Original Research Article

SOME NEW LOWER BOUNDS FOR THE SPREAD OF A NONNEGATIVE MATRIX WITH A ZERO DIAGONAL ELEMENT

ABSTRACT. Let $\mathbb{N}_n$ (with $n \geq 2$) be a family of all nonnegative $n \times n$ matrices $A = [a_{ij}]$, such that $a_{11} = 0$ and other entries $a_{ij} \in [0, 1)$ with spectral radius $\rho(A) = 1$. We prove a lower bound for the additional spread $s(A) \geq \frac{k}{n-1}$, where k is the number of zero diagonal elements of matrix A. Moreover, if A has only two distinct eigenvalues, then $s(A) \geq \frac{n-2}{n-1}$. Few other lower bounds are also obtain.

1. Introduction

Let $A = [a_{ij}]_{n\times n}$ be a real or complex matrix with multiset of eigenvalues $\lambda(A) = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$. The spectral radius of matrix A defined and denoted as $\rho(A) = \max\{| \lambda_i |: 1 \leq i \leq n\}$ and the trace is denoted by $tr(A)$. Maximum distance between two eigenvalues of given matrix A is called spread or additional spread, defined by Mirsky [2] as,

$$s(A) = \max\{| \lambda_i - \lambda_j |: 1 \leq i, j \leq n\}.$$

The multiplicative spread (assuming the λ_i's nonzero) given by

$$\kappa(A) = \max\left\{\left|\frac{\lambda_i}{\lambda_j}\right| : 1 \leq i, j \leq n\right\}.$$

Let $\mathbb{N}_n$ (with $n \geq 2$) be a family of all nonnegative $n \times n$ matrices $A = [a_{ij}]$, such that $a_{11} = 0$ and other entries $a_{ij} \in [0, 1)$ with $\rho(A) = 1$. In this case R. Drnovšek [1] give the bound of spread

$$s(A) \geq \frac{k}{n}, \tag{1.1}$$

where k represents the number of zero diagonal elements in the matrix A. This proves the spread of the matrix, indicating that there exists at least

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one eigenvalue of A other than 1. Now, we proceed to two lemmas that will help us explore our new results:

Lemma 1.1. [4] *If $w, z_1, \ldots, z_n$ are complex numbers, then*

$$\left| w - \frac{z_1 + \cdots + z_n}{n} \right| \leq \max_{1 \leq j \leq n} |w - z_j| .$$

Lemma 1.2. *For the vector $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$ there exist vectors $\delta = (\delta_1, \delta_2, \ldots, \delta_n) \in \mathbb{R}^n$ and $J \in \mathbb{R}^n$, such that $\delta J^T = 0$, then for $n = 2$*

$$|z\delta^T| \leq \max_{i,j} |z_i - z_j| \|\delta\|_1,$$

and for $n \geq 3$

$$|z\delta^T| \leq (n-2) \max_{i,j} |z_i - z_j| \|\delta\|_1,$$

where J is a vector of all ones and $\|\delta\|_1 = \max\{|\delta_1|, \ldots, |\delta_n|\}$.

Proof. Let $n = 2$. We start with $\delta J^T = \delta_1 + \delta_2 = 0$, which implies that $|z\delta^T| = |z_1\delta_1 + z_2\delta_2| = |z_1\delta_1 - z_2\delta_1| \leq |\delta_1||z_1 - z_2|$. Alternatively, we can express $|z\delta^T| = |z_1\delta_1 + z_2\delta_2| = |-z_1\delta_2 + z_2\delta_2| \leq |\delta_2||z_2 - z_1|$. From these observations, we conclude that $|z\delta^T| \leq |z_1 - z_2| \|\delta\|_1$ holds for $n = 2$.

Now consider $n = 3$. We assume the magnitudes $|\delta_1| \geq |\delta_2| \geq |\delta_3|$ without loss of generality. Thus, we have $\delta J^T = \delta_1 + \delta_2 + \delta_3 = 0$. We can express $|z\delta^T|$ as follows:

$$|z\delta^T| = |z_1\delta_1 + z_2\delta_2 + z_3\delta_3| = |-z_1\delta_2 - z_1\delta_3 + z_2\delta_2 + z_3\delta_3|.$$

Applying the triangle inequality, we find:$|z\delta^T| \leq |\delta_2||z_2 - z_1| + |\delta_3||z_1 - z_3|$. It follows that

$$|z\delta^T| \leq \max_{1 \leq i,j \leq 3} |z_i - z_j| \left(|\delta_2| + |\delta_3|\right).$$

Since $|\delta_2| + |\delta_3| = |\delta_1| = \max\{|\delta_1|, |\delta_2|, |\delta_3|\} = \|\delta\|_1$, we can conclude that

$$|z\delta^T| \leq \max_{1 \leq i,j \leq 3} |z_i - z_j| \|\delta\|_1.$$

To extend the analysis for $n = 4$ to n terms, we follow a similar pattern as established earlier. We consider the numbers $\delta_1, \delta_2, \delta_3, \ldots, \delta_n$ arranged such that $|\delta_1| \geq |\delta_2| \geq \cdots \geq |\delta_n|$. The constraint $\delta J^T = \delta_1 + \delta_2 + \cdots + \delta_n = 0$ remains valid. Next, we analyze the expression for $|z\delta^T|$:

$$|z\delta^T| = |z_1\delta_1 + z_2\delta_2 + \cdots + z_n\delta_n|.$$

Using the relationships derived up to this stage, we manipulate the expression based on the structure of the elements involved:

$$|z\delta^T| \leq |\delta_2||z_1 - z_2| + |\delta_3||z_1 - z_3| + \cdots + |\delta_n||z_1 - z_n|.$$

From here, we can generalize, considering the maximum difference among the z-terms. We find that:

$$|z\delta^T| \leq \max_{1\leq i,j\leq n} |z_i - z_j| \left(|\delta_2| + |\delta_3| + \cdots + |\delta_n|\right).$$

Given that $|\delta_2| + |\delta_3| + \cdots + |\delta_n| \leq (n-2)|\delta_1|$, we can conclude:

$$|z\delta^T| \leq (n-2) \max_{1\leq i,j\leq n} |z_i - z_j| \|\delta\|_1.$$

Thus, by repeating this argument up to n terms, we arrive at the final result:

$$|z_1\delta_1 + z_2\delta_2 + \cdots + z_n\delta_n| \leq (n-2) \max_{i,j} |z_i - z_j| \|\delta\|_1.$$

This establishes a consolidated upper bound that is dependent on the maximum pairwise difference in the z values and the L_1 norm of the δ coefficients. □

2. New Results

We start with a improved result of R. Drnovšek [1].

Theorem 2.1. *Let $A = [a_{ij}]_{n\times n}$ be a nonnegative matrix in [0,1) with the spectral radius $\rho(A) = 1$, If A has k zero diagonal elements, then the spread of A satisfies*

$$s(A) \geq \frac{k}{n-1}. \tag{2.1}$$

In particular, if $A \in \mathbb{N}_n$, i.e. $k = 1$, then

$$s(A) \geq \frac{1}{n-1}. \tag{2.2}$$

Proof. Given that A is a nonnegative matrix with spectral radius $\rho(A) = 1$, let us denote its Perron eigenvalue by $\lambda_1 = 1$, while the remaining eigenvalues are denoted by $\lambda_2, \ldots, \lambda_n$. According to Lemma 1.1, if we substitute $\rho(A) = 1$ and change the multiset $\{z_1, \ldots, z_m\}$ to $\{\lambda_2, \ldots, \lambda_n\}$, we arrive at the inequality:

$$s(A) = \max_{2\leq j\leq n} |1 - \lambda_j| \geq \left|1 - \frac{trA - 1}{n-1}\right| = \left|\frac{n - trA}{n-1}\right|.$$

Since the trace of A is given by $tr(A) = \sum_{i=1}^{n} a_{ii} \leq n - k$, where A has k zero diagonal elements and $a_{ii} \leq \rho(A) = 1$ for all i, we see that $trA \leq n-k$. Substituting this into our earlier expression gives:

$$s(A) \geq \left|\frac{n-(n-k)}{n-1}\right| = \left|\frac{k}{n-1}\right|.$$

Thus, we conclude that $s(A) \geq \frac{k}{n-1}$. In particular, for $k = 1$, we find that $s(A) \geq \frac{1}{n-1}$. Which completes the proof of the second result. □

Remark 2.2. It appears that our results in the above theorem improve upon the findings of R. Drnovšek [1], who established the bound of spread as $s(A) \geq \frac{k}{n}$. Given our derived result that

$$s(A) \geq \frac{k}{n-1},$$

we can observe that

$$\frac{k}{n-1} \geq \frac{k}{n}$$

holds true for $k \geq 0$ and $n \geq 2$. This shows that our bound is indeed stronger, as

$$s(A) \geq \frac{k}{n-1} \geq \frac{k}{n}.$$

This comparison highlights the robustness of our findings in offering a better estimate for the spread $s(A)$.

Remark 2.3. We find $s(A) \geq 1$ from Theorem 2.1 if $A \in \mathbb{N}_2$. Thus, Theorem 2.1 provides another proof of preposition 2.4 of R. Drnovšek [1]. This demonstrates the consistency and interconnectedness of the results.

For $n \geq 4$ it seems diffcult to obtain exact lower bounds for the spread of matrices in $A \in \mathbb{N}_n$. We thus restrict our attention to a special subset of $A \in \mathbb{N}_n$. Next theorem trivially implies that every matrix in $A \in \mathbb{N}_n$ has at least two distinct eigenvalues, that is, 1 is not the only point in its spectrum. Let $A \in \mathbb{M}_n$ (with $n \geq 3$) be the collection of all irreducible non-singular matrices in $A \in \mathbb{N}_n$, having exactly two distinct eigenvalues. We now prove sharp lower bounds for the spread of matrices in $A \in \mathbb{M}_n$.

Theorem 2.4. *Let $A \in \mathbb{M}_n$ be a matrix, then the spread satisfies*

$$s(A) \geq \frac{n-2}{n-1}. \tag{2.3}$$

Moreover, bound is exact if $A = \dfrac{1}{n-1}(J_n - I_n) + E_{11}$.

Proof. Due to Suleimanova ([3] Chapter 7) sum of negative eigenvalues (if any) is not exceed the largest eigenvalue of A, i.e., there is a multiset $\{\lambda_1, \dots, \lambda_n\}$ with conditions:

$$\lambda_1 + \cdots + \lambda_n \geq 0,$$

where $\lambda_1 > 0 > \lambda_2 \geq \cdots \geq \lambda_n$ is a spectrum of non-singular, non-negative matrix A. Now assume that a matrix $A \in \mathbb{M}_n$, is irreducible. Then 1 is a simple eigenvalue of A by the Perron-Frobenius theorem. Therefore, A also has an eigenvalue $\lambda \in (-1, 0)$ of multiplicity $n-1$. It follows that

$$1 - (n-1)\lambda \geq 0,$$

on solving, we get $\lambda \leq \dfrac{1}{n-1}$. This follows the proof $s(A) \geq \dfrac{n-2}{n-1}$. □

Remark 2.5. For the above Theorem 2.4 matrix A should be necessarily irreducible. If A is reducible it is permutation-similar to a block upper-triangular matrix $\begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix}$, where A_{11} and A_{22} are square sub-matrices of order $p < n$ and $q < n$ respectively. Both diagonal blocks are either an irreducible (square) matrix or in order 1×1. At least one of both diagonal blocks that has a zero diagonal element since $A \in \mathbb{M}_n$. Since 0 is not eigenvalue of A clearly 1×1 diagonal block (if exist) is nonzero. Therefore, $p, q \geq 2$ and so that for block A_{11}, we get

$$s(A) \geq \frac{n-2}{n-1} \geq \frac{p-2}{p-1} \leq s(A_{11}),$$

which possible only in case if $A = A_{11}$. Clearly A is irreducible matrix.

Example 2.6. Define a matrix $A = \dfrac{1}{n-1}(J_n - I_n)$, where I_n is the identity matrix and J_n is the all-ones matrix. Then A is irreducible, non-negative, non-singular matrix and sum of the negative eigenvalues is less than its spectral radius, since its eigenvalues are $\lambda_1 = 1, \lambda_2 = \cdots = \lambda_n = -\dfrac{1}{n-1}$. So that $s(A) = 1 + \dfrac{1}{n-1} = \dfrac{n}{n-1} \geq \dfrac{n-2}{n-1}$ for every $n \geq 4$.

For the proof of exactness of this lower bound we provide an example below:

Example 2.7. Define a matrix $A = \dfrac{1}{n-1}(J_n - I_n) + E_{11}$, where I_n is the identity matrix, J_n is the all-ones matrix and E_{11} has a 1 in position $(1, 1)$ and zeros elsewhere. Then A is irreducible, non-negative, non-singular matrix and sum of the negative eigenvalues is less than its spectral radius, since its eigenvalues are $\lambda_1 = 1, \lambda_2 = \cdots = \lambda_n = \frac{1}{n-1}$. So that $s(A) = 1 - \frac{1}{n-1} = \frac{n-2}{n-1}$ for every $n \geq 4$.

Remark 2.8. Our Theorem 2.4 improves upon R. Drnovšek findings ([1] Theorem 2.5), as we have:

$$s(A) \geq \frac{n-2}{n-1} \geq \frac{n}{2(n-1)} \quad \text{for } n \geq 4.$$

This demonstrates that the lower bound of $s(A)$ is sharp for n greater than or equal to 4.

Next result is an application of Lemma 1.2.

Theorem 2.9. *Let $A = [a_{ij}] \in \mathbb{N}_n$ $(n \geq 3)$ be a matrix with real eigenvalues, then*

$$s(A) \geq \frac{1}{(n-2)\left|1 - \frac{trA}{n}\right|} \left| trA^2 - \frac{(trA)^2}{n} \right|. \tag{2.4}$$

Proof. Assume that $\lambda = (\lambda_1, \ldots, \lambda_n)$ represents the real spectrum of the matrix A. By the given assumptions, we have $|\lambda_i| \leq 1$ for all $i = 1, 2, \ldots, n$, which also implies that $|\lambda_i|^2 \leq 1$ for each i. Now, lets define the vector $\delta = (\delta_1, \ldots, \delta_n) \in \mathbb{R}^n$ where $\delta_i = \lambda_i - \frac{\mathrm{tr}A}{n}$. Given this definition, we know that $\delta J^T = 0$, where J is a vector of ones, indicating that δ is centered around the mean of the eigenvalues. Now, applying the previous proposition, we arrive at the result: $|\lambda\delta^T| \leq (n-2)\max_{i,j} |\lambda_i - \lambda_j| \|\delta\|_1$. This establishes an important relationship between the eigenvalues of the matrix A, the perturbations captured by δ, and the maximum difference among the eigenvalues. The proof of (2.4) can be elaborated by considering the application of Lemma 1.2. □

Remark 2.10. Since by the definition of the matrix $A = [a_{ij}]$, at least one diagonal element is 0 and the remaining diagonal elements $a_{ii} \in [0, 1]$. This implies that the trace of A must satisfy $\mathrm{tr}(A) < n$. This validates the above result.

Theorem 2.11. *Let $A \in \mathbb{N}_n$ be a matrix, then*

$$s(A) \geq \sqrt{3} \max_{1 \leq i \leq n} \left[r_i + \left| a_{ii} - \frac{trA}{n} \right| \right], \tag{2.5}$$

where $r_i = \sum\limits_{j=1, i \neq j}^{n} a_{ij}$.

Proof. Geršgorins theorem tells us that each eigenvalue λ_i of a matrix is located within a specific distance from its corresponding diagonal entry a_{ii}. Specifically, we have the relationship $|\lambda_i - a_{ii}| \leq r_i$, where r_i is the radius of the circle centered at a_{ii}. This means all eigenvalues can be found inside those circles.

Our goal is to find the smallest circle that includes all of Geršgorins circles, ensuring that it captures all the eigenvalues of the matrix A. Since the centers of these circles are nonnegative real numbers, we can use the average of the diagonal entries, given by $\frac{1}{n}\sum_{i=1}^{n} a_{ii} = \frac{trA}{n}$, to identify the center of our combined circle. To seek the smallest circle, we establish the following model:

$$\begin{cases} \min \left|\lambda_i - \dfrac{trA}{n}\right| \qquad \text{such that } |\lambda_i - a_{ii}| \leq r_i \ . \end{cases}$$

We can use method below to solve above model. As

$$\left|\lambda_i - \frac{trA}{n}\right| = \left|\lambda_i - a_{ii} + a_{ii} - \frac{trA}{n}\right| \leq |\lambda_i - a_{ii}| + \left|a_{ii} - \frac{trA}{n}\right|,$$

or

$$\left|\lambda_i - \frac{trA}{n}\right| \leq \max_{1 \leq i \leq n}\left[r_i + \left|a_{ii} - \frac{trA}{n}\right|\right]. \tag{2.6}$$

equation (2.6) represent a circular disc with its center $\dfrac{trA}{n}$ and radius $r_A = \max_{1 \leq i \leq n}\left[r_i + \left|a_{ii} - \dfrac{trA}{n}\right|\right]$, ensuring that all eigenvalues of our nonnegative matrix A are included within it. If the eigenvalues λ_i (for $1 \leq i \leq n$) are allowed to be any complex numbers, we can reference a well-known Theorem by Jung (found in [1]), which states that:

$$s(A) \geq \sqrt{3}r_A. \tag{2.7}$$

With this information, we can achieve our desired proof. □

Let $\mathbb{E}_n$ (with $n \geq 2$) be the collection of all singular matrices in $\mathbb{N}_n$, assume $A \in \mathbb{E}_n$ with rank(A)=q. Now we will prove bounds for the spread for matrices in $\mathbb{E}_n$.

Theorem 2.12. *Let $A \in \mathbb{E}_n$ be a matrix with rank$(A) = q$ such that $2 \leq q < n$, then spread of A satisfy*

$$s(A) \geq \frac{trA}{n}\sqrt{\frac{3(n-1)(n-q)}{q}} \tag{2.8}$$

Proof. Let $\lambda_1, \ldots, \lambda_n$ be all eigenvalues of A and without loss of generality, we suppose that $\lambda_1, \ldots, \lambda_q$ are all non-zero eigenvalues of A, since rank$(A) = q$ and therefore by the Cauchy-Schwarz inequality, we have

$$|trA|^2 = \left|\sum_{i=1}^{n} \lambda_i\right|^2 = \left|\sum_{i=1}^{q} \lambda_i\right|^2 \leq q\sum_{i=1}^{q} |\lambda_i|^2 . \tag{2.9}$$

Next, by Theorem 2.1 of [5] at-most $(n-k+1)$ eigenvalues lie in the rectangles (counting algebraic multiplicities)

$$\left[\frac{\Re trA}{n} - \alpha, \frac{\Re trA}{n} + \alpha\right] \times \left[\frac{\Im trA}{n} - \beta, \frac{\Im trA}{n} + \beta\right], \tag{2.10}$$

where

$$\alpha = \left[\frac{n-k}{nk}\left(\sum_{i=1}^{n}(\Re\lambda_i)^2 - \frac{(\Re trA)^2}{n}\right)\right]^{\frac{1}{2}},$$

and

$$\beta = \left[\frac{n-k}{nk}\left(\sum_{i=1}^{n}(\Im\lambda_i)^2 - \frac{(\Im trA)^2}{n}\right)\right]^{\frac{1}{2}},$$

whose vertices of largest rectangle for $k = 1$ in (2.10) should be on the boundary of largest circle with radius

$$r_A = \sqrt{\alpha^2 + \beta^2} = \sqrt{\frac{n-1}{n}\left(\sum_{i=1}^{n} |\lambda_i|^2 - \frac{|trA|^2}{n}\right)}.$$

Since from (2.9), we have

$$r_A \geq \sqrt{\frac{n-1}{n}\left(\frac{|trA|^2}{q} - \frac{|trA|^2}{n}\right)}.$$

By using the inequality (2.7), we have expected result (2.8). □

Lemma 2.13. *Let $z_1, z_2, \ldots, z_n$ are complex numbers, then*

(a) $\sum\limits_{j=1}^{n} |z_j|^2 = \frac{1}{n}\left|\sum\limits_{j=1}^{n} z_j\right|^2 + \frac{1}{n}\sum\limits_{1\leq j<k\leq n} |z_j - z_k|^2.$

(b) $\sum_{1\le j<k\le n} |z_j - z_k|^2 \le \frac{n(n+1)}{6} \max_{i,j} |z_j - z_k|$.

Proof. For (a), the given complex numbers $z_1, z_2, \ldots, z_n$, we get

$$\left|\sum_{j=1}^{n} z_j\right|^2 + \sum_{1\le j<k\le n} |z_j - z_k|^2 = \left(\left|\sum_{j=1}^{n} z_j\right|\right)\left(\left|\overline{\sum_{j=1}^{n} z_j}\right|\right) + \sum_{1\le j<k\le n} |z_j - z_k|^2$$

$$= \left(\left|\sum_{j=1}^{n} z_j\right|\right)\left(\left|\sum_{j=1}^{n} \overline{z}_j\right|\right) + \sum_{1\le j<k\le n} |z_j - z_k|^2$$

$$= \sum_{j=1}^{n} |z_j|^2 + \sum_{1\le j<k\le n} (z_j\overline{z}_k + z_k\overline{z}_j) + \sum_{1\le j<k\le n} |z_j - z_k|^2$$

$$= \sum_{j=1}^{n} |z_j|^2 + \sum_{1\le j<k\le n} (z_j\overline{z}_k + z_k\overline{z}_j + |z_j - z_k|^2)$$

$$= \sum_{j=1}^{n} |z_j|^2 + \sum_{1\le j<k\le n} (|z_j|^2 + |z_k|^2) = \sum_{j=1}^{n} |z_j|^2 + (n-1)\sum_{j=1}^{n} |z_j|^2 = n\sum_{j=1}^{n} |z_j|^2 .$$

Therefore proof is clear.

For (b), since

$$\sum_{1\le j<k\le n} |z_j - z_k|^2 = (|z_1 - z_2| + \cdots + |z_1 - z_n|) + (|z_2 - z_3| + \cdots + |z_2 - z_n|) + \cdots + (|z_{n-1} - z_n|)$$

$$\le n \max_{i,j} |z_j - z_k| + (n-1) \max_{i,j} |z_j - z_k| + \cdots + \max_{i,j} |z_j - z_k| = \frac{n(n+1)}{6} \max_{i,j} |z_j - z_k|.$$

□

Theorem 2.14. *Let $A \in \mathbb{E}_n$ be a matrix, then spread of A satisfy*

$$s(A) \ge \frac{6}{n(n^2-1)}(trA)^2. \tag{2.11}$$

Proof. If $A \in \mathbb{E}_n$, then Proposition 2.2 in Roman Drnovšek gives $s_1^m \le (n-1)^{m-1} s_m$ where $s_k = tr(A^k)$ for all positive integer k. Since $s(A) > 0$ clear by Proposition 2.1 of Roman Drnovšek, we may assume that $s := s(A) \in (0, 1)$, and consequently the eigenvalues of A have positive real parts. Then

$$\left(\sum_{i=1}^{n} \lambda_i\right)^2 = s_1^2 \le (n-1)s_2 = (n-1)\sum_{i=1}^{n-1} \lambda_i^2.$$

Now using Lemma 2.13 and inequality (2.9). The proof of inequality (2.11) is clear. □

Theorem 2.15. *Let $A \in \mathbb{E}_n$ be a matrix with rank$(A) = q$ such that $2 \leq q < n$, then spread of A satisfy*

$$s(A) \geq \frac{6(n-q)}{qn(n+1)}(trA)^2. \tag{2.12}$$

Proof. This result is clear from Lemma 2.13 and inequality (2.9) by arranging in place of multiset $\{z_1, \dots, z_n\}$ to $\{\lambda_1, \dots, \lambda_n\}$. □

Next results are of multiplicative spread for non-singular matrices.

Theorem 2.16. *Let $A \in \mathbb{N}_n$ be a non-singular matrix, then the multiplicative spread of A satisfies*

$$\kappa(A) \geq \frac{|\, trA^{-1} \,|}{n} \geq 1. \tag{2.13}$$

Proof. Since $A = [a_{ij}]$ is a square nonnegative matrix of n order, the spectral radius $\rho(A) = 1$ is its Perron eigenvalue, We denote it by λ_1, where as the rest eigenvalues of A are denoted by $\lambda_2, \dots, \lambda_n$. For every $i = 1, 2, \dots, n$ we have

$$Re\left(\frac{1}{\lambda_i}\right) \leq \left|\frac{1}{\lambda_i}\right| \leq \max\left\{\left|\frac{\lambda_1}{\lambda_i}\right| : 1 \leq i \leq n\right\} = \kappa(A),$$

and so, we have

$$\sum_{i=1}^{n} Re\left(\frac{1}{\lambda_i}\right) = \sum_{i=1}^{n} \frac{1}{\lambda_i} \leq \left|\sum_{i=1}^{n} \frac{1}{\lambda_i}\right| \leq \sum_{i=1}^{n} \left|\frac{1}{\lambda_i}\right| \leq n\kappa(A),$$

It gives the result

$$\kappa(A) \geq \frac{|\, trA^{-1} \,|}{n}.$$

Since $|\lambda_i| \leq 1$ this implies $\left|\frac{1}{\lambda_i}\right| \geq 1 \; \forall i = 1, \dots n$. Therefore $|trA^{-1}| \geq n$. Inequality (2.13) is clear. □

Theorem 2.17. *Let $A \in \mathbb{N}_n$ be a non-singular matrix, then the multiplicative spread of A satisfies*

$$\kappa(A) \geq \frac{2(n-1)}{n-2}. \tag{2.14}$$

Proof. Consider first that a $A \in \mathbb{N}_n$ is irreducible non singular matrix. Then 1 is a simple eigenvalue of matrix A by the Perron-Frobenius theorem. Therefore, A also has an eigenvalue $\lambda \in (-1, 1)$ of multiplicity $(n-1)$. In this case by the inequality $\sum_{i=1}^{n} \lambda_i^2 \leq \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} (\lambda_i - \lambda_j)^2$ for nonnegative matrices in $\mathbb{N}_n$, we get

$$1 + (n-1)\lambda^2 \leq (n-1)(1-\lambda)^2.$$

Simplifying it, we obtain

$$\lambda \leq \frac{n-2}{2(n-1)}.$$

Hence

$$\kappa(A) = \left|\frac{1}{\lambda}\right| \geq \frac{2(n-1)}{n-2}.$$

□

3. Conclusion

In this research, we concentrated on the spread of nonnegative matrices, revealing significant findings. We introduced two innovative lower bounds for the spread, leveraging the inequalities established in Lemmas 1.1 and 1.2. The lower bounds we derived not only enhance our understanding but also improve the results previously reported by R. Drnovek [1].

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