Original Research Article

On Dual Lorentzian Vectors and Angles with Leonardo Number Sequences

Abstract

This study introduces dual Lorentzian vectors whose components are derived from Leonardo number sequences. By embedding these sequences into a three-dimensional dual Lorentzian space, the research investigates their geometric properties and establishes the mathematical framework for dual Lorentzian angles.

Keywords:Dual vector; dual angle; recurrences; integer sequences; Leonardo numbers.

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1 Introduction

Sequences of positive integers have long attracted mathematicians due to their elegant properties and diverse applications. Among these, the Fibonacci and Lucas sequences are particularly prominent, known for their deep mathematical structure and utility across various scientific disciplines. These sequences not only play an essential role in number theory and combinatorics but also emerge in biology, physics, computer science, and beyond. Their recursive nature and connections to algebraic identities make them a central focus in both theoretical investigations and applied problem-solving (Bicknell et al., 1972; Koshy, 2001; Park et al., 2020; Candan, 2022; Vajda, 1989; Verner and Hoggatt, 1969; Halıcı, 2012; Özkan and Akkuş, 2024).

In pursuit of broader generalizations, many researchers have introduced and studied extensions and analogs of these classical sequences. For instance, Horadam explored a generalized Fibonacci sequence $H_n(a, b; p, q)$, where $H_0 = a$, $H_1 = b$, and

$$H_n = pH_{n-1} + qH_{n-2}, \quad n \geqslant 2$$

(Horadam, 1961, 1963). This formulation allows for the construction of various new integer sequences by adjusting the parameters a, b, p, and q, encompassing numerous well-known recurrences as special cases.

One such notable sequence is the Leonardo numbers denoted by L_n which are defined by the recurrence relation

$$\mathcal{L}e_n = \mathcal{L}e_{n-1} + \mathcal{L}e_{n-2} + 1, \quad \mathcal{L}e_0 = 1, \mathcal{L}e_1 = 1, \quad n \geq 2. \tag{1.1}$$

This sequence begins as

$$1, 1, 3, 5, 9, 15, 25, 41, 67, 109, 177, \ldots$$

and finds applications in computational theory, particularly in data structure analysis such as smoothsort, where it models the size of heaps in a heap-based sorting algorithm. The recurrence of Leonardo numbers can also be interpreted as a variant of the Fibonacci sequence with an additional increment term, yielding a slower-growing but structurally rich sequence. The Binet's formula for Leonardo number sequence is

$$\mathcal{L}e_n = \frac{2\varphi^{n+1} - 2\psi^{n+1}}{\varphi - \psi} - 1, \quad n \geq 0 \tag{1.2}$$

and

$$\mathcal{L}e_n = 2F_{n+1} - 1, \quad n \geq 0 \tag{1.3}$$

where $\varphi = \frac{1+\sqrt{5}}{2}$, $\psi = \frac{1-\sqrt{5}}{2}$, F_n is n^{th} Fibonacci number ($F_0 = 0$, $F_1 = 1$, ...) and $F_n = F_{n-1} + F_{n-2}$. In other words,

$$L_n = L_{n-1} + L_{n-2}, \quad n \geq 2$$

where L_n is n^{th} Lucas number ($L_0 = 2$, $L_1 = 1$, ...), and $L_n = \varphi^n + \psi^n$. The following equations can be written:

$$L_n = F_{n+1} + F_{n-1},$$

$$F_m F_{n+1} - F_{m+1} F_n = (-1)^n F_{m-n} \tag{1.4}$$

(Halıcı, 2019; Catarino and Borges, 2020; Karataş, 2022).

In the context of algebraic structures, dual numbers introduced by Clifford and further developed in geometry and kinematics by Kotelnikov (1895) and Study (1901) offer an intriguing extension to real numbers. A dual number is an expression of the form $\gamma = \gamma_0 + \gamma_1\varepsilon$, where $\gamma_0, \gamma_1 \in \mathbb{R}$, and ε is a dual unit satisfying $\varepsilon^2 = 0$, $\varepsilon \neq 0$. The set of all such numbers is denoted by

$$\mathbb{D} = \{\gamma = \gamma_0 + \gamma_1\varepsilon \mid \gamma_0, \gamma_1 \in \mathbb{R}, \varepsilon^2 = 0, \varepsilon \neq 0\}.$$

The square root of a dual number γ is defined as

$$\sqrt{\gamma} = \sqrt{\gamma_0} + \frac{\gamma_1}{2\sqrt{\gamma_0}}\varepsilon. \tag{1.5}$$

Similarly, a dual vector takes the form

$$\vec{\gamma} = \vec{\gamma}_0 + \vec{\gamma}_1\varepsilon,$$

where $\vec{\gamma}_0$ and $\vec{\gamma}_1$ are real vectors. Dual vectors and dual numbers are widely used in representing motions and orientations in Euclidean space, particularly in the study of screws, rotations, and transformations (Veldkamp, 1976; Guggenheimer, 1963).

In this paper, we explore a dual number extension of the Leonardo number sequence, investigate its algebraic and combinatorial properties, and derive new identities and formulas based on this generalized structure.

2 Dual Lorentzian Space $\mathbb{D}_1^3$

Let $\mathbb{R}^3 = \{(a_1, a_2, a_3) \mid a_1, a_2, a_3 \in \mathbb{R}\}$ be a 3-dimensional real vector space given by the vectors $\vec{a} = (a_1, a_2, a_3)$ and $\vec{b} = (b_1, b_2, b_3)$ in $\mathbb{R}^3$. The Lorentzian scalar product of $\vec{a}$ and $\vec{b}$ is defined by

$$\langle \vec{a}, \vec{b} \rangle_{\mathbb{L}} = -a_1 b_1 + a_2 b_2 + a_3 b_3. \tag{2.1}$$

$(\mathbb{R}^3, \langle \cdot, \cdot \rangle)$ is called the 3-dimensional Lorentzian space, or Minkowski 3-space, denoted by $\mathbb{R}_1^3$, and the Lorentzian vector product of $\vec{a}$ and $\vec{b}$ is defined by

$$\begin{aligned} \vec{a} \wedge_{\mathbb{L}} \vec{b} &= \begin{vmatrix} -\vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \\ &= \left(-a_2 b_3 + a_3 b_2, -a_1 b_3 + a_3 b_1, a_1 b_2 - a_2 b_1\right), \end{aligned} \tag{2.2}$$

where $\vec{e_1} \wedge \vec{e_2} = \vec{e_3}$, $\vec{e_2} \wedge \vec{e_3} = -\vec{e_1}$, and $\vec{e_3} \wedge \vec{e_1} = \vec{e_2}$ (Akutagawa and Nishikawa, 1990).

Also, $\vec{a}$ in $\mathbb{R}_1^3$ is called a spacelike vector, a lightlike vector or a timelike vector if

$$\langle \vec{a}, \vec{a} \rangle_{\mathbb{L}} > 0 \text{ or } \vec{a} = \vec{0}, \quad \langle \vec{a}, \vec{a} \rangle_{\mathbb{L}} = 0 \text{ and } \vec{a} \neq \vec{0}, \quad \text{or } \langle \vec{a}, \vec{a} \rangle_{\mathbb{L}} < 0,$$

respectively.

The Lorentzian scalar product and vector product of dual vectors $\vec{\mathcal{A}} = \vec{a}_0 + \varepsilon \vec{a}_0^*$ and $\vec{\mathcal{B}} = \vec{b}_0 + \varepsilon \vec{b}_0^*$ in $\mathbb{D}_1^3$ are given by

$$\langle \vec{\mathcal{A}}, \vec{\mathcal{B}} \rangle_{\mathbb{L}} = \langle \vec{a_0}, \vec{b_0} \rangle_{\mathbb{L}} + \varepsilon \left(\langle \vec{a_0}, \vec{b_0^*} \rangle_{\mathbb{L}} + \langle \vec{a_0^*}, b_0 \rangle_{\mathbb{L}} \right), \tag{2.3}$$

and

$$\vec{\mathcal{A}} \wedge_{\mathbb{L}} \vec{\mathcal{B}} = \vec{a_0} \wedge_{\mathbb{L}} \vec{b_0} + \varepsilon (\vec{a_0} \wedge_{\mathbb{L}} \vec{b_0^*} + \vec{a_0^*} \wedge_{\mathbb{L}} \vec{b_0}), \tag{2.4}$$

where we call the dual space $\mathbb{D}^3$ together with this Lorentzian inner product as dual Lorentzian space and denote it by $\mathbb{D}_1^3$ (Uğurlu and Çalışkan, 1996).

Definition 2.1. Let $\vec{\mathcal{A}} \in \mathbb{D}_1^3$. The dual vector $\vec{\mathcal{A}}$ is said to be spacelike if the vector $\vec{a}_0$ is spacelike, timelike if the vector $\vec{a}_0$ is timelike, and lightlike (dual null) if the vector $\vec{a}_0$ is lightlike, respectively.

Definition 2.2. The norm of a dual vector $\vec{\mathcal{A}}$ in $\mathbb{D}_1^3$ is given by

$$\left| \vec{\mathcal{A}} \right|_{\mathbb{L}} = \sqrt{\langle \vec{\mathcal{A}}, \vec{\mathcal{A}} \rangle_{\mathbb{L}}} = |\vec{a_0}| + \varepsilon \frac{\langle \vec{a_0}, \vec{a_0} \rangle}{|\vec{a_0}|}. \tag{2.5}$$

Let $h(x_0 + x_1 \varepsilon)$ be a dual function Fike and Alonso (2009). Then the Taylor series expansion of this dual function $x_0 + x_1 \varepsilon = a_0 + a_1 \varepsilon$ can be given as:

$$h(a_0 + a_1 \varepsilon) = h(a_0) + a_1 h'(a_0) \varepsilon, \tag{2.6}$$

where the prime represents differentiation with respect to x, i.e.

$$h'(x) = h'(x + \varepsilon 0) = \frac{d}{dx}h(x). \tag{2.7}$$

In this paper, we introduce the dual Leonardo numbers, which generalize the classical Leonardo number sequence (Karataş, 2022, 2023; Halıcı, 2019; Halıcı and Curuk, 2019; Alp and Koçer, 2021; Catarino and Borges, 2020; Babadağ and Atasoy, 2024; Babadağ et al., 2024; Atasoy, 2025) by employing the concept of dual numbers. We investigate fundamental properties of these numbers and construct new geometric entities, namely, the dual Leonardo vector and dual Leonardo angle within the framework of dual Lorentzian geometry.

3 Dual Leonardo Number Sequences

In this section, the focus is on dual Leonardo number sequences, with an exposition of their fundamental characteristics (Karataş, 2023; Alp and Koçer, 2021; Babadağ and Atasoy, 2024). The following terms are defined as follows: identities and properties.

Definition 3.1. The dual number of the form

$$\mathcal{DL}e_n = \mathcal{L}e_n + \varepsilon \mathcal{L}e_{n+1} \tag{3.1}$$

is called the n^{th} dual Leonardo number *and* $\varepsilon^2 = 0,\ \varepsilon \neq 0$, where $\mathcal{L}e_n$ is n^{th} Leonardo number.

From definition, the following recurrence relation can be prove

$$\mathcal{DL}e_n = \mathcal{DL}e_{n-1} + \mathcal{DL}e_{n-2} + 1 + \varepsilon, \quad n \geq 2.$$

The few dual Leonardo numbers are given as

$$\mathcal{DL}e_1 = 1 + 3\varepsilon,\ \mathcal{DL}e_2 = 3 + 5\varepsilon, ...$$

Theorem 3.1 (Karataş (2023))**.** *The Binet's like formula for dual Leonardo numbers is*

$$\mathcal{DL}e_n = 2\frac{\varphi^{n+1}\underline{\varphi} - \psi^{n+1}\underline{\psi}}{\sqrt{5}} - 1 - \varepsilon \tag{3.2}$$

where

$$\underline{\varphi} = 1 + \varepsilon\varphi \text{ and } \underline{\psi} = 1 + \varepsilon\psi. \tag{3.3}$$

Proof. From (1.2), (3.1) and (3.3), we find that

$$\begin{aligned}
\mathcal{DL}e_n &= \mathcal{L}e_n + \varepsilon \mathcal{L}e_{n+1} \\
&= 2\frac{\varphi^{n+1} - \psi^{n+1}}{\varphi - \psi} - 1 + \varepsilon\left(2\frac{\varphi^{n+2} - \psi^{n+2}}{\varphi - \psi} - 1\right) \\
&= \frac{2\varphi^{n+1}(1 + \varphi\varepsilon) - 2\psi^{n+1}(1 + \psi\varepsilon)}{\varphi - \psi} - 1 - \varepsilon \\
&= 2\frac{\varphi^{n+1}\underline{\varphi} - \psi^{n+1}\underline{\psi}}{\sqrt{5}} - 1 - \varepsilon.
\end{aligned}$$

This is the desired result. □

For example, for $n = 1$,

$$\begin{aligned}
\mathcal{DL}e_1 &= 2\frac{\varphi^2\underline{\varphi} - \psi^2\underline{\psi}}{\sqrt{5}} - 1 - \varepsilon \\
&= 2\frac{(\varphi^2 - \psi^2) + \varepsilon(\varphi^3 - \psi^3)}{\sqrt{5}} - 1 - \varepsilon \\
&= 2\frac{\sqrt{5} + 2\varepsilon\sqrt{5}}{\sqrt{5}} - 1 - \varepsilon \\
&= 1 + 3\varepsilon.
\end{aligned}$$

Checking result: $\mathcal{L}e_1 = 1$, $\mathcal{L}e_2 = 3$ and $\mathcal{DL}e_1 = \mathcal{L}e_1 + \varepsilon\mathcal{L}e_2 = 1 + 3\varepsilon$.

Theorem 3.2. *Let $\mathcal{DL}e_n$ be the n^{th} dual Leonardo number. Then,*

$$\mathcal{DL}e_n = \mathcal{DL}e_{n-1} + \mathcal{DL}e_{n-2} + (1 + \varepsilon), \tag{3.4}$$

$$\mathcal{DL}e_n - \mathcal{DL}e_{n+1}\varepsilon = \mathcal{L}e_n, \tag{3.5}$$

$$\begin{aligned}
\mathcal{DL}e_n\mathcal{DL}e_m + \mathcal{DL}e_{n+1}\mathcal{DL}e_{m+1} &= (\mathcal{L}e_n\mathcal{L}e_m + \mathcal{L}e_{n+1}\mathcal{L}e_{m+1}) \\
&\quad + (\mathcal{L}e_n\mathcal{L}e_{m+1} + \mathcal{L}e_{n+1}\mathcal{L}e_m \\
&\quad + \mathcal{L}e_{n+1}\mathcal{L}e_{m+2} + \mathcal{L}e_{n+2}\mathcal{L}e_{m+1})\varepsilon.
\end{aligned} \tag{3.6}$$

Proof. (3.4): Using the definition of dual Leonardo numbers, we have

$$\begin{aligned}
\mathcal{DL}e_{n-1} + \mathcal{DL}e_{n-2} &= (\mathcal{L}e_{n-1} + \mathcal{L}e_n\varepsilon) + (\mathcal{L}e_{n-2} + \mathcal{L}e_{n-1}\varepsilon) \\
&= (\mathcal{L}e_{n-1} + \mathcal{L}e_{n-2}) + (\mathcal{L}e_n + \mathcal{L}e_{n-1})\varepsilon.
\end{aligned}$$

Since Leonardo numbers satisfy

$$\mathcal{L}e_n = \mathcal{L}e_{n-1} + \mathcal{L}e_{n-2} + 1$$

and

$$\mathcal{L}e_{n+1} = \mathcal{L}e_n + \mathcal{L}e_{n-1} + 1$$

we obtain

$$\begin{aligned}
\mathcal{DL}e_{n-1} + \mathcal{DL}e_{n-2} &= (\mathcal{L}e_n - 1) + (\mathcal{L}e_{n+1} - 1)\varepsilon \\
&= \mathcal{DL}e_n - (1 + \varepsilon).
\end{aligned}$$

Thus,

$$\mathcal{DL}e_n = \mathcal{DL}e_{n-1} + \mathcal{DL}e_{n-2} + (1 + \varepsilon).$$

(3.5): From (4.1),

$$\begin{aligned}
\mathcal{DL}e_n - \mathcal{DL}e_{n+1}\varepsilon &= (\mathcal{L}e_n + \mathcal{L}e_{n+1}\varepsilon) - (\mathcal{L}e_{n+1} + \mathcal{L}e_{n+2}\varepsilon)\varepsilon \\
&= \mathcal{L}e_n + \mathcal{L}e_{n+1}\varepsilon - \mathcal{L}e_{n+1}\varepsilon - \mathcal{L}e_{n+2}\varepsilon^2 \\
&= \mathcal{L}e_n.
\end{aligned}$$

(3.6): From (4.1),

$$\begin{aligned}
\mathcal{DL}e_n\mathcal{DL}e_m &= (\mathcal{L}e_n + \mathcal{L}e_{n+1}\varepsilon)(\mathcal{L}e_m + \mathcal{L}e_{m+1}\varepsilon) \\
&= \mathcal{L}e_n\mathcal{L}e_m + (\mathcal{L}e_n\mathcal{L}e_{m+1} + \mathcal{L}e_{n+1}\mathcal{L}e_m)\varepsilon.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\mathcal{DL}e_{n+1}\mathcal{DL}e_{m+1} &= (\mathcal{L}e_{n+1} + \mathcal{L}e_{n+2}\varepsilon)(\mathcal{L}e_{m+1} + \mathcal{L}e_{m+2}\varepsilon) \\
&= \mathcal{L}e_{n+1}\mathcal{L}e_{m+1} + (\mathcal{L}e_{n+1}\mathcal{L}e_{m+2} + \mathcal{L}e_{n+2}\mathcal{L}e_{m+1})\varepsilon.
\end{aligned}$$

Adding both expressions gives

$$\begin{aligned}&\mathcal{D}\mathcal{L}e_n\mathcal{D}\mathcal{L}e_m+\mathcal{D}\mathcal{L}e_{n+1}\mathcal{D}\mathcal{L}e_{m+1}\\&=\left(\mathcal{L}e_n\mathcal{L}e_m+\mathcal{L}e_{n+1}\mathcal{L}e_{m+1}\right)\\&+\left(\mathcal{L}e_n\mathcal{L}e_{m+1}+\mathcal{L}e_{n+1}\mathcal{L}e_m+L_{n+1}\mathcal{L}e_{m+2}+\mathcal{L}e_{n+2}\mathcal{L}e_{m+1}\right)\varepsilon.\end{aligned}$$

The proof is completed. ☐

4 Dual Leonardo Vector and Angle in Dual Lorentzian Space

In this section, we obtain dual Leonardo vector and dual Lorentzian angle by using Leonardo number sequences in dual Lorentzian spaces $\mathbb{D}_1^3$ (see Fig. 1).

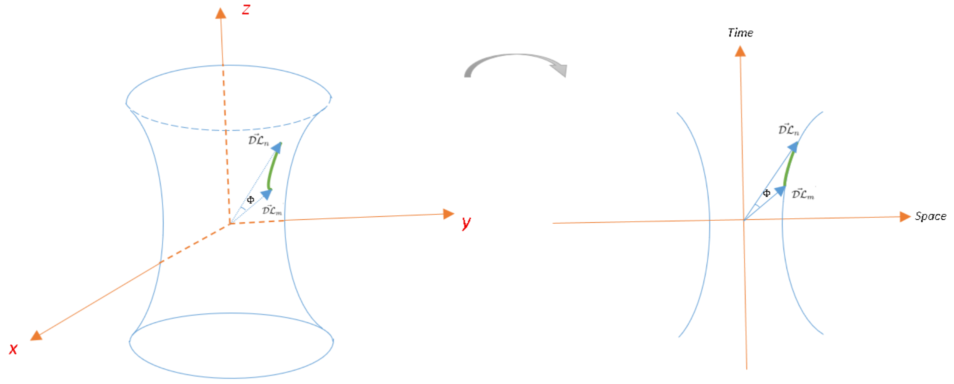

Figure 1: Vector and angle on Lorentzian sphere.

Definition 4.1. The n^{th} dual Leonardo Lorentzian vector $\mathcal{D}\vec{\mathcal{L}}e_n$ is defined as:

$$\mathcal{D}\vec{\mathcal{L}}e_n=\vec{\mathcal{L}}e_n+\varepsilon\vec{\mathcal{L}}e_{n+1}$$

where $\vec{\mathcal{L}}e_n=(\mathcal{L}e_n,\mathcal{L}e_{n+1},\mathcal{L}e_{n+2})$ is n^{th} Leonardo Lorentzian vector.

Theorem 4.1. *Let $\mathcal{D}\vec{\mathcal{L}}e_n$ and $\mathcal{D}\vec{\mathcal{L}}e_m$ be dual Leonardo Lorentzian vectors in dual Lorentzian spaces $\mathbb{D}_1^3$, the Lorentzian scalar product and the Lorentzian vectoral product are given as:*

$$\begin{aligned}\left\langle\mathcal{D}\vec{\mathcal{L}}e_n,\mathcal{D}\vec{\mathcal{L}}e_m\right\rangle_{\mathbb{L}}&=\frac{4}{5}(-1)^{n+1}L_{m-n}+8L_{m+n+1}+\frac{24}{5}L_{m+n}-4F_{n+2}-4F_{m+2}+1\\&+\left(\frac{4}{5}(-1)^{n+1}L_{m-n}+16L_{m+n+2}+\frac{48}{5}L_{m+n+1}-4F_{n+4}-4F_{m+4}+2\right)\varepsilon.\end{aligned}\tag{4.1}$$

and

$$\mathcal{D}\vec{\mathcal{L}}e_n\wedge_{\mathbb{L}}\mathcal{D}\vec{\mathcal{L}}e_m=\begin{pmatrix}4(-1)^nF_{m-n}-2F_{n+1}+2F_{m+1}\\4(-1)^nF_{m-n}-2F_{n+2}+2F_{m+2}\\4(-1)^nF_{m-n}+2F_n-2F_m\end{pmatrix}+\varepsilon\begin{pmatrix}4(-1)^nF_{m-n}-2F_{n+3}+2F_{m+3}\\4(-1)^nF_{m-n}-2F_{n+5}+2F_{m+5}\\4(-1)^nF_{m-n}+2F_{n+2}-2F_{m+2}\end{pmatrix}.$$

Proof. Using (2.1) for the Leonardo Lorentzian vectors $\vec{\mathcal{L}e}_n$ and $\vec{\mathcal{L}e}_m$, the Lorentzian scalar product is as follows:

$$\begin{aligned}
\left\langle \vec{\mathcal{L}e}_n, \vec{\mathcal{L}e}_m \right\rangle_{\mathbb{L}} &= -\mathcal{L}e_n\mathcal{L}e_m + \mathcal{L}e_{n+1}\mathcal{L}e_{m+1} + \mathcal{L}e_{n+2}\mathcal{L}e_{m+2} \\
&= -(\frac{2\varphi^{n+1} - 2\psi^{n+1}}{\varphi - \psi} - 1)(\frac{2\varphi^{m+1} - 2\psi^{m+1}}{\varphi - \psi} - 1) \\
&+ (\frac{2\varphi^{n+2} - 2\psi^{n+2}}{\varphi - \psi} - 1)(\frac{2\varphi^{m+2} - 2\psi^{m+2}}{\varphi - \psi} - 1) \\
&+ (\frac{2\varphi^{n+3} - 2\psi^{n+3}}{\varphi - \psi} - 1)(\frac{2\varphi^{m+3} - 2\psi^{m+3}}{\varphi - \psi} - 1) \\
&= \frac{4}{5}(-1)^{n+1}(\varphi^{m-n} + \psi^{m-n}) + \frac{4}{5}\psi^{m+n}(-\psi^2 + \psi^4 + \psi^6) \\
&+ \frac{4}{5}\varphi^{m+n}(-\varphi^2 + \varphi^4 + \varphi^6) + \frac{2}{\sqrt{5}}\psi^m(-\psi + \psi^2 + \psi^3) + \frac{2}{\sqrt{5}}\varphi^m(\varphi - \varphi^2 - \varphi^3) \\
&+ \frac{2}{\sqrt{5}}\psi^n(-\psi + \psi^2 + \psi^3) + \frac{2}{\sqrt{5}}\varphi^n(\varphi - \varphi^2 - \varphi^3) + 1 \\
&= \frac{4}{5}(-1)^{n+1}L_{m-n} + \frac{4}{5}\psi^{m+n}(10\psi + 6) + \frac{4}{5}\varphi^{m+n}(10\varphi + 6) \\
&+ \frac{2}{\sqrt{5}}(\psi^m + \psi^n)(2\psi^2) + \frac{2}{\sqrt{5}}(\varphi^m + \varphi^n)(-2\varphi^2) + 1 \\
&= \frac{4}{5}(-1)^{n+1}L_{m-n} + 8L_{m+n+1} + \frac{24}{5}L_{m+n} - 4F_{n+2} - 4F_{m+2} + 1 \qquad (4.2)
\end{aligned}$$

where L_n is n^{th} Lucas number. Similarly,

$$\begin{aligned}
\left\langle \vec{\mathcal{L}e}_n, \vec{\mathcal{L}e}_{m+1} \right\rangle_{\mathbb{L}} &= \frac{4}{5}(-1)^{n+1}L_{m-n+1} + 8L_{m+n+2} \\
&+ \frac{24}{5}L_{m+n+1} - 4F_{n+2} - 4F_{m+3} + 1 \qquad (4.3)
\end{aligned}$$

and

$$\begin{aligned}
\left\langle \vec{\mathcal{L}e}_{n+1}, \vec{\mathcal{L}e}_m \right\rangle_{\mathbb{L}} &= \frac{4}{5}(-1)^{n+2}L_{m-n-1} + 8L_{m+n+2} \\
&+ \frac{24}{5}L_{m+n+1} - 4F_{n+3} - 4F_{m+2} + 1 \qquad (4.4)
\end{aligned}$$

By using (2.3), (4.2), (4.3) and (4.4), we obtain the dual Lorentzian scalar product as follows:

$$\begin{aligned}
\left\langle \mathcal{D}\vec{\mathcal{L}e}_n, \mathcal{D}\vec{\mathcal{L}e}_m \right\rangle_{\mathbb{L}} &= \left\langle \vec{\mathcal{L}e}_n, \vec{\mathcal{L}e}_m \right\rangle_{\mathbb{L}} + \varepsilon\left(\left\langle \vec{\mathcal{L}e}_n, \vec{\mathcal{L}e}_{m+1} \right\rangle_{\mathbb{L}} + \left\langle \vec{\mathcal{L}e}_{n+1}, \vec{\mathcal{L}e}_m \right\rangle_{\mathbb{L}}\right) \\
&= \frac{4}{5}(-1)^{n+1}L_{m-n} + 8L_{m+n+1} + \frac{24}{5}L_{m+n} - 4F_{n+2} - 4F_{m+2} + 1 \\
&+ \left(\frac{4}{5}(-1)^{n+1}L_{m-n} + 16L_{m+n+2} + \frac{48}{5}L_{m+n+1} - 4F_{n+4} - 4F_{m+4} + 2\right)\varepsilon.
\end{aligned}$$

If we use (2.2) and identity (1.4), the Lorentzian vectoral product is obtained

$$\begin{aligned}
\overrightarrow{\mathcal{L}e}_n \wedge_{\mathbb{L}} \overrightarrow{\mathcal{L}e}_m &= \begin{vmatrix} -\vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ \mathcal{L}e_n & \mathcal{L}e_{n+1} & \mathcal{L}e_{n+2} \\ \mathcal{L}e_m & \mathcal{L}e_{m+1} & \mathcal{L}e_{m+2} \end{vmatrix} \\
&= (-\mathcal{L}e_{n+1}\mathcal{L}e_{m+2} + \mathcal{L}e_{n+2}\mathcal{L}e_{m+1}, -\mathcal{L}e_n\mathcal{L}e_{m+2} + \mathcal{L}e_{n+2}\mathcal{L}e_m, \\
&- \mathcal{L}e_{n+1}\mathcal{L}e_m + \mathcal{L}e_{m+1}\mathcal{L}e_n) \\
&= [4(-1)^n F_{m-n} - 2F_{n+1} + 2F_{m+1}]\ \vec{e}_1 \\
&+ [4(-1)^n F_{m-n} - 2F_{n+2} + 2F_{m+2}]\ \vec{e}_2 \\
&+ [4(-1)^n F_{m-n} + 2F_n - 2F_m]\ \vec{e}_3.
\end{aligned}$$

Similarly

$$\begin{aligned}
\overrightarrow{\mathcal{L}e}_{n+1} \wedge_{\mathbb{L}} \overrightarrow{\mathcal{L}e}_m &= [4(-1)^{n+1} F_{m-n-1} - 2F_{n+2} + 2F_{m+1}]\ \vec{e}_1 \\
&+ [4(-1)^{n+1} F_{m-n-1} - 2F_{n+3} + 2F_{m+2}]\ \vec{e}_2 \\
&+ [4(-1)^{n+1} F_{m-n-1} + 2F_{n+1} - 2F_m]\ \vec{e}_3.
\end{aligned}$$

and

$$\begin{aligned}
\overrightarrow{\mathcal{L}e}_n \wedge_{\mathbb{L}} \overrightarrow{\mathcal{L}e}_{m+1} &= [4(-1)^n F_{m-n+1} - 2F_{n+1} + 2F_{m+2}]\ \vec{e}_1 \\
&+ [4(-1)^n F_{m-n+1} - 2F_{n+2} + 2F_{m+3}]\ \vec{e}_2 \\
&+ [4(-1)^n F_{m-n+1} + 2F_n - 2F_{m+1}]\ \vec{e}_3.
\end{aligned}$$

where $\vec{e}_1,\ \vec{e}_2,\ \vec{e}_3$ are unit direction vectors and

$$\begin{aligned}
&\mathcal{D}\overrightarrow{\mathcal{L}e}_n \wedge_{\mathbb{L}} \mathcal{D}\overrightarrow{\mathcal{L}e}_m = \overrightarrow{\mathcal{L}e}_n \wedge_{\mathbb{L}} \overrightarrow{\mathcal{L}e}_m + \varepsilon(\overrightarrow{\mathcal{L}e}_n \wedge_{\mathbb{L}} \overrightarrow{\mathcal{L}e}_{m+1} + \overrightarrow{\mathcal{L}e}_{n+1} \wedge_{\mathbb{L}} \overrightarrow{\mathcal{L}e}_m) \\
&= \begin{pmatrix} 4(-1)^n F_{m-n} - 2F_{n+1} + 2F_{m+1} \\ 4(-1)^n F_{m-n} - 2F_{n+2} + 2F_{m+2} \\ 4(-1)^n F_{m-n} + 2F_n - 2F_m \end{pmatrix} + \varepsilon \begin{pmatrix} 4(-1)^n F_{m-n} - 2F_{n+3} + 2F_{m+3} \\ 4(-1)^n F_{m-n} - 2F_{n+5} + 2F_{m+5} \\ 4(-1)^n F_{m-n} + 2F_{n+2} - 2F_{m+2} \end{pmatrix}.
\end{aligned}$$

Thus, the proof is completed. ☐

Corollary 4.2. *The norm of* $\mathcal{D}\overrightarrow{\mathcal{L}e}_n$ *is*

$$\begin{aligned}
\|\mathcal{D}\overrightarrow{\mathcal{L}e}_n\|^2 &= \langle \mathcal{D}\overrightarrow{\mathcal{L}e}_n, \mathcal{D}\overrightarrow{\mathcal{L}e}_n \rangle_{\mathbb{L}} \\
&= \frac{8}{5}(-1)^{n+1} + 8L_{2n+1} + \frac{24}{5}L_{2n} - 8F_{n+2} + 1 \\
&+ \left(\frac{8}{5}(-1)^{n+1} + 16L_{2n+2} + \frac{48}{5}L_{2n+1} - 8F_{n+4} + 2\right)\varepsilon.
\end{aligned}$$

Proof. The proof is clear from taking $m = n$ in (4.1). ☐

Proposition 4.1 (Kotelnikov (1895); Study (1901))**.** *Let* $\vec{\mathcal{A}} = \vec{a} + \varepsilon\vec{a}^*$ *be a unit dual Lorentzian vector, then the directed line* d *that corresponds with* $\vec{\mathcal{A}}$ *has the equation of the form*

$$\vec{r} = \vec{a} \wedge_{\mathbb{L}} \vec{a}^* + \mu\ \vec{a} \tag{4.5}$$

where $0 \leq \mu \leq 1$.

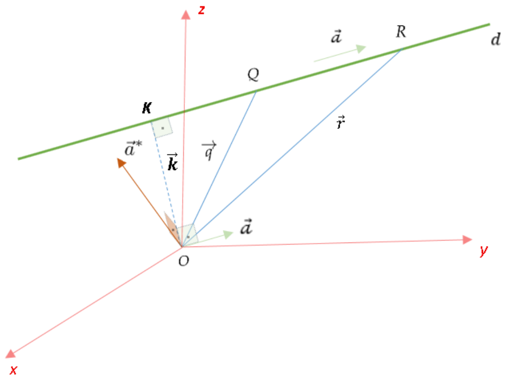

Figure 2: E. Study mapping in $\mathbb{D}^3$, Kotelnikov (1895); Study (1901).

Proof. Let $\vec{\mathcal{A}} = \vec{a} + \varepsilon \vec{a}^*$ be the unit dual vector and if Q and R are points on the corresponding line d, and O is the origin (see Fig. 2), then

$$\vec{OR} = \vec{OQ} + \vec{QR},$$
$$\vec{r} = \vec{q} + \lambda\, \vec{a},$$

where λ is a real parameter. A point R is on the line of vectors $\vec{a}$ and $\vec{a}^*$ if and only if

$$\vec{a} \wedge_{\mathbb{L}} \vec{a}^* = \vec{a} \wedge_{\mathbb{L}} (\vec{q} \wedge_{\mathbb{L}} \vec{a}) = \langle \vec{a}, \vec{a} \rangle_{\mathbb{L}} \vec{q} - \langle \vec{a}, \vec{q} \rangle_{\mathbb{L}} \vec{a}$$

and then

$$\vec{q} = \vec{a} \wedge_{\mathbb{L}} \vec{a}^* + \langle \vec{a}, \vec{q}\, \rangle_{\mathbb{L}}\, \vec{a}$$

and

$$\vec{r} = \vec{a} \wedge_{\mathbb{L}} \vec{a}^* + \langle \vec{a}, \vec{q}\, \rangle_{\mathbb{L}}\, \vec{a} + \lambda\, \vec{a}.$$

By taking $\mu = \langle \vec{a}, \vec{q}\, \rangle_{\mathbb{L}} + \lambda$, we get the result as

$$\vec{r} = \vec{a} \wedge_{\mathbb{L}} \vec{a}^* + \mu\, \vec{a}.$$

□

Theorem 4.3. *Suppose that $\mathcal{D}\vec{\mathcal{L}}e_n$ is a dual Leonardo Lorentzian vector and let*

$$\mathcal{D}\vec{\mathcal{L}}e_n = \vec{\mathcal{L}}e_n + \varepsilon \vec{\mathcal{L}}e_{n+1} = \vec{\mathcal{L}}e_n + \varepsilon \vec{\mathcal{L}}e_n^{\,*} \tag{4.6}$$

be its unitized vector, then the equation of corresponding dual line is

$$\begin{aligned} \vec{r}_n = {} & [4(-1)^n F_{m-n} - 2F_{n+1} + 2F_{m+1} + \mu\, L_n]\, \vec{e}_1 \\ & + [4(-1)^n F_{m-n} - 2F_{n+2} + 2F_{m+2} + \mu\, L_{n+1}]\, \vec{e}_2 \\ & + [4(-1)^n F_{m-n} + 2F_n - 2F_m + \mu\, L_{n+2}]\, \vec{e}_3. \end{aligned}$$

Proof. By using (4.5), we obtain

$$\begin{aligned}
\vec{r_n} &= \vec{\mathcal{L}e}_n \wedge_{\mathbb{L}} \vec{\mathcal{L}e}_{n+1} + \mu\, \vec{\mathcal{L}e}_n, \\
&= \begin{vmatrix} -\vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ \mathcal{L}e_n & \mathcal{L}e_{n+1} & \mathcal{L}e_{n+2} \\ \mathcal{L}e_{n+1} & \mathcal{L}e_{n+2} & \mathcal{L}e_{n+3} \end{vmatrix} + \mu\, (\mathcal{L}e_n\vec{e}_1 + \mathcal{L}e_{n+1}\vec{e}_2 + \mathcal{L}e_{n+2}\vec{e}_3) \\
&= [4(-1)^n F_{m-n} - 2F_{n+1} + 2F_{m+1} + \mu\, \mathcal{L}e_n]\, \vec{e}_1 \\
&+ [4(-1)^n F_{m-n} - 2F_{n+2} + 2F_{m+2} + \mu\, \mathcal{L}e_{n+1}]\, \vec{e}_2 \\
&+ [4(-1)^n F_{m-n} + 2F_n - 2F_m + \mu\, \mathcal{L}e_{n+2}]\, \vec{e}_3.
\end{aligned}$$

☐

Definition 4.2. (**Dual center angle**) (Uğurlu and Çalışkan (1996)) Let $\vec{\mathcal{A}} = \vec{a} + \varepsilon\vec{a}^*$ and $\vec{\mathcal{B}} = \vec{b} + \varepsilon\vec{b}^*$ be dual spacelike unit vectors in $\mathbb{D}^3_1$. Given the center angle ϕ and the shortest distance ϕ^*between the directional spacelike lines corresponding to these vectors, the dual number

$$\Phi = \phi + \varepsilon\phi^*$$

is called the dual center angle between these vectors (see Fig. 3).

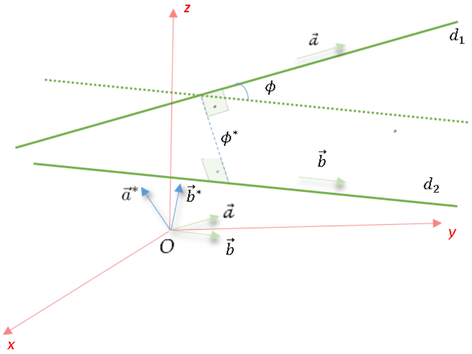

Figure 3: Dual Central Angle between Spacelike Lines

Definition 4.3. (Aslan et al., 2020) Let $\vec{\mathcal{A}} = \vec{a} + \varepsilon\vec{a}^*$ and $\vec{\mathcal{B}} = \vec{b} + \varepsilon\vec{b}^*$ be the spacelike dual vectors in $\mathbb{D}^3_1$. The Lorentzian scalar product of these vectors is

$$\langle \vec{\mathcal{A}}, \vec{\mathcal{B}} \rangle_{\mathbb{L}} = \cos\Phi = \cos\phi - \varepsilon\phi^* \sin\phi \tag{4.7}$$

where $\Phi = \phi + \varepsilon\phi^*$ is the dual Lorentzian angle between them.

We will have the following cases for $\vec{\mathcal{A}} = \vec{a} + \varepsilon\vec{a}^* = \mathcal{D}\vec{\mathcal{L}}e_n$ and $\vec{\mathcal{B}} = \vec{b} + \varepsilon\vec{b}^* = \mathcal{D}\vec{\mathcal{L}}e_{\mathrm{m}}$.

Corollary 4.4. *Let* $\mathcal{D}\vec{\mathcal{L}}e_n$ *and* $\mathcal{D}\vec{\mathcal{L}}e_{\mathrm{m}}$ *be dual Leonardo vectors in dual Lorentzian spaces, by using* (4.1), *Definition*(4.2) *and Definition*(4.3)

$$\begin{aligned}
\left\langle \mathcal{D}\vec{\mathcal{L}}e_n, \mathcal{D}\vec{\mathcal{L}}e_m \right\rangle_{\mathbb{L}} &= \frac{4}{5}(-1)^{n+1} L_{m-n} + 8L_{m+n+1} + \frac{24}{5} L_{m+n} - 4F_{n+2} - 4F_{m+2} + 1 \\
&+ \left(\frac{4}{5}(-1)^{n+1} L_{m-n} + 16L_{m+n+2} + \frac{48}{5} L_{m+n+1} - 4F_{n+4} - 4F_{m+4} + 2 \right) \varepsilon
\end{aligned}$$

where if we take

$$\cos\phi = \frac{4}{5}(-1)^{n+1}L_{m-n} + 8L_{m+n+1} + \frac{24}{5}L_{m+n} - 4F_{n+2} - 4F_{m+2} + 1$$

and

$$-\phi^* \sin\phi = \frac{4}{5}(-1)^{n+1}L_{m-n} + 16L_{m+n+2} + \frac{48}{5}L_{m+n+1} - 4F_{n+4} - 4F_{m+4} + 2.$$

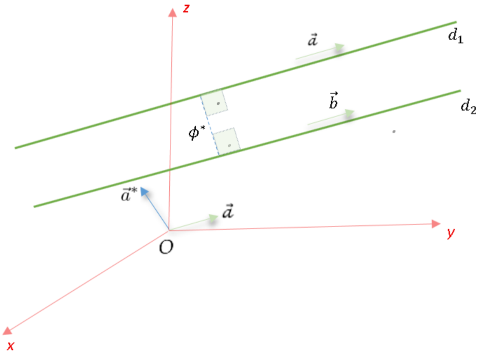

Figure 4: Parallel of dual lines d_1 and d_2

Case 4.5. *If* $\phi = 0$ *and* $\phi^* \neq 0$, *then*

$$\left\langle \mathcal{D}\vec{\mathcal{L}}e_n, \mathcal{D}\vec{\mathcal{L}}e_m \right\rangle_{\mathbb{L}} = \cos\Phi = \cos\phi = 1$$

which gives

$$\frac{4}{5}(-1)^{n+1}L_{m-n} + 8L_{m+n+1} + \frac{24}{5}L_{m+n} - 4F_{n+2} - 4F_{m+2} = 0.$$

Thus, corresponding dual Leonardo vectors $\mathcal{D}\vec{\mathcal{L}}e_n$ *and* $\mathcal{D}\vec{\mathcal{L}}e_{\mathrm{m}}$ *are parallel (see Fig. 4).*

Case 4.6. *If* $\phi^* = 0$ *and* $\phi \neq 0$, *then we obtain*

$$\left\langle \mathcal{D}\vec{\mathcal{L}}e_n, \mathcal{D}\vec{\mathcal{L}}e_m \right\rangle_{\mathbb{L}} = \cos\Phi = \cos\phi$$

which gives

$$\phi = arccos\left(\frac{4}{5}(-1)^{n+1}L_{m-n} + 8L_{m+n+1} + \frac{24}{5}L_{m+n} - 4F_{n+2} - 4F_{m+2} + 1\right)$$

and

$$\frac{4}{5}(-1)^{n+1}L_{m-n} + 16L_{m+n+2} + \frac{48}{5}L_{m+n+1} = 4F_{n+4} + 4F_{m+4} - 2.$$

Thus, corresponding dual Leonardo vectors $\mathcal{D}\vec{\mathcal{L}}e_n$ *and* $\mathcal{D}\vec{\mathcal{L}}e_{\mathrm{m}}$ *intersect each other (see Fig. 5).*

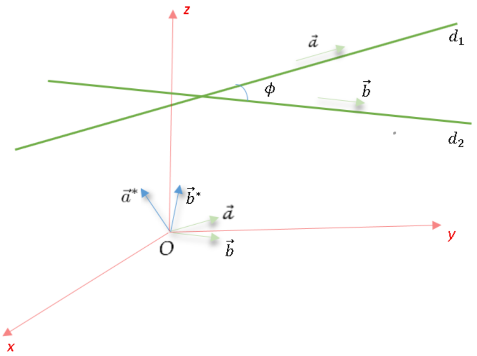

Figure 5: Intersection of dual lines d_1 and d_2

5 Conclusion

This study introduced dual Leonardo sequences, extending classical Leonardo numbers into the dual number system. We established fundamental recurrence relations and algebraic identities. By integrating these sequences into three-dimensional dual Lorentzian space, we defined dual Lorentzian vectors and analyzed their geometric behavior through the formulation of dual Lorentzian angles. The results provide a robust foundation for future research and offer potential applications in kinematics, theoretical physics, and geometric modeling.

Disclaimer (Artificial Intelligence)

Author(s) hereby declare that NO generative AI technologies such as Large Language Models (ChatGPT, COPILOT, etc) and text-to-image generators have been used during writing or editing of manuscripts.

Competing Interests

Authors have declared that no competing interests exist.

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