UNDER PEER REVIEW

Spectra of Lower Triangular Double-Band Infinite Matrices with Oscillatory Entries on ℓ_1

Abstract

Motivated by the work of Mahto et al. (Kragujevac Journal of Mathematics, 46(3), 369–381, 2022), we study the spectra of a lower triangular infinite matrix on the sequence space ℓ_1, where the diagonal and sub-diagonal entries consist of two oscillatory sequences with distinct limit points.

Keywords: Sequence Space, Infinite Matrix, Spectrum, Point Spectrum, Residual Spectrum, Continuous Spectrum

1. Introduction

Spectral theory is an important branch of functional analysis and plays a significant role in various areas of mathematics and physics, particularly in classical quantum mechanics. The spectral characteristics of a linear operator often reveal important information about its structure and behavior. In many situations, operators such as infinite matrices and differential operators can be better understood through the investigation of their spectra.

In recent years, considerable attention has been devoted to studying the fine spectra of infinite matrices acting on different sequence spaces. One of the earlier studies in this direction was carried out by Wenger [1], who determined the fine spectrum of the Hölder summability operator on the space of convergent sequences c. Later, Rhoades [2, 3] investigated the fine spectra of weighted mean operators. The spectral properties of the Cesàro operator have also attracted much interest. Gonzalez [4] determined the fine spectrum of the Cesàro operator on the space l_p for $1 < p < \infty$, whereas Reade [5] studied its spectrum on the space of null sequences c_0. Furthermore, Yildirim [6] examined the spectrum of Rhaly operators over the spaces c_0 and c.

Further investigations extended these studies to other operators and sequence spaces. Altay and Başar [7, 8] considered the difference operators Δ and $B(r,s)$ over the spaces c_0 and c. Later, Furkan and Bilgiç [9] investigated the operator $B(r,s)$ on the sequence spaces l_p and bv_p. Later on Mahto et.al. [10] extended these results to the double band lower triangular infinite matrices in which diagonal and subdiagonal entries consist of oscillatory sequences with distinct limit points.

Motivated by these developments, in the present paper we study the spectrum and fine spectra of the generalized difference matrix A. The matrix A is a lower triangular infinite matrix consisting of two oscillatory sequences $p = (p_k)$ and $q = (q_k)$ in its diagonal and subdiagonal. The sequence $p = (p_k)$ is oscillating between the points p_i, $i \in \{0, 1, 2, 3\}$ and the sequence $q = (q_k)$ is oscillating between the points q_i, $i \in \{0, 1, 2, \ldots, 6\}$. Our aim is to determine the spectrum and the fine spectra of this matrix and to analyze the spectral structure of the corresponding operator.

2. Preliminaries

Let X be a Banach space and T be a linear operator defined on a subset of X. Then the operator $(T - \lambda I)^{-1}$ is called a resolvent operator, where I is the identity operator. A regular value [11] is a complex number λ for which the resolvent operator exists, bounded and is defined on a set which is dense in X. The set of all regular values is called the resolvent set. The complement of the resolvent set is called the spectrum of the operator T and is denoted by $\sigma(T, X)$. The spectrum is partitioned into three disjoint subsets point spectrum, continuous spectrum and residual spectrum. The point spectrum is the set of all complex numbers for which the resolvent operator does not exist. The continuous spectrum is the set at which resolvent operator exists and its domain is dense in X, but it is not bounded. At residual spectrum, the domain of the resolvent operator is not dense in X.

Let $T_\lambda = T - \lambda I$ and $R(T_\lambda)$ be the range of the operator T_λ. Then the Goldberg's subdivision of spectrum of T is given by the Table 1.

Theorem 2.1. *[12] Let T be a bounded linear operator on a normed linear space X, then T has a bounded inverse if and only if T^* is onto.*

Lemma 2.2. *[13] $A = (a_{nk}) \in B(\ell_1)$ if and only if the supremum of the l_1 norms of the columns is bounded and $\|A\| = \sup_k\{\sum_n |a_{nk}|\}$.*

	(I)$R(T_\lambda) = X$	(II)$R(T_\lambda) \neq X$, but $\overline{R(T_\lambda)} = X$	(III)$\overline{R(T_\lambda)} \neq X$
(1) T_λ^{-1} exists	$\rho(T, X)$	—	$\sigma_r(T, X)$
(2) T_λ^{-1} exists but not bounded	$\sigma_c(T, X)$	$\sigma_c(T, X)$	$\sigma_r(T, X)$
(3) T_λ^{-1} does not exists	$\sigma_p(T, X)$	$\sigma_p(T, X)$	$\sigma_p(T, X)$

Table 1: Goldberg's classification of spectrum

3. Main results

Let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. We define $\gamma_n(k) \in \{0, 1, \ldots, n-1\}$ as the least non negative remainder obtained after dividing k by n. Next, we consider an infinite matrix $A = (a_{nk})$. The diagonal and subdiagonal entries of A consist of oscillatory sequences $p = (p_k)$ and $q = (q_k)$, respectively, where $k \in \mathbb{N}_0$. The sequence $p = (p_k)$, $p_k = p_{\gamma_4(k)}$, has four limit points and the sequence $q = (q_k)$, $q_k = p_{\gamma_6(k)}$, has six limit points. The entries of the matrix $A = (a_{nk})$ can be represented as follows:

$$a_{nk} = \begin{cases} p_{\gamma_4(k)}, & \text{if } n = k \\ q_{\gamma_6(k)}, & \text{if } n = k+1 \\ 0, & \text{otherwise.} \end{cases} \tag{1}$$

Corollary 3.1. *The matrix* $A \in B(\ell_1)$ *and* $\|A\|_{(\ell_1:\ell_1)} \leq \max_{0 \leq i \leq 11} \{|p_{\gamma_4(i)}| + |q_{\gamma_6(i)}|\}$.

Next, we claim that the set $\left\{\lambda \in \mathbb{C} : \prod_{i=0}^{3} |p_i - \lambda|^3 \leq \prod_{i=0}^{5} |q_i|^2\right\}$ is the spectrum of A.

Theorem 3.2. $\sigma(A, \ell_1) = \left\{\lambda \in \mathbb{C} : \prod_{i=0}^{3} |p_i - \lambda|^3 \leq \prod_{i=0}^{5} |q_i|^2\right\}$.

Proof. Let $\lambda \notin S = \left\{\lambda \in \mathbb{C} : \prod_{i=0}^{3} |p_i - \lambda|^3 \leq \prod_{i=0}^{5} |q_i|^2\right\}$ be a complex number. Then $\prod_{i=0}^{3} |p_i - \lambda|^3 > \prod_{i=0}^{5} |q_i|^2$. Since each q_i, $i \in \{0, 1, \ldots, 5\}$ is nonzero, none of the values p_i, $i \in \{0, 1, 2, 3\}$, equals λ. That is, the matrix $A - \lambda I$ is a triangle with nonzero diagonal entries and hence invertible. Let n

and k be nonnegative integers and r and s be the remainders obtained when $n-k$ and k are divided by 12, respectively. That is, $n-k=12m+r$ and $k=12l+s$ for some nonnegative integers m and l. We write

$$\alpha_{r,s}=\begin{cases}\frac{1}{p_{\gamma_4(s)}-\lambda}, & \text{if } r=0\\ \frac{\prod\limits_{i=0}^{r-1} q_{\gamma_6(s+i)}}{\prod\limits_{i=0}^{r}(p_{\gamma_4(s+i)}-\lambda)}, & \text{if } r\in\{1,2,\dots 11\}\end{cases} \tag{2}$$

and

$$\delta=\frac{\prod\limits_{i=0}^{5} q_i^2}{\prod\limits_{i=0}^{3}(p_i-\lambda)^3}. \tag{3}$$

Evidently, $|\delta|<1$ and the entries of $(A-\lambda I)^{-1}=(c_{nk})$ are given by

$$c_{nk}=\begin{cases}\alpha_{r,s}\delta^m, & \text{if } n\geq k\\ 0, & \text{otherwise.}\end{cases} \tag{4}$$

Next,

$$\sup_k\sum_{n=0}^{\infty}|c_{nk}|=\sup_k\sum_{r=0}^{11}\sum_{m=0}^{\infty}|\alpha_{r,s}||\delta|^m \tag{5}$$

$$\leq 12\max_{0\leq r,s\leq 11}|\alpha_{r,s}|\sum_{m=0}^{\infty}|\delta|^m \tag{6}$$

$$<12\max_{0\leq r,s\leq 11}|\alpha_{r,s}|<\infty. \tag{7}$$

This implies that $\lambda\in\rho(A,\ell_1)$. We have shown that $S^c\subseteq\rho(A,\ell_1)$. Taking complement, we get $\sigma(A,\ell_1)\subseteq S$.

Conversely, let $\lambda\in S$, then $|\delta|\geq 1$ and $(A-\lambda I)^{-1}\notin B(\ell_1)$. Hence, $\lambda\in\sigma(A,\ell_1)$, that is, $S\subseteq\sigma(A,\ell_1)$. Combining both parts of the proof, we conclude that S is the spectrum of A acting on ℓ_1. □

In the next theorem, we show that the point spectrum of A when acting on ℓ_1 is the empty set.

Theorem 3.3. $\sigma_p(A,\ell_1)=\varnothing$.

Proof. Assume that $\lambda \in \sigma_p(A, \ell_1)$. Then there exists a nonzero sequence $x = (x_k) \in \ell_1$ satisfying $Ax = \lambda x$. Consequently, for each $k \in \mathbb{N}_0$, the eigenvalue equation can be written componentwise as

$$q_{\gamma_6(k-1)} x_{k-1} + p_{\gamma_4(k)} x_k = \lambda x_k, \tag{8}$$

where $x_{-1} = 0$. Let x_{k_0} be the first nonzero entry of the sequence $x = (x_k)$. Then from (8), we obtain that $\lambda = p_{\gamma_4(k_0)}$. Setting $k = k_0 + 4$ in (8), we get

$$q_{\gamma_6(k_0+3)} x_{k_0+3} + p_{\gamma_4(k_0+4)} x_{k_0+4} = \lambda x_{k_0+4}. \tag{9}$$

As $\lambda = p_{\gamma_4(k_0)}$ and $q_i \neq 0$ for all i, from (9), we get $x_{k_0+3} = 0$. Similarly, setting $k = k_0+3$ and $x_{k_0+3} = 0$ in (8), we obtain that $x_{k_0+2} = 0$. Continuing this process for $k = k_0 + 2$ and $k = k_0 + 1$, we get $x_{k_0+1} = 0$ and $x_{k_0} = 0$, respectively. This contradicts the assumption that $x_{k_0} \neq 0$. Hence, point spectrum of A acting on ℓ_1 is the empty set. □

As $A \in B(\ell_1)$ and the dual of ℓ_1 is ℓ_∞, the adjoint $A^* \in B(\ell_\infty)$ can be given by the transpose of the matrix A. Let $A^* = (a^*_{nk})$. Then

$$a^*_{nk} = \begin{cases} p_{\gamma_4(n)}, & \text{if } n = k \\ q_{\gamma_6(n)}, & \text{if } n+1 = k \\ 0, & \text{otherwise} \end{cases} \tag{10}$$

Next, we show that the point spectrum of A^* acting on ℓ_∞ is same as that of the spectrum of A acting on ℓ_1.

Theorem 3.4. $\sigma_p(A^*, \ell_1^* \cong \ell_\infty) = \left\{\lambda \in \mathbb{C} : \prod_{i=0}^{3} |p_i - \lambda|^3 \leq \prod_{i=0}^{5} |q_i|^2\right\}$.

Proof. Let $k = 12j + l$, $l \in \{0, 1, \ldots, 11\}$ and $x_0 \neq 0$. Then solving the equation $A^*x = \lambda x$, we get

$$x_k = \begin{cases} \prod\limits_{i=0}^{l-1} \frac{p_{\gamma_4(i)}-\lambda}{\gamma_6(i)} \left(\frac{1}{\delta}\right)^j x_0, & \text{if } l \geq 1 \\ \left(\frac{1}{\delta}\right)^j x_0, & \text{if } l = 0, \end{cases} \tag{11}$$

where δ is given by (3). Evidently, $x = (x_k) \in \ell_\infty$ if and only if $|\delta| \geq 1$. That is, $\lambda \in \sigma_p(A^*, \ell_\infty)$ if and only if $\prod_{i=0}^{3} |p_i - \lambda|^3 \leq \prod_{i=0}^{5} |q_i|^2$. This proves the theorem. □

Theorem 3.5. $\sigma_r(A, \ell_1) = \left\{\lambda \in \mathbb{C} : \prod_{i=0}^{3} |p_i - \lambda|^3 \leq \prod_{i=0}^{5} |q_i|^2\right\}$.

Proof. Because A is a bounded linear operator on the Banach space ℓ_1, its residual spectrum satisfies $\sigma_r(A,\ell_1) = \sigma_p(A^*,\ell_1^*) \setminus \sigma_p(A,\ell_1)$. Combining this relation with Theorems 3.3 and 3.4, we conclude that $\sigma_r(A,\ell_1) = \left\{\lambda \in \mathbb{C} : \prod_{i=0}^{3} |p_i - \lambda|^3 \leq \prod_{i=0}^{5} |q_i|^2\right\}$. □

Theorem 3.6. $\sigma_c(A,\ell_1) = \varnothing$

Proof. Since the spectrum of a bounded linear operator on a Banach space can be decomposed into the disjoint union of its point spectrum, residual spectrum and continuous spectrum, it follows from Theorems 3.3, 3.2 and 3.5 that the continuous spectrum A on ℓ_1 is empty. □

Lemma 3.7. $\{p_0, p_1, p_2, p_3\} \subseteq III_1(A,\ell_1)$

Proof. Theorem 3.5 shows that $p_i \in \sigma_r(A,\ell_1)$ for all $i \in \{0,1,2,3\}$. Thus, for each $i \in \{0,1,2,3\}$, the operator $A - p_iI$ is not dense in ℓ_1. Next, by Theorem 2.1, it suffices to prove that the operator $(A-p_iI)^* : \ell_\infty \to \ell_\infty$ is onto for each $i \in \{0,1,2,3\}$. Without loss of generality, we first consider the operator $(A-p_2I)^*$. For this, let $y = (y_k) \in \ell_\infty$. That is, $\sup_k |y_k| = M < \infty$. Solving $(A-p_2I)^*x = y$, we obtain that

$$
\begin{aligned}
x_{4n+2} &= \frac{1}{q_{\gamma_6(4n+1)}} y_{4n+1} + \frac{p_2 - p_1}{q_{\gamma_6(4n)} q_{\gamma_6(4n+1)}} y_{4n} + \frac{(p_2-p_0)(p_2-p_1)}{q_{\gamma_6(4n-1)} q_{\gamma_6(4n)} q_{\gamma_6(4n+1)}} y_{4n-1} \\
&\quad + \frac{(p_2-p_3)(p_2-p_0)(p_2-p_1)}{q_{\gamma_6(4n-2)} q_{\gamma_6(4n-1)} q_{\gamma_6(4n)} q_{\gamma_6(4n+1)}} y_{4n-2}, \\
x_{4n+1} &= \frac{1}{q_{\gamma_6(4n)}} y_{4n} + \frac{p_2 - p_0}{q_{\gamma_6(4n-1)} q_{\gamma_6(4n)}} y_{4n-1} + \frac{(p_2-p_3)(p_2-p_0)}{q_{\gamma_6(4n-2)} q_{\gamma_6(4n-1)} q_{\gamma_6(4n)}} y_{4n-2}, \\
x_{4n} &= \frac{1}{q_{\gamma_6(4n-1)}} y_{4n-1} + \frac{p_2 - p_3}{q_{\gamma_6(4n-2)} q_{\gamma_6(4n-1)}} y_{4n-2}, \\
x_{4n+3} &= \frac{1}{q_{\gamma_6(4n+2)}} y_{4n+2}.
\end{aligned}
$$

Evidently, the sequence $x = (x_k)$ is bounded if and only if all its subsequences (x_{4n+2}), (x_{4n+1}), (x_{4n}) and (x_{4n+3}) are bounded. Let M_1, M_2, M_3 and M_4 be the positive numbers such that

$$\begin{aligned}
&\frac{1}{q_{\gamma_6(4n+1)}} \leq \max\left\{\frac{1}{q_1}, \frac{1}{q_3}, \frac{1}{q_5}\right\} = M_1\\
&\frac{1}{q_{\gamma_6(4n)}q_{\gamma_6(4n+1)}} \leq \max\left\{\frac{1}{q_0q_1}, \frac{1}{q_2q_3}, \frac{1}{q_4q_5}\right\} = M_2\\
&\frac{1}{q_{\gamma_6(4n-1)}q_{\gamma_6(4n)}q_{\gamma_6(4n+1)}} \leq \max\left\{\frac{1}{q_5q_0q_1}, \frac{1}{q_1q_2q_3}, \frac{1}{q_3q_4q_5}\right\} = M_3\\
&\frac{1}{q_{\gamma_6(4n-2)}q_{\gamma_6(4n-1)}q_{\gamma_6(4n)}q_{\gamma_6(4n+1)}} \leq \left\{\frac{1}{q_4q_5q_0q_1}, \frac{1}{q_0q_1q_2q_3}, \frac{1}{q_2q_3q_4q_5}\right\} = M_4.
\end{aligned}$$

Then,

$$\begin{aligned}
\sup_n |x_{4n+2}| &\leq M_1|y_{4n+1}| + M_2(p_2-p_1)|y_{4n}| + M_3(p_2-p_0)(p_2-p_1)|y_{4n-1}|\\
&\quad + M_4(p_2-p_3)(p_2-p_0)(p_2-p_1)|y_{4n-2}|\\
&\leq M\{M_1 + M_2(p_2-p_1) + M_3(p_2-p_0)(p_2-p_1)\\
&\quad + M_4(p_2-p_3)(p_2-p_0)(p_2-p_1)\} < \infty.
\end{aligned}$$

Thus, the subsequence (x_{4n+2}) is bounded. Similarly, we can show that the subsequences (x_{4n}), (x_{4n+1}) and (x_{4n+3}) are also bounded. Consequently, the sequence $x = (x_k)$ is bounded and hence the operator $(A - p_2 I)^*$ is onto. On the similar lines, it can be shown that the operators $(A - p_i I)^*$ for all $i \in 0, 1, 3$ are also onto. This proves the theorem. □

Theorem 3.8. $III_2(A, \ell_2) = \sigma_r(A, \ell_1) \setminus \{p_0, p_1, p_2, p_3\}$.

Proof. From Table 1, it is evident that $\sigma_r(A, \ell_1)$ is the disjoint union of $III_1(A, \ell_1)$ and $III_2(A, \ell_2)$. Therefore, taking complement of the inclusion relation of Lemma 3.7 in $\sigma_r(A, \ell_1)$, we get $III_2(A, \ell_1) \subseteq \sigma_r(A, \ell_1) \setminus \{p_0, p_1, p_2, p_3\}$. Conversly, suppose $\lambda \in \sigma_r(A, \ell_1) \setminus \{p_0, p_1, p_2, p_3\}$, then from (7), we get $(A - \lambda)^{-1}$ is not bounded. That is, $\lambda \in III_2(A, \ell_1)$. Hence, $\sigma_r(A, \ell_1) \setminus \{p_0, p_1, p_2, p_3\} \subseteq III_2(A, \ell_2)$. This proves the theorem. □

Theorem 3.9. $III_1(A, \ell_1) = \{p_0, p_1, p_2, p_3\}$

Proof. Taking complement of the result of Theorem 3.8 in $\sigma_r(A, \ell_1)$, we get $III_1(A, \ell_1) = \{p_0, p_1, p_2, p_3\}$. □

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