH(G) = \frac{4n}{5+5} + \frac{2(4)}{5+4} + \frac{8(n-1)}{5+1} + \frac{4}{4+1} + \frac{2(n-2)}{2}$$
$$= \frac{2n}{5} + \frac{8}{9} + \frac{4(n-1)}{3} + \frac{4}{5} + n - 2$$
$$RSO(G,X) = \frac{1}{2n(50)+4(41)+4(n-1)(26)+2(17)+(n-2)(2)} X^{[2n(50)+4(41)+4(n-1)(26)+2(17)+(n-2)(2)]}$$
$$= \frac{1}{2(103n+45)} X^{2(103n+45)}$$

□

Example: For $n = 3$ in the Double Quadrilateral Snake Graph (DQS_n), we have $|V| = 16$ vertices and $|E| = 21$ edges. Then, the indices are given by $RSO(G) = 31\sqrt{2} + 4\sqrt{41} + 8\sqrt{26} + 2\sqrt{17}$, $REUSO(G) = 30\sqrt{3} + 4\sqrt{61} + 8\sqrt{31} + 2\sqrt{21}$, $RESO(G) = 302\sqrt{2} + 36\sqrt{41} + 48\sqrt{26} + 10\sqrt{17}$, $RH(G) = \frac{59}{9}$, and $RSO(G,X) = \frac{1}{708}X^{708}$.

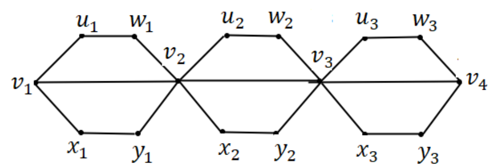

Figure 6: Double Quadrilateral snake graph (DQS_3). Adapted from[30]

3.4 Alternating Double Quadrilateral Snake Graph

An Alternate Double Quadrilateral snake graph consists of two alternate quadrilateral snakes that have a common path and is denoted by $ADQS_n$ see [30]. let n denote the number of alternating double quadrilateral units in the graph. Hence, the Alternating Double Quadrilateral Snake Graph ($ADQS_n$) has $|V| = 6(n+1)$ vertices and $|E| = 8n+7$ edges.

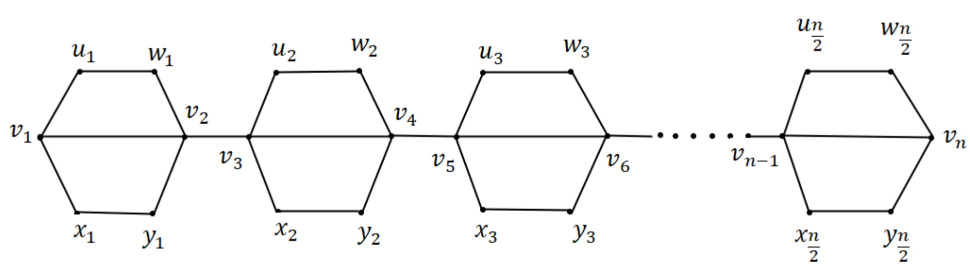

Figure 7: Alternating Double Quadrilateral Snake Graph. Adapted from[30]

Theorem 3.4. *Let G be the alternating double quadrilateral snake graph ($ADQS_n$). Then*

$$RSO(G) = 6\sqrt{2}(n+1) + 4\sqrt{13} + 4\sqrt{10}n + 2\sqrt{5} + (2n-1)\sqrt{2}$$
$$REUSO(G) = 6\sqrt{3}(n+1) + 4\sqrt{19} + 4\sqrt{13}n + 2\sqrt{7} + (2n-1)\sqrt{3}$$
$$RESO(G) = 36\sqrt{2}(n+1) + 20\sqrt{13} + 16\sqrt{10}n + 6\sqrt{5} + 2\sqrt{2}(2n-1)$$
$$RH(G) = \frac{2(n+1)}{3} + 4n + \frac{29}{15}$$
$$RSO(G,X) = \frac{1}{16(5n+6)} X^{16(5n+6)}$$

Proof. Let G be the alternating double quadrilateral snake graph $ADQ(S_n)$ with $|V| = 6(n+1)$ vertices and $|E| = 8n+7$ edges. The edge set can be partitioned as,

$$\begin{aligned}
E_1 &= \{xy \in E(G)|d_G(x) = d_G(y) = 2\}, |E_1| = 2(n+1)\\
E_2 &= \{xy \in E(G)|d_G(x) = 2, d_G(y) = 3\}, |E_2| = 4\\
E_3 &= \{xy \in E(G)|d_G(x) = 2, d_G(y) = 4\}, |E_3| = 4n\\
E_4 &= \{xy \in E(G)|d_G(x) = 3, d_G(y) = 4\}, |E_4| = 2\\
E_5 &= \{xy \in E(G)|d_G(x) = d_G(y) = 4\}, |E_5| = 2n-1
\end{aligned}$$

Here $\Delta(G) = 4$ and then, $c_G(x) = 5 - d_G(x)$

$$\begin{aligned}
CE_1 &= \{xy \in E(G)|c_G(x) = c_G(y) = 3\}, |CE_1| = 2(n+1)\\
CE_2 &= \{xy \in E(G)|c_G(x) = 3, c_G(y) = 2\}, |CE_2| = 4\\
CE_3 &= \{xy \in E(G)|c_G(x) = 3, c_G(y) = 1\}, |CE_3| = 4n\\
CE_4 &= \{xy \in E(G)|c_G(x) = 2, c_G(y) = 1\}, |CE_4| = 2\\
CE_5 &= \{xy \in E(G)|c_G(x) = c_G(y) = 1\}, |CE_5| = 2n-1
\end{aligned}$$

Using the reverse edge partition of graph $ADQ(S_n)$ and indices formula, we get

$$\begin{aligned}
RSO(G) &= 2(n+1)\sqrt{3^2+3^2} + 4\sqrt{3^2+2^2} + 4n\sqrt{3^2+1} + 2\sqrt{2^2+1} + (2n-1)\sqrt{1+1}\\
&= 6\sqrt{2}(n+1) + 4\sqrt{13} + 4\sqrt{10}n + 2\sqrt{5} + (2n-1)\sqrt{2}\\
REUSO(G) &= 2(n+1)\sqrt{3^2+3^2+3.3} + 4\sqrt{3^2+2^2+3.2} + 4n\sqrt{3^2+1+3} +\\
&\quad 2\sqrt{2^2+1+2} + (2n-1)\sqrt{1+1+1}\\
&= 6\sqrt{3}(n+1) + 4\sqrt{19} + 4\sqrt{13}n + 2\sqrt{7} + (2n-1)\sqrt{3}\\
RESO(G) &= 2(n+1)(3+3)\sqrt{3^2+3^2} + 4(3+2)\sqrt{3^2+2^2} + 4n(3+1)\sqrt{3^2+1} +\\
&\quad 2(2+1)\sqrt{2^2+1} + (2n-1)(1+1)\sqrt{1+1}\\
&= 36\sqrt{2}(n+1) + 20\sqrt{13} + 16\sqrt{10}n + 6\sqrt{5} + 2\sqrt{2}(2n-1)\\
RH(G) &= \frac{4(n+1)}{3+3} + \frac{2(4)}{3+2} + \frac{8n}{3+1} + \frac{4}{2+1} + \frac{2(2n-1)}{2}\\
&= \frac{2(n+1)}{3} + 4n + \frac{29}{15}\\
RSO(G,X) &= \frac{1}{2(n+1)(18) + 4(13) + 4n(10) + 2(5) + (2n-1)(2)} X^{[2(n+1)(18)+4(13)+4n(10)+2(5)+(2n-1)(2)]}\\
&= \frac{1}{16(5n+6)} X^{16(5n+6)}
\end{aligned}$$

□

Example: For $n = 1$ in the Alternating Double Quadrilateral Snake Graph ($ADQS_n$), we have $|V| = 12$ vertices and $|E| = 15$ edges. Then, the indices are given by $RSO(G) = 13\sqrt{2} + 4\sqrt{13} + 4\sqrt{10} + 2\sqrt{5}$, $REUSO(G) = 13\sqrt{3} + 4\sqrt{19} + 4\sqrt{13} + 2\sqrt{7}$, $RESO(G) = 74\sqrt{2} + 20\sqrt{3} + 16\sqrt{10} + 6\sqrt{5}$, $RH(G) = \frac{109}{15}$, and $RSO(G,X) = \frac{1}{176}X^{176}$.

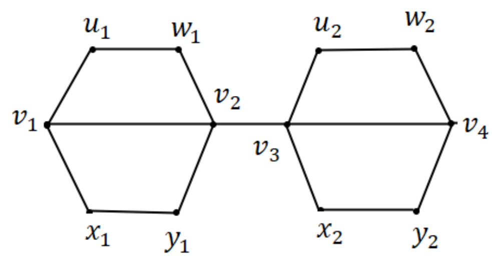

Figure 8: Alternate Double Quadrilateral snake graph ($ADQS_1$). Adapted from[30]

3.5 Cycle Quadrilateral Snake Graph

A Cycle Quadrilateral snake graph is derived from a cycle $v_1, v_2, v_3, v_4, ..., v_n$ by connecting new vertices u_i and w_i to v_i and v_{i+1} respectively and then connecting u_i and w_i. It is denoted by CQS_n. let n denote the number of cycle quadrilateral units in the graph. Hence, the Cycle Quadrilateral Snake Graph (CQS_n) has $|V| = 3n$ vertices and $|E| = 4n$ edges.

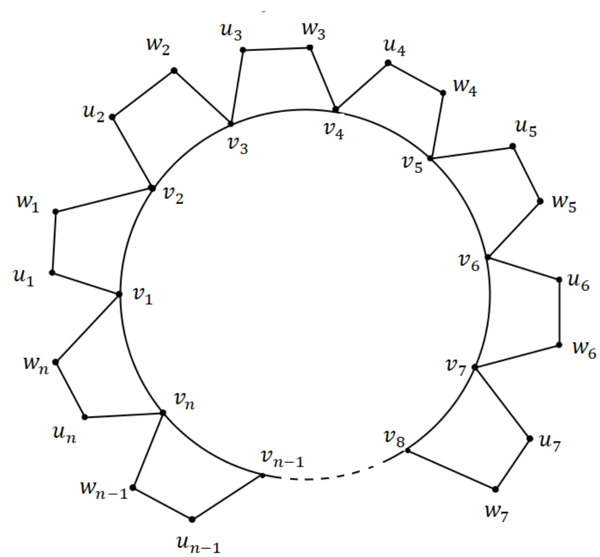

Figure 9: Cycle Quadrilateral Snake Graph. Adapted from[30]

Theorem 3.5. *Let G be the cycle quadrilateral snake graph (CQS_n). Then*

$$\begin{aligned}
RSO(G) &= 3\sqrt{2}n + 2\sqrt{10}n + \sqrt{2}n \\
REUSO(G) &= 3\sqrt{3}n + 2\sqrt{13}n + \sqrt{3}n \\
RESO(G) &= 18\sqrt{2}n + 8\sqrt{10}n + 2\sqrt{2}n \\
RH(G) &= \frac{7n}{3} \\
RSO(G, X) &= \frac{1}{40n} X^{40n}
\end{aligned}$$

Proof. Let G be the cycle quadrilateral snake graph (CQS_n) with $|V| = 3n$ vertices and $|E| = 4n$ edges. The edge set can be partitioned as,

$$\begin{aligned}
E_1 &= \{xy \in E(G) | d_G(x) = d_G(y) = 2\}, |E_1| = n \\
E_2 &= \{xy \in E(G) | d_G(x) = 2, d_G(y) = 4\}, |E_2| = 2n \\
E_3 &= \{xy \in E(G) | d_G(x) = d_G(y) = 4\}, |E_3| = n
\end{aligned}$$

Here $\Delta(G) = 4$ and then, $c_G(x) = 5 - d_G(x)$

$$\begin{aligned} CE_1 &= \{xy \in E(G) | c_G(x) = c_G(y) = 3\}, |CE_1| = n \\ CE_2 &= \{xy \in E(G) | c_G(x) = 3, c_G(y) = 1\}, |CE_2| = 2n \\ CE_3 &= \{xy \in E(G) | c_G(x) = c_G(y) = 1\}, |CE_3| = n \end{aligned}$$

Using the reverse edge partition of graph $CQ(S_n)$ and indices formula, we get

$$\begin{aligned} RSO(G) &= n\sqrt{3^2+3^2} + 2n\sqrt{3^2+1^2} + n\sqrt{1+1} \\ &= 3\sqrt{2}n + 2\sqrt{10}n + \sqrt{2}n \\ REUSO(G) &= n\sqrt{3^2+3^2+3.3} + 2n\sqrt{3^2+1^2+3} + n\sqrt{1+1+1} \\ &= 3\sqrt{3}n + 2\sqrt{13}n + \sqrt{3}n \\ RESO(G) &= n(3+3)\sqrt{3^2+3^2} + 2n(3+1)\sqrt{3^2+1^2} + n(1+1)\sqrt{1+1} \\ &= 18\sqrt{2}n + 8\sqrt{10}n + 2\sqrt{2}n \\ RH(G) &= \frac{2n}{3+3} + \frac{2(2n)}{3+1} + \frac{2n}{1+1} \\ &= \frac{7n}{3} \\ RSO(G,X) &= \frac{1}{18n+2n(10)+n(2)} X^{[18n+2n(10)+n(2)]} \\ &= \frac{1}{40n} X^{40n} \end{aligned}$$

□

Example: For $n = 3$ in the Cycle Quadrilateral Snake Graph (CQS_n), we have $|V| = 9$ vertices and $|E| = 12$ edges. Then, the indices are given by $RSO(G) = 12\sqrt{2} + 6\sqrt{10}$, $REUSO(G) = 12\sqrt{3} + 6\sqrt{13}$, $RESO(G) = 60\sqrt{2} + 24\sqrt{10}$, $RH(G) = 7$, and $RSO(G,X) = \frac{1}{120}X^{120}$.

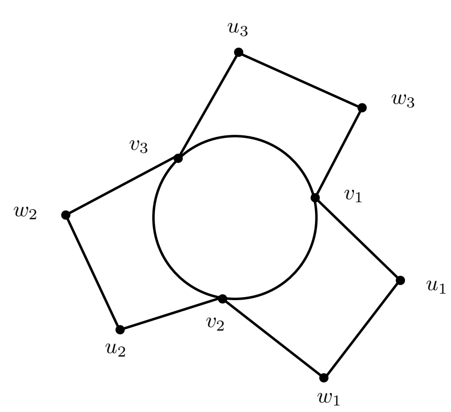

Figure 10: Cycle Quadrilateral Snake Graph CQS_3

4 Conclusion

In this paper, we have studied multiple variants of quadrilateral snake graphs and derived closed form expressions for several reverse degree-based topological indices. The reverse Sombor, reverse Euler Sombor, reverse Elliptic Sombor, and reverse Harmonic indices were computed systematically using edge partition methods. Furthermore, the reverse Sombor polynomial was established for each graph class, providing an extended view of their structural properties. The obtained results contribute to the theoretical development of reverse topological indices and offer a foundation for further studies on related graph structures and their applications.

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