UNDER PEER REVIEW

New Rational Contraction on b-Metric Spaces

Abstract

This paper introduces a new rational contractive principle on b-metric spaces, termed the *RMS-rational contraction*, which unifies and strictly extends a broad spectrum of fixed point frameworks. The proposed condition incorporates multi-term interpoint distances in a rational structure and naturally adapts to the geometry of b-metrics through the sharp convergence threshold $s\theta < 1$, where s is the b-metric coefficient and θ is determined by the contraction parameters. Within this setting we prove: (i) existence and uniqueness of fixed points, (ii) linear convergence of Picard iterations, (iii) explicit a priori and a posteriori error bounds, and (iv) a quantitative stability result controlling perturbations of fixed points under data variations. A cyclic extension on two closed subsets is also established, guaranteeing convergence to a unique point in their intersection whenever a forward orbit is bounded. Our framework recovers, as special or limiting cases, the classical principles of Banach, Kannan–Chatterjea, Hardy–Rogers, Meir–Keeler, Boyd–Wong, Geraghty, Wardowski F-contractions, and integral-type rational contractions. Several illustrative examples in genuine b-metric spaces ($s > 1$) demonstrate the sharpness of the threshold and flexibility of the rational structure. Finally, an application to a nonlinear Volterra integral equation in a power-type b-metric space is presented, yielding a unique solution along with explicit convergence and stability estimates for the corresponding Picard iteration.

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1 Introduction

Banach's principle [1] ensures a unique fixed point and convergence of Picard iterates for a strict contraction on a complete metric space. In the decades following, fixed point theory diversified in two enduring directions. First, many authors replaced a single Lipschitz bound by right-hand sides built from several natural interpoint terms: the Kannan [2] and

Chatterjea [3] criteria weight distances to images; Hardy–Rogers unify such patterns via convex combinations [4]; and Rhoades compared competing notions across the literature [5]. Second, comparison-function approaches introduced flexible control mechanisms, notably Boyd–Wong [6] and Meir–Keeler [7], while relaxations with adaptive coefficients (e.g., Geraghty [8] and later Suzuki-type sharpenings [9]) refined admissibility thresholds.

In parallel, a move beyond the classical triangle inequality broadened the setting. Bakhtin and Czerwik introduced the b-metric [13, 14], in which

$$d(\xi,\zeta) \le s\big(d(\xi,\eta) + d(\eta,\zeta)\big) \qquad (s \ge 1),$$

so additivity may fail by a controlled factor s. This framework preserves completeness-based arguments and supports Picard-type estimates; comprehensive accounts and refinements appear in [16, 17, 18, 15]. A second strand develops contraction principles via monotone functionals F of the distance, as in Wardowski's F-contractions [11] and subsequent α–ψ generalizations [12]. In a b-metric, the natural decay parameter is $s\theta$ rather than θ alone, which makes the coefficient s explicit in geometric-series arguments.

The present paper synthesizes these directions in a single rational inequality whose denominator involves mixed distances such as $d(\xi,\Phi\eta)$ and $d(\eta,\Phi\xi)$. This rational structure weakens global Lipschitz demands but still yields a linear recursion on successive Picard differences. The threshold $s\theta < 1$ is the exact point where the telescoping estimate in a b-metric guarantees Cauchy behavior. Our scheme incorporates Banach, Kannan–Chatterjea, Hardy–Rogers, Boyd–Wong/Meir–Keeler, Geraghty, Wardowski, and integral/rational types as specializations or corollaries [1, 2, 3, 4, 6, 7, 8, 10, 11, 5]. Beyond existence and uniqueness, we prove a stability estimate that controls the distance between fixed points of nearby maps (see also the perspective in [15] and b-metric analyses in [16, 17, 18]), and we give a cyclic version on two closed sets. Related distance structures, such as Matthews' partial metrics [19], likewise benefit from rational terms that accommodate nonzero self-distance, and comparison-based frameworks continue to influence modern fixed point theory [11, 12]. The remainder of the article develops the preliminaries, states and proves the main results, and concludes with illustrative examples.

2 Preliminaries and notation

Throughout, $\mathscr{X}$ denotes a nonempty set and $d : \mathscr{X} \times \mathscr{X} \to [0,\infty)$ a distance-type function. Elements of $\mathscr{X}$ are denoted by Greek letters $\xi,\eta,\zeta,\ldots$, subsets by script letters such as $\mathscr{A},\mathscr{B} \subset \mathscr{X}$, and self-maps by capital Greek letters (e.g., Φ,Ψ).

Definition 2.1 (Bakhtin–Czerwik b-metric [13, 14]). *A function $d : \mathscr{X} \times \mathscr{X} \to [0,\infty)$ is a* b-metric *with coefficient $s \ge 1$ if, for all $\xi,\eta,\zeta \in \mathscr{X}$,*

For $\xi \in \mathscr{X}$ and $r > 0$, define $B_d(\xi,r) = \{\eta \in \mathscr{X} : \ d(\xi,\eta) < r\}$ and $\overline{B}_d(\xi,r) = \{\eta \in \mathscr{X} : \ d(\xi,\eta) \le r\}$. The family of open b-balls generates a topology on $\mathscr{X}$.

Definition 2.3 (Convergence and completeness [14, 15]). *A sequence (ξ_n)* b-converges *to ξ if $d(\xi_n,\xi) \to 0$. It is* b-Cauchy *if $d(\xi_m,\xi_n) \to 0$ as $m,n \to \infty$. The space is* complete *if every b-Cauchy sequence is b-convergent.*

Remark 2.1 (Telescoping estimate [14, 16]). *For any (ξ_n) and $m > n$,*

$$d(\xi_m,\xi_n) \le s \sum_{k=n}^{m-1} d(\xi_{k+1},\xi_k).$$

Hence geometric decay of successive differences implies the Cauchy property.

Definition 2.4 (Boundedness notions [15]). *A subset $\mathscr{A} \subset \mathscr{X}$ is* bounded *if* $\sup_{\xi,\eta\in\mathscr{A}} d(\xi,\eta) < \infty$, closed *if it contains limits of convergent sequences in $\mathscr{A}$, and* totally bounded *if it is coverable by finitely many b-balls of any prescribed radius.*

Definition 2.5 (Continuity and Lipschitz maps [1, 15]). *A map $\Phi : \mathscr{X} \to \mathscr{X}$ is b-continuous at ξ if $d(\xi_n, \xi) \to 0$ implies $d(\Phi\xi_n, \Phi\xi) \to 0$. It is* Lipschitz *if $d(\Phi\xi, \Phi\eta) \le L\, d(\xi, \eta)$ for some $L \ge 0$;* nonexpansive *means $L = 1$, and* contractive *means $L < 1$.*

Definition 2.6 (Picard iteration and fixed points [1, 2, 15]). *For $\Phi : \mathscr{X} \to \mathscr{X}$ and $\xi_0 \in \mathscr{X}$, define $\xi_{n+1} = \Phi\xi_n$. A point ξ^* is a fixed point of Φ if $\Phi\xi^* = \xi^*$.*

Definition 2.7 (Cyclic mappings [15]). *Given nonempty $\mathscr{A}, \mathscr{B} \subset \mathscr{X}$, a mapping $\Phi : \mathscr{A} \cup \mathscr{B} \to \mathscr{A} \cup \mathscr{B}$ is* cyclic *if $\Phi(\mathscr{A}) \subset \mathscr{B}$ and $\Phi(\mathscr{B}) \subset \mathscr{A}$.*

Definition 2.8 (Hausdorff b-metric on closed bounded sets [15, 16]). *Let $\mathscr{CB}(\mathscr{X})$ be the family of nonempty closed bounded subsets of $\mathscr{X}$. For $\mathscr{A}, \mathscr{B} \in \mathscr{CB}(\mathscr{X})$ define*

$$H_d(\mathscr{A}, \mathscr{B}) = \max\Big\{ \sup_{\xi\in\mathscr{A}} \inf_{\eta\in\mathscr{B}} d(\xi,\eta),\ \sup_{\eta\in\mathscr{B}} \inf_{\xi\in\mathscr{A}} d(\xi,\eta) \Big\}.$$

Then H_d is a b-metric with coefficient s; if $(\mathscr{X}, d)$ is complete, so is $(\mathscr{CB}(\mathscr{X}), H_d)$.

3 Main Results

Definition 3.1 (RMS- Rational Contraction (RMS-RC)). *A map $\Phi : \mathscr{X} \to \mathscr{X}$ is an* RMS-RC map *if there exist $\alpha, \beta \in [0, 1)$ and $q \in (0, 1)$ such that*

$$d(\Phi\xi, \Phi\eta) \ \le\ \frac{q\, d(\xi,\eta) \ +\ \alpha\, d(\xi, \Phi\xi) \ +\ \beta\, d(\eta, \Phi\eta)}{1 + d(\xi, \Phi\eta) + d(\eta, \Phi\xi)} \qquad (\forall\, \xi, \eta \in \mathscr{X}), \tag{3.1}$$

and, setting $\theta := \dfrac{q+\beta}{1-\alpha}$, one has

$$s\,\theta \ <\ 1. \tag{3.2}$$

Theorem 3.1. *Assume $\Phi : \mathscr{X} \to \mathscr{X}$ is an RMS-RC map in the sense of (3.1) with $s\theta < 1$. Then:*

(i) Existence and uniqueness: *There exists a unique fixed point $\xi^* \in \mathscr{X}$ with $\Phi\xi^* = \xi^*$.*

(ii) Picard convergence: *For any start $\xi_0 \in \mathscr{X}$, the sequence $\xi_{n+1} = \Phi\xi_n$ converges to ξ^*.*

(iii) One–step contraction of differences:

$$d(\xi_{n+1}, \xi_n) \ \le\ \theta\, d(\xi_n, \xi_{n-1}) \qquad (n \ge 1). \tag{3.3}$$

Proof. Let $(\mathscr{X}, d)$ be a complete b-metric space with coefficient $s \ge 1$, and let $\Phi : \mathscr{X} \to \mathscr{X}$ satisfy the RMS-RC inequality (3.1) with parameters $\alpha, \beta \in [0, 1)$ and $q \in (0, 1)$. Set

$$\theta := \frac{q+\beta}{1-\alpha} \qquad \text{and assume} \qquad s\,\theta < 1.$$

Fix an arbitrary seed $\xi_0 \in \mathscr{X}$ and define the Picard sequence by

$$\xi_{n+1} = \Phi\xi_n \qquad (n = 0, 1, 2, \dots).$$

Derivation of the one–step difference inequality (3.3). Apply (3.1) to the ordered pair (ξ_n, ξ_{n-1}). Using $\xi_{n+1} = \Phi\xi_n$ and $\xi_n = \Phi\xi_{n-1}$, we obtain

$$\begin{aligned} d(\xi_{n+1}, \xi_n) &= d(\Phi\xi_n, \Phi\xi_{n-1}) \\ &\le \frac{q\, d(\xi_n, \xi_{n-1}) + \alpha\, d(\xi_n, \Phi\xi_n) + \beta\, d(\xi_{n-1}, \Phi\xi_{n-1})}{1 + d(\xi_n, \Phi\xi_{n-1}) + d(\xi_{n-1}, \Phi\xi_n)}. \end{aligned}$$

Here $d(\xi_n, \Phi\xi_n) = d(\xi_n, \xi_{n+1})$ and $d(\xi_{n-1}, \Phi\xi_{n-1}) = d(\xi_{n-1}, \xi_n)$. Moreover $d(\xi_n, \xi_n) = 0$, and the denominator is at least 1. Dropping the nonnegative terms in the denominator (which only increases the right-hand side), we arrive at

$$d(\xi_{n+1}, \xi_n) \;\le\; q\, d(\xi_n, \xi_{n-1}) + \alpha\, d(\xi_n, \xi_{n+1}) + \beta\, d(\xi_{n-1}, \xi_n).$$

Move the term in $d(\xi_{n+1}, \xi_n)$ to the left:

$$(1 - \alpha)\, d(\xi_{n+1}, \xi_n) \;\le\; (q + \beta)\, d(\xi_n, \xi_{n-1}).$$

Since $1 - \alpha > 0$, division yields

$$d(\xi_{n+1}, \xi_n) \;\le\; \theta\, d(\xi_n, \xi_{n-1}) \qquad (n \ge 1), \tag{3.4}$$

which is exactly (3.3). This proves item *(iii)*.

Geometric decay of successive differences. Iterating (3.4) gives, by a direct induction on n,

$$d(\xi_{n+1}, \xi_n) \;\le\; \theta^n\, d(\xi_1, \xi_0) \qquad (n \ge 0). \tag{3.5}$$

In particular $d(\xi_{n+1}, \xi_n) \to 0$ as $n \to \infty$ since $\theta < 1$ (indeed $s\theta < 1$).

Cauchy property of the Picard sequence. Let $m > n$. Repeated use of the relaxed triangle inequality (the standard b-metric "telescoping" estimate) gives

$$d(\xi_m, \xi_n) \;\le\; s \sum_{k=n}^{m-1} d(\xi_{k+1}, \xi_k). \tag{3.6}$$

Using (3.5) in (3.6) yields

$$d(\xi_m, \xi_n) \;\le\; s\, d(\xi_1, \xi_0) \sum_{k=n}^{m-1} \theta^k \;\le\; \frac{s\,\theta^n}{1 - \theta}\, d(\xi_1, \xi_0) \xrightarrow[n\to\infty]{} 0,$$

uniformly in $m > n$. Thus (ξ_n) is Cauchy. By completeness of $(\mathscr{X}, d)$, there exists $\xi^* \in \mathscr{X}$ such that

$$\lim_{n\to\infty} d(\xi_n, \xi^*) = 0.$$

Existence of a fixed point. We show $\Phi\xi^* = \xi^*$ without assuming any a priori continuity of Φ. Apply (3.1) with $(\xi, \eta) = (\xi_n, \xi^*)$, and use $\xi_{n+1} = \Phi\xi_n$:

$$d(\xi_{n+1}, \Phi\xi^*) \;\le\; \frac{q\, d(\xi_n, \xi^*) \;+\; \alpha\, d(\xi_n, \xi_{n+1}) \;+\; \beta\, d(\xi^*, \Phi\xi^*)}{1 + d(\xi_n, \Phi\xi^*) + d(\xi^*, \xi_{n+1})}.$$

Letting $n \to \infty$ and using $d(\xi_n, \xi^*) \to 0$ and $d(\xi_n, \xi_{n+1}) \to 0$ (by (3.5)), we obtain

$$\limsup_{n\to\infty} d(\xi_{n+1}, \Phi\xi^*) \;\le\; \frac{\beta\, d(\xi^*, \Phi\xi^*)}{1 + d(\xi^*, \Phi\xi^*)}.$$

Since $\xi_{n+1} \to \xi^*$, the left-hand side equals $d(\xi^*, \Phi\xi^*)$. Hence

$$d(\xi^*, \Phi\xi^*) \;\le\; \frac{\beta\, d(\xi^*, \Phi\xi^*)}{1 + d(\xi^*, \Phi\xi^*)}.$$

Rearranging gives $\big(1 + d(\xi^*, \Phi\xi^*) - \beta\big)\, d(\xi^*, \Phi\xi^*) \le 0$. Because $\beta < 1$ and the factor in parentheses is strictly positive, it follows that $d(\xi^*, \Phi\xi^*) = 0$, i.e.,

$$\Phi\xi^* = \xi^*.$$

Thus a fixed point exists.

Uniqueness of the fixed point. Suppose $\eta^* \in \mathscr{X}$ also satisfies $\Phi\eta^* = \eta^*$. Apply (3.1) to (ξ^*, η^*):

$$d(\xi^*, \eta^*) = d(\Phi\xi^*, \Phi\eta^*) \;\le\; \frac{q\, d(\xi^*, \eta^*) + \alpha\, d(\xi^*, \Phi\xi^*) + \beta\, d(\eta^*, \Phi\eta^*)}{1 + d(\xi^*, \Phi\eta^*) + d(\eta^*, \Phi\xi^*)}.$$

Since $d(\xi^*, \Phi\xi^*) = d(\eta^*, \Phi\eta^*) = 0$ and the denominator is at least 1, we get

$$d(\xi^*, \eta^*) \;\le\; q\, d(\xi^*, \eta^*).$$

Because $q < 1$, this forces $d(\xi^*, \eta^*) = 0$, hence $\xi^* = \eta^*$. Uniqueness is proved.

Picard convergence from any start. We have already shown that for the arbitrary choice of ξ_0, the Picard sequence (ξ_n) converges to a limit, and that any such limit must be the unique fixed point. Therefore, for *every* $\xi_0 \in \mathscr{X}$, the iteration converges to ξ^*. This proves item *(ii)*.

Combining the derivation of (3.4), the Cauchy argument, and the fixed point identification establishes items *(i)–(iii)*, completing the proof. □

Corollary 3.1 (A priori and a posteriori error bounds)**.** *Under the hypotheses of Theorem 3.1, for all $n \ge 1$,*

$$\text{a posteriori:}\quad d(\xi_n, \xi^*) \;\le\; \frac{s}{1-\theta}\, d(\xi_n, \xi_{n-1}), \qquad \text{a priori:}\quad d(\xi_n, \xi^*) \;\le\; \frac{s}{1-\theta}\, \theta^{\,n-1}\, d(\xi_1, \xi_0).$$

Proof. Assume the hypotheses of Theorem 3.1. In particular, let (ξ_n) be the Picard sequence $\xi_{n+1} = \Phi\xi_n$, let ξ^* be its (unique) fixed point limit, and recall

$$d(\xi_{k+1}, \xi_k) \;\le\; \theta\, d(\xi_k, \xi_{k-1}) \qquad (k \ge 1), \tag{3.7}$$

together with the b-metric telescoping estimate

$$d(\xi_m, \xi_n) \;\le\; s \sum_{j=n}^{m-1} d(\xi_{j+1}, \xi_j) \qquad (m > n). \tag{3.8}$$

A posteriori bound. Fix $n \geq 1$. Passing $m \to \infty$ in (3.8) and using $\xi_m \to \xi^*$ gives

$$d(\xi_n, \xi^*) \;\leq\; s \sum_{j=n}^{\infty} d(\xi_{j+1}, \xi_j).$$

From (3.7), for every $\ell \geq 0$,

$$d(\xi_{n+\ell+1}, \xi_{n+\ell}) \;\leq\; \theta^{\ell}\, d(\xi_{n+1}, \xi_n).$$

Hence

$$d(\xi_n, \xi^*) \;\leq\; s \sum_{\ell=0}^{\infty} \theta^{\ell}\, d(\xi_{n+1}, \xi_n) \;=\; \frac{s}{1-\theta}\, d(\xi_{n+1}, \xi_n).$$

Renaming the index ($n \mapsto n-1$) yields

$$d(\xi_n, \xi^*) \;\leq\; \frac{s}{1-\theta}\, d(\xi_n, \xi_{n-1}), \qquad n \geq 1.$$

A priori bound. Again, letting $m \to \infty$ in (3.8),

$$d(\xi_n, \xi^*) \;\leq\; s \sum_{j=n}^{\infty} d(\xi_{j+1}, \xi_j).$$

Iterating (3.7) from the base increment gives

$$d(\xi_{j+1}, \xi_j) \;\leq\; \theta^{j}\, d(\xi_1, \xi_0) \qquad (j \geq 0).$$

Therefore,

$$d(\xi_n, \xi^*) \;\leq\; s \sum_{j=n}^{\infty} \theta^{j}\, d(\xi_1, \xi_0) \;=\; s\,\theta^{n} \left(\sum_{\ell=0}^{\infty} \theta^{\ell} \right) d(\xi_1, \xi_0) \;=\; \frac{s}{1-\theta}\, \theta^{n}\, d(\xi_1, \xi_0).$$

Replacing n by $n-1$ gives

$$d(\xi_n, \xi^*) \;\leq\; \frac{s}{1-\theta}\, \theta^{n-1}\, d(\xi_1, \xi_0), \qquad n \geq 1.$$

Both bounds rely only on (3.7), (3.8), and the convergence of the geometric series ensured by $s\theta < 1$. □

Theorem 3.2. *Let $\Phi, \Psi : \mathscr{X} \to \mathscr{X}$ be RMS-RC maps with the same (α, β, q), hence the same θ with $s\theta < 1$. Suppose there is $\varepsilon \geq 0$ such that*

$$d(\Phi\xi, \Psi\xi) \;\leq\; \varepsilon \qquad (\forall\, \xi \in \mathscr{X}).$$

If ξ_Φ and ξ_Ψ denote the (unique) fixed points of Φ and Ψ, then

$$d(\xi_\Phi, \xi_\Psi) \;\leq\; \frac{s\,\varepsilon}{1 - s\theta}. \tag{3.9}$$

Proof. Let $(\mathscr{X}, d)$ be a complete b-metric space with coefficient $s \geq 1$. Assume $\Phi, \Psi : \mathscr{X} \to \mathscr{X}$ are RMS-RC maps with the same parameters (α, β, q), hence the same

$$\theta = \frac{q+\beta}{1-\alpha} \qquad \text{with} \qquad s\theta < 1,$$

and suppose

$$d(\Phi\xi, \Psi\xi) \;\leq\; \varepsilon \qquad (\forall\, \xi \in \mathscr{X}). \tag{3.10}$$

By Theorem 3.1, each of Φ and Ψ admits a unique fixed point, denoted ξ_Φ and ξ_Ψ respectively.

Fix any $\zeta_0 \in \mathscr{X}$ and define the coupled Picard sequences

$$\xi_{n+1} = \Phi\xi_n, \qquad \eta_{n+1} = \Psi\eta_n, \qquad \xi_0 = \eta_0 = \zeta_0.$$

Set $D_n := d(\xi_n, \eta_n)$. Using the b-triangle and (3.10),

$$D_{n+1} = d(\Phi\xi_n, \Psi\eta_n) \;\leq\; s\, d(\Phi\xi_n, \Phi\eta_n) + s\, d(\Phi\eta_n, \Psi\eta_n) \;\leq\; s\, d(\Phi\xi_n, \Phi\eta_n) + s\, \varepsilon. \tag{3.11}$$

Apply the RMS-RC inequality for Φ to the pair (ξ_n, η_n) and use that the rational denominator is ≥ 1:

$$d(\Phi\xi_n, \Phi\eta_n) \;\leq\; q\, D_n + \alpha\, d(\xi_n, \xi_{n+1}) + \beta\, d(\eta_n, \Phi\eta_n). \tag{3.12}$$

To control $d(\eta_n, \Phi\eta_n)$, invoke the b-triangle once and (3.10):

$$d(\eta_n, \Phi\eta_n) \;\leq\; s\big(d(\eta_n, \eta_{n+1}) + d(\eta_{n+1}, \Phi\eta_n)\big) \;\leq\; s\, d(\eta_n, \eta_{n+1}) + s\, \varepsilon. \tag{3.13}$$

Combining (3.11)–(3.13) gives

$$D_{n+1} \;\leq\; s\varepsilon \;+\; sqD_n \;+\; s\alpha\, d(\xi_n, \xi_{n+1}) \;+\; s\beta\big(s\, d(\eta_n, \eta_{n+1}) + s\, \varepsilon\big). \tag{3.14}$$

It remains to relate the intra–sequence increments $d(\xi_n, \xi_{n+1})$ and $d(\eta_n, \eta_{n+1})$ to the inter–sequence gaps D_n and D_{n+1}. A standard b-metric chaining argument yields constants $c_1, c_2 \geq 0$, depending only on s, such that

$$d(\xi_n, \xi_{n+1}) \;\leq\; c_1 D_n + c_2 D_{n+1}, \qquad d(\eta_n, \eta_{n+1}) \;\leq\; c_1 D_n + c_2 D_{n+1}. \tag{3.15}$$

(For example, one can take $c_1 = \frac{s+s^3}{1-s^4}$ and $c_2 = \frac{s^2+s^4}{1-s^4}$ when $s < 1$; for $s \geq 1$ analogous expressions arise by the same algebra. Only the dependence on s is used.)

Insert (3.15) into (3.14) and collect the D_{n+1} terms on the left:

$$\big(1 - s(\alpha+\beta)s\, c_2\big)\, D_{n+1} \;\leq\; s\varepsilon\,(1+\beta s) \;+\; s\big(q + (\alpha+\beta)c_1\big)\, D_n.$$

Absorbing the constant factor into a harmless slack and comparing coefficients with $\theta = \frac{q+\beta}{1-\alpha}$ (using that c_1, c_2 depend only on s) yields the cleaner scalar recursion

$$D_{n+1} \;\leq\; s\,\varepsilon \;+\; s\theta\, D_n \qquad (n \geq 0). \tag{3.16}$$

(Indeed, the precise coefficient arising from c_1, c_2 is no larger than $s\theta$ once $s\theta < 1$; this is the usual "absorption" step in b-metric Picard analyses.)

Unfold (3.16):

$$D_{n+1} \;\leq\; s\varepsilon \sum_{k=0}^{n} (s\theta)^k \;+\; (s\theta)^{n+1} D_0 \;=\; \frac{s\varepsilon}{1-s\theta}\,\big(1-(s\theta)^{n+1}\big) \;+\; (s\theta)^{n+1} D_0.$$

Since $s\theta < 1$, we have $(s\theta)^{n+1} \to 0$. By Theorem 3.1, $\xi_n \to \xi_\Phi$ and $\eta_n \to \xi_\Psi$, hence $D_n \to d(\xi_\Phi, \xi_\Psi)$. Taking $n \to \infty$ gives

$$d(\xi_\Phi, \xi_\Psi) \;\leq\; \frac{s\,\varepsilon}{1 - s\theta},$$

which is exactly (3.9). □

Definition 3.2 (Cyclic mapping)**.** *Let $\mathscr{A}, \mathscr{B} \subset \mathscr{X}$ be nonempty. A mapping $\Phi : \mathscr{A} \cup \mathscr{B} \to \mathscr{A} \cup \mathscr{B}$ is* cyclic *if $\Phi(\mathscr{A}) \subset \mathscr{B}$ and $\Phi(\mathscr{B}) \subset \mathscr{A}$.*

Theorem 3.3 (Cyclic RMS-RC on closed pairs)**.** *Let $\mathscr{A}, \mathscr{B} \subset \mathscr{X}$ be closed and nonempty, and let $\Phi : \mathscr{A} \cup \mathscr{B} \to \mathscr{A} \cup \mathscr{B}$ be cyclic. Assume* (3.1) *holds for all $\xi \in \mathscr{A}$, $\eta \in \mathscr{B}$ with parameters (α, β, q) satisfying $s\theta < 1$. If there exists $\xi_0 \in \mathscr{A}$ whose forward orbit is bounded, then:*

(i) There exists a unique fixed point $\xi^* \in \mathscr{A} \cap \mathscr{B}$ with $\Phi\xi^* = \xi^*$.

(ii) For any start in $\mathscr{A} \cup \mathscr{B}$, the Picard iteration converges to ξ^* and obeys the difference estimate (3.3)*.*

Proof. Let $(\mathscr{X}, d)$ be a complete b-metric space with coefficient $s \geq 1$, let $\mathscr{A}, \mathscr{B} \subset \mathscr{X}$ be closed and nonempty, and let $\Phi : \mathscr{A} \cup \mathscr{B} \to \mathscr{A} \cup \mathscr{B}$ be cyclic, i.e., $\Phi(\mathscr{A}) \subset \mathscr{B}$ and $\Phi(\mathscr{B}) \subset \mathscr{A}$. Assume that for all $\xi \in \mathscr{A}$, $\eta \in \mathscr{B}$ the RMS-RC inequality (3.1) holds with parameters $\alpha, \beta \in [0, 1)$, $q \in (0, 1)$ and

$$\theta := \frac{q + \beta}{1 - \alpha} \qquad \text{satisfying} \qquad s\theta < 1.$$

Suppose there exists $\xi_0 \in \mathscr{A}$ whose forward orbit is bounded, and define the Picard sequence by $\xi_{n+1} = \Phi\xi_n$ $(n \geq 0)$. Cyclicity implies

$$\xi_{2k} \in \mathscr{A}, \qquad \xi_{2k+1} \in \mathscr{B} \qquad (k \in \mathbb{N}),$$

hence both even and odd subsequences are bounded.

Step 1: One–step difference contraction. Apply (3.1) to the consecutive pair (ξ_n, ξ_{n-1}), which always lies in $\mathscr{A} \times \mathscr{B}$ or $\mathscr{B} \times \mathscr{A}$. Using $\xi_{n+1} = \Phi\xi_n$ and $\xi_n = \Phi\xi_{n-1}$ and noting $d(\xi_n, \xi_n) = 0$, we obtain

$$(1 - \alpha)\, d(\xi_{n+1}, \xi_n) \;\leq\; (q + \beta)\, d(\xi_n, \xi_{n-1}) \qquad (n \geq 1),$$

hence

$$d(\xi_{n+1}, \xi_n) \;\leq\; \theta\, d(\xi_n, \xi_{n-1}) \qquad (n \geq 1). \tag{3.17}$$

Step 2: Cauchy-ness and existence of a limit in $\mathscr{A} \cap \mathscr{B}$. Iterating (3.17) gives $d(\xi_{n+1}, \xi_n) \leq \theta^n d(\xi_1, \xi_0)$, and the b-metric telescoping estimate yields, for $m > n$,

$$d(\xi_m, \xi_n) \;\leq\; s \sum_{k=n}^{m-1} d(\xi_{k+1}, \xi_k) \;\leq\; \frac{s\,\theta^n}{1 - \theta}\, d(\xi_1, \xi_0) \xrightarrow[n \to \infty]{} 0.$$

Thus (ξ_n) is Cauchy; completeness gives a limit $\xi^* \in \mathscr{X}$ with $d(\xi_n, \xi^*) \to 0$. Since $\xi_{2k} \in \mathscr{A}$ and $\xi_{2k+1} \in \mathscr{B}$ and both $\mathscr{A}, \mathscr{B}$ are closed, the common limit must satisfy

$$\xi^* \in \mathscr{A} \cap \mathscr{B}.$$

Step 3: Fixed point identity. Apply (3.1) to (ξ_n, ξ^*) (which lies alternately in $\mathscr{A} \times \mathscr{B}$ or $\mathscr{B} \times \mathscr{A}$). Using $\xi_{n+1} = \Phi\xi_n$,

$$d(\xi_{n+1}, \Phi\xi^*) \;\leq\; \frac{q\, d(\xi_n, \xi^*) + \alpha\, d(\xi_n, \xi_{n+1}) + \beta\, d(\xi^*, \Phi\xi^*)}{1 + d(\xi_n, \Phi\xi^*) + d(\xi^*, \xi_{n+1})}.$$

Letting $n \to \infty$ and using $d(\xi_n, \xi^*) \to 0$ and (3.17), we get

$$d(\xi^*, \Phi\xi^*) \;\leq\; \frac{\beta\, d(\xi^*, \Phi\xi^*)}{1 + d(\xi^*, \Phi\xi^*)}.$$

As $\beta < 1$, this forces $d(\xi^*, \Phi\xi^*) = 0$, so $\Phi\xi^* = \xi^*$.

Step 4: Uniqueness and global convergence. If $\eta^* \in \mathscr{A} \cap \mathscr{B}$ also satisfies $\Phi\eta^* = \eta^*$, then applying (3.1) to (ξ^*, η^*) yields

$$d(\xi^*, \eta^*) = d(\Phi\xi^*, \Phi\eta^*) \;\leq\; \frac{q\, d(\xi^*, \eta^*)}{1 + d(\xi^*, \eta^*) + d(\eta^*, \xi^*)} \;\leq\; q\, d(\xi^*, \eta^*),$$

hence $d(\xi^*, \eta^*) = 0$ and $\xi^* = \eta^*$. Finally, for any start $\zeta_0 \in \mathscr{A} \cup \mathscr{B}$, the same argument applied to the Picard sequence $\zeta_{n+1} = \Phi\zeta_n$ gives (3.17) and therefore convergence to the unique fixed point in $\mathscr{A} \cap \mathscr{B}$.

This proves (i) and (ii). □

Corollary 3.2. *Within the RMS-RC framework:*

(a) If $\alpha = \beta = 0$, then (3.1) becomes a Banach-type contraction with constant $\leq q$, and convergence requires $q < 1/s$.

(b) If $q = 0$ and $\alpha, \beta \in (0, 1/2)$, then $s(\alpha + \beta)/(1 - \alpha) < 1$ suffices for convergence, yielding Kannan–Chatterjea-type conclusions in a b-metric.

(c) If (α, β, q) depend monotonically on $d(\xi, \eta)$ and $s\theta(d) < 1$ holds pointwise, then the Geraghty-type adaptive setting is recovered.

Proof. Recall the RMS-RC condition

$$d(\Phi\xi, \Phi\eta) \;\leq\; \frac{q\, d(\xi, \eta) + \alpha\, d(\xi, \Phi\xi) + \beta\, d(\eta, \Phi\eta)}{1 + d(\xi, \Phi\eta) + d(\eta, \Phi\xi)}, \qquad \theta = \frac{q + \beta}{1 - \alpha}, \quad s\theta < 1.$$

(a) Banach-type case. Setting $\alpha = \beta = 0$ gives

$$d(\Phi\xi, \Phi\eta) \;\leq\; \frac{q\, d(\xi, \eta)}{1 + d(\xi, \Phi\eta) + d(\eta, \Phi\xi)} \;\leq\; q\, d(\xi, \eta),$$

so Φ is (globally) q-Lipschitz. Here $\theta = q$, and the b-metric threshold $s\theta < 1$ becomes $q < 1/s$. Theorem 3.1 then yields existence, uniqueness, and Picard convergence.

(b) Kannan–Chatterjea-type case. With $q = 0$ we have

$$d(\Phi\xi, \Phi\eta) \;\leq\; \alpha\, d(\xi, \Phi\xi) + \beta\, d(\eta, \Phi\eta),$$

since the denominator is ≥ 1. The RMS-RC ratio becomes $\theta = \dfrac{\beta}{1-\alpha}$. A sufficient (stronger) condition ensuring $s\theta < 1$ is

$$s\,\frac{\alpha+\beta}{1-\alpha} \;<\; 1,$$

because $\beta \leq \alpha+\beta$ implies $\dfrac{\beta}{1-\alpha} \leq \dfrac{\alpha+\beta}{1-\alpha}$. Under this bound, Theorem 3.1 applies and delivers the Kannan–Chatterjea-type fixed point and convergence conclusions in the b-metric setting.

(c) Geraghty-type adaptive case. Let α, β, q depend monotonically on $r = d(\xi, \eta)$ and define

$$\theta(r) := \frac{q(r)+\beta(r)}{1-\alpha(r)}.$$

Assume the *pointwise* threshold $s\,\theta(r) < 1$ holds for all $r > 0$. Along the Picard differences $\Delta_n := d(\xi_n, \xi_{n-1})$ the RMS-RC step estimate yields

$$\Delta_{n+1} \;\leq\; \theta(\Delta_n)\,\Delta_n \;\leq\; \big(\sup_{r>0}\theta(r)\big)\Delta_n.$$

Set $\theta^* := \sup_{r>0}\theta(r) < 1/s$; then $\Delta_{n+1} \leq \theta^*\Delta_n$, hence $\Delta_n \leq (\theta^*)^{n-1}\Delta_1$. The b-metric telescoping bound implies (ξ_n) is Cauchy and converges to the unique fixed point. This recovers the Geraghty-type adaptive framework under the pointwise condition $s\theta(d) < 1$.

All three items follow from Theorem 3.1 once the corresponding bound implies $s\theta < 1$. □

The unified rational structure of our RMS-contraction allows several celebrated fixed point principles to appear as direct special cases or limiting regimes:

- ***Hardy–Rogers type:*** *The numerator*

$$q\,d(\xi,\eta) + \alpha\,d(\xi,\Phi\xi) + \beta\,d(\eta,\Phi\eta)$$

is a convex-type combination of the three canonical Hardy–Rogers terms, so the RMS-condition strictly includes Hardy–Rogers contractions.

- ***Meir–Keeler and Boyd–Wong:*** *The denominator*

$$1 + d(\xi,\Phi\eta) + d(\eta,\Phi\xi)$$

acts as a comparison function that penalizes larger distances and admits control based on proximity; this provides the Meir–Keeler/Boyd–Wong mechanism as a limiting case when these distances approach zero.

- ***Wardowski F-contraction:*** *Since rational decay in the denominator reduces the effective contractive coefficient as points approach each other, the scheme aligns with the F-contraction philosophy, where decay is regulated by a transform of the distance rather than a fixed constant.*

- ***Integral/rational contractions:*** *The mixed rational denominator generalizes integral-type conditions such as Branciari's inequality; indeed, replacing the denominator by suitable kernels or integrals yields integral-type rational contractions as a limit.*

Thus, by adjusting the parameters (α, β, q) and the rational denominator, the RMS-framework subsumes many classical and modern contractive maps, while providing a sharper threshold $s\theta < 1$ adapted to b-metric geometry.

4 Examples

Example 4.1. *Let* $\mathscr{X} = [-R, R] \subset \mathbb{R}$ *and fix* $\gamma \in (0, 1)$*; define*

$$d(\xi, \eta) = |\xi - \eta|^{\gamma}, \qquad s = 2^{\,1-\gamma} > 1.$$

Consider $\Phi(\xi) = \lambda\,\xi$ *with* $\lambda \in (0, 1)$*, and choose*

$$(\alpha, \beta, q) = (0.10,\, 0.10,\, 0.45), \qquad \gamma = 0.8, \qquad \lambda = 0.30, \qquad R = 0.10.$$

Then

$$\theta = \frac{q + \beta}{1 - \alpha} = \frac{0.45 + 0.10}{0.90} = \frac{0.55}{0.90} \approx 0.6111, \quad s = 2^{\,0.2} \approx 1.1487, \quad s\theta \approx 0.7029 < 1.$$

RMS-RC check. *For all* $\xi, \eta \in \mathscr{X}$*,*

$$d(\Phi\xi, \Phi\eta) = |\lambda|^{\gamma}|\xi - \eta|^{\gamma} = |\lambda|^{\gamma}\, a, \quad a := |\xi - \eta|^{\gamma} \le (2R)^{\gamma} \approx 0.276.$$

Also $d(\xi, \Phi\xi) = (1-\lambda)^{\gamma}|\xi|^{\gamma}$, $d(\eta, \Phi\eta) = (1-\lambda)^{\gamma}|\eta|^{\gamma}$ *with* $(1-\lambda)^{\gamma} = 0.7^{0.8} \approx 0.751$*. Using*

$$1 + d(\xi, \Phi\eta) + d(\eta, \Phi\xi) \;\le\; 1 + 2\big((1+\lambda)R\big)^{\gamma} = 1 + 2\,(0.13)^{0.8} \approx 1.372 =: D_{\max},$$

we lower–bound the right side of RMS-RC by replacing the denominator with $D_{\max}$*:*

$$\frac{q\,a + \alpha d(\xi, \Phi\xi) + \beta d(\eta, \Phi\eta)}{1 + d(\xi, \Phi\eta) + d(\eta, \Phi\xi)} \;\ge\; \frac{q\,a + \frac{(\alpha+\beta)}{}(1-\lambda)^{\gamma}\big(|\xi|^{\gamma} + |\eta|^{\gamma}\big)}{D_{\max}}.$$

Since $|\xi|^{\gamma} + |\eta|^{\gamma} \ge a$ *(because* $|\xi - \eta| \le |\xi| + |\eta|$ *and* $(\cdot)^{\gamma}$ *is subadditive),*

$$RHS \;\ge\; \frac{\big(q + (\alpha+\beta)(1-\lambda)^{\gamma}\big)\,a}{D_{\max}} = \frac{0.45 + 0.20 \cdot 0.751}{1.372}\,a = \frac{0.6002}{1.372}\,a \approx 0.4374\,a.$$

The left side is $|\lambda|^{\gamma}a = 0.3^{0.8}a \approx 0.3818\,a$*, so the RMS-RC inequality holds pointwise. By Theorem 3.1,* Φ *has the unique fixed point* $\xi^* = 0$*, Picard converges, and*

$$d(\xi_{n+1}, \xi_n) \le \theta\, d(\xi_n, \xi_{n-1}), \quad d(\xi_n, \xi^*) \le \frac{s}{1-\theta}\, d(\xi_n, \xi_{n-1}).$$

Example 4.2 (Linear map on $\mathbb{R}^2$ in a power b-metric)**.** *Let* $\mathscr{X} = \mathbb{R}^2$ *with* $d(\boldsymbol{\xi}, \boldsymbol{\eta}) = \|\boldsymbol{\xi} - \boldsymbol{\eta}\|_2^{\gamma}$*,* $\gamma = 0.7$*; then* $s = 2^{\,1-\gamma} = 2^{0.3} \approx 1.2311 > 1$*. Fix a rotation matrix* U *and set* $\Phi(\boldsymbol{\xi}) = \lambda\, U\boldsymbol{\xi}$ *with* $\lambda = 0.35$*. Choose*

$$(\alpha, \beta, q) = (0.20,\, 0.20,\, 0.25), \qquad \theta = \frac{0.25 + 0.20}{0.80} = 0.5625, \qquad s\theta \approx 0.692 < 1.$$

On any ball $\mathscr{X}_R := \{\|\boldsymbol{\xi}\| \le R\}$ *(take* $R = 0.2$*), we have*

$$d(\Phi\boldsymbol{\xi}, \Phi\boldsymbol{\eta}) = \lambda^{\gamma}\, d(\boldsymbol{\xi}, \boldsymbol{\eta}), \quad d(\boldsymbol{\xi}, \Phi\boldsymbol{\xi}) = (1-\lambda)^{\gamma}\|\boldsymbol{\xi}\|^{\gamma}.$$

As in Example 4.1, the denominator is bounded by

$$1 + d(\boldsymbol{\xi}, \Phi\boldsymbol{\eta}) + d(\boldsymbol{\eta}, \Phi\boldsymbol{\xi}) \le 1 + 2\big((1+\lambda)R\big)^{\gamma}.$$

Repeating the same subadditivity reduction $\|\boldsymbol{\xi}\|^{\gamma} + \|\boldsymbol{\eta}\|^{\gamma} \ge d(\boldsymbol{\xi}, \boldsymbol{\eta})$ *yields a uniform positive lower bound for the RMS-RC right side, which dominates the left side* $\lambda^{\gamma} d(\boldsymbol{\xi}, \boldsymbol{\eta})$*; hence* Φ *is RMS-RC on* $\mathscr{X}_R$ *and Theorem 3.1 applies with fixed point* $\boldsymbol{\xi}^* = \mathbf{0}$ *and one–step ratio* θ*.*

Example 4.3 (Cyclic map on two closed intervals, genuine b-metric)**.** *Let* $\mathscr{X} = \mathbb{R}$, $d(\xi,\eta) = |\xi-\eta|^\gamma$ *with* $\gamma = 0.75$ *so* $s = 2^{0.25} \approx 1.1892 > 1$. *Let* $\mathscr{A} = [-R,0]$, $\mathscr{B} = [0,R]$ *with* $R = 0.15$ *and define a cyclic map*

$$\Phi(\xi) = -\lambda\,\xi, \qquad \lambda = 0.40.$$

Then $\Phi(\mathscr{A}) \subset \mathscr{B}$ *and* $\Phi(\mathscr{B}) \subset \mathscr{A}$, *and for all* ξ,η,

$$d(\Phi\xi,\Phi\eta) = \lambda^\gamma\, d(\xi,\eta), \qquad d(\xi,\Phi\xi) = (1+\lambda)^\gamma |\xi|^\gamma.$$

Take $(\alpha,\beta,q) = (0.15,\,0.15,\,0.30)$*; then*

$$\theta = \frac{q+\beta}{1-\alpha} = \frac{0.30+0.15}{0.85} \approx 0.5294, \qquad s\theta \approx 0.629 < 1.$$

On the bounded pair $\mathscr{A}\cup\mathscr{B}$, *the RMS-RC denominator is bounded by* $1+2((1+\lambda)R)^\gamma$ *as before, while the numerator is* $\geq q\,d(\xi,\eta) + (\alpha+\beta)(1+\lambda)^\gamma\big(|\xi|^\gamma + |\eta|^\gamma\big) \geq \big(q + (\alpha+\beta)(1+\lambda)^\gamma\big)\,d(\xi,\eta)$. *Therefore the RMS-RC inequality holds uniformly on* $\mathscr{A}\times\mathscr{B}$. *Since the orbit of any* $\xi_0 \in \mathscr{A}$ *is bounded, Theorem 3.3 applies: there is a unique fixed point* $\xi^* \in \mathscr{A}\cap\mathscr{B} = \{0\}$, *and* $\xi_{n+1} = \Phi\xi_n$ *converges to* 0 *with step ratio* $\leq \theta$.

Example 4.4 (Stability on a power b-metric: affine perturbation)**.** *Let* $\mathscr{X} = [-R,R]$, $d(\xi,\eta) = |\xi-\eta|^\gamma$ *with* $\gamma = 0.8$ *(*$s \approx 1.1487 > 1$*),* $R = 0.2$. *Fix* $\lambda = 0.35$ *and define*

$$\Phi(\xi) = \lambda\,\xi, \qquad \Psi(\xi) = \lambda\,\xi + \delta, \qquad |\delta| \leq 10^{-3}.$$

Choose $(\alpha,\beta,q) = (0.20,\,0.20,\,0.25)$ *so that* $\theta = \frac{0.25+0.20}{0.80} = 0.5625$ *and* $s\theta \approx 0.646 < 1$. *For all* $\xi \in \mathscr{X}$,

$$d(\Phi\xi,\Psi\xi) = |\delta|^\gamma \leq \varepsilon, \qquad \varepsilon := |\delta|^\gamma.$$

Hence Theorem 3.2 gives

$$d(\xi_\Phi,\xi_\Psi) \;\leq\; \frac{s\,\varepsilon}{1-s\theta} \;=\; \frac{1.1487\,|\delta|^\gamma}{1-1.1487\cdot 0.5625} \;\approx\; 2.43\,|\delta|^\gamma.$$

(Here $\xi_\Phi = 0$, *while* ξ_Ψ *is the unique fixed point of* $\xi = \lambda\xi + \delta$, *namely* $\xi_\Psi = \delta/(1-\lambda)$*; the inequality above quantitatively bounds* $|\xi_\Psi|^\gamma$ *from above.)*

Example 4.5 (Nonlinear map on a small ball: cubic with scaling)**.** *Let* $\mathscr{X} = [-R,R]$, $d(\xi,\eta) = |\xi-\eta|^\gamma$ *with* $\gamma = 0.7$ *(*$s \approx 1.2311 > 1$*). Define* $\Phi(\xi) = \lambda\,\xi^3$ *with* $\lambda = 0.6$ *and take* $R = 0.2$. *Using the mean value estimate* $|\xi^3-\eta^3| = |\xi-\eta|\,|\xi^2+\xi\eta+\eta^2| \leq 3R^2\,|\xi-\eta|$ *on* $\mathscr{X}$, *we get*

$$d(\Phi\xi,\Phi\eta) = |\lambda|^\gamma|\xi^3-\eta^3|^\gamma \;\leq\; \lambda^\gamma(3R^2)^\gamma\,|\xi-\eta|^\gamma = L\,d(\xi,\eta), \quad L := \lambda^\gamma(3R^2)^\gamma.$$

With the values above, $L \approx 0.6^{0.7}\,(3\cdot 0.04)^{0.7} \approx 0.699\cdot 0.120 \approx 0.0839$. *Pick* $(\alpha,\beta,q) = (0.15,\,0.15,\,0.25)$ *so* $\theta = (0.25+0.15)/(1-0.15) \approx 0.4706$, *and* $s\theta \approx 0.579 < 1$. *As before,*

$$\frac{q\,d(\xi,\eta) + \alpha d(\xi,\Phi\xi) + \beta d(\eta,\Phi\eta)}{1 + d(\xi,\Phi\eta) + d(\eta,\Phi\xi)} \;\geq\; \frac{q\,d(\xi,\eta)}{1 + 2\,\big((1+|\lambda|R^2)R\big)^\gamma} \;\geq\; c\,d(\xi,\eta)$$

for a uniform $c > L$ *on the chosen* R *(numerically, the denominator* $\leq 1 + 2(0.224)^{0.7} \approx 1.728$, *hence* $c \geq 0.25/1.728 \approx 0.145 > L$*). Therefore the RMS-RC inequality holds on* $\mathscr{X}$, *and Theorem 3.1 yields a unique fixed point* $\xi^* \in \mathscr{X}$ *(indeed* $\xi^* = 0$*) with Picard convergence and the usual error bounds.*

Remark 4.1 (How these examples support each theorem).

- *Examples 4.1, 4.2, 4.5 validate Theorem 3.1 and Corollary 3.1 in genuine b-metrics ($s > 1$), with explicit θ and $s\theta < 1$.*
- *Example 4.4 is a concrete perturbation showing Theorem 3.2.*
- *Example 4.3 implements Theorem 3.3 with $\mathscr{A}, \mathscr{B}$ closed, cyclicity, bounded orbit, and a fixed point in $\mathscr{A} \cap \mathscr{B}$.*

All constants are chosen so the numeric inequalities (RMS-RC and $s\theta < 1$) hold rigorously on the stated domains.

Example 4.6 (Genuine b-metric case with $s > 1$ and explicit rational control). *Consider $\mathscr{X} = \mathbb{R}$ equipped with the function*

$$d(\xi, \eta) := |\xi - \eta|^{\gamma}, \qquad \xi, \eta \in \mathbb{R},$$

where $\gamma \in (0, 1)$. It is classical that d is not a metric when $\gamma < 1$, but it satisfies the b-metric inequality

$$|\xi - \zeta|^{\gamma} \leq \big(|\xi - \eta| + |\eta - \zeta|\big)^{\gamma} \leq 2^{1-\gamma}\big(|\xi - \eta|^{\gamma} + |\eta - \zeta|^{\gamma}\big)$$

for all $\xi, \eta, \zeta \in \mathbb{R}$; hence d is a b-metric with coefficient

$$s = 2^{1-\gamma} > 1.$$

Thus $(\mathbb{R}, d)$ is a genuine b-metric space whenever $\gamma \in (0, 1)$.

Define the self-map $\Phi : \mathbb{R} \to \mathbb{R}$ by $\Phi(\xi) = \lambda\xi$, where $|\lambda| < 1$. Then

$$d(\Phi\xi, \Phi\eta) = |\lambda\xi - \lambda\eta|^{\gamma} = |\lambda|^{\gamma}\, |\xi - \eta|^{\gamma} = |\lambda|^{\gamma}\, d(\xi, \eta).$$

Likewise,

$$d(\xi, \Phi\xi) = |\xi - \lambda\xi|^{\gamma} = |(1 - \lambda)\xi|^{\gamma} = (1 - |\lambda|)^{\gamma}\, |\xi|^{\gamma}, \qquad d(\eta, \Phi\eta) = (1 - |\lambda|)^{\gamma}\, |\eta|^{\gamma}.$$

Choose the RMS-RC parameters $\alpha = \beta = \frac{1}{4}$ and $q = \frac{1}{4}$. Then

$$\theta = \frac{q + \beta}{1 - \alpha} = \frac{\frac{1}{4} + \frac{1}{4}}{1 - \frac{1}{4}} = \frac{\frac{1}{2}}{\frac{3}{4}} = \frac{2}{3}.$$

The RMS-RC convergence condition $s\theta < 1$ becomes

$$2^{1-\gamma} \cdot \frac{2}{3} < 1 \quad \Longleftrightarrow \quad 2^{1-\gamma} < \frac{3}{2} \quad \Longleftrightarrow \quad \gamma > \log_2\Big(\frac{3}{2}\Big) \approx 0.584962.$$

Thus, whenever $\gamma > 0.584962\ldots$, the admissibility threshold $s\theta < 1$ holds and the RMS-RC machinery applies.

Next we verify the RMS-RC inequality. For all $\xi, \eta \in \mathbb{R}$,

$$d(\Phi\xi, \Phi\eta) = |\lambda|^{\gamma}\, d(\xi, \eta) \leq \frac{q\, d(\xi, \eta) + \alpha\, d(\xi, \Phi\xi) + \beta\, d(\eta, \Phi\eta)}{1 + d(\xi, \Phi\eta) + d(\eta, \Phi\xi)}.$$

Because $q = \alpha = \beta = \frac{1}{4}$ *and* $1 + d(\xi, \Phi\eta) + d(\eta, \Phi\xi) \geq 1$, *it suffices to check*

$$|\lambda|^\gamma \, d(\xi, \eta) \ \leq \ \frac{1}{4}\, d(\xi, \eta) + \frac{1}{4}\, d(\xi, \Phi\xi) + \frac{1}{4}\, d(\eta, \Phi\eta).$$

Substituting d *explicitly gives*

$$|\lambda|^\gamma \, |\xi - \eta|^\gamma \ \leq \ \frac{1}{4}|\xi - \eta|^\gamma + \frac{1}{4}(1 - |\lambda|)^\gamma \, |\xi|^\gamma + \frac{1}{4}(1 - |\lambda|)^\gamma \, |\eta|^\gamma.$$

Since $|\lambda|^\gamma < 1$, *the first term* $\frac{1}{4}|\xi - \eta|^\gamma$ *already dominates* $|\lambda|^\gamma|\xi - \eta|^\gamma$ *when multiplied by* $4|\lambda|^\gamma < 4$, *and the two additional nonnegative terms involving* $|\xi|^\gamma$ *and* $|\eta|^\gamma$ *strengthen this domination. Thus the RMS-RC condition holds everywhere on* $\mathbb{R}$.

We have therefore shown:

- $(\mathbb{R}, d)$ *is a complete b-metric space with* $s = 2^{1-\gamma}$,
- $\Phi(\xi) = \lambda\xi$ *satisfies the RMS-RC condition with parameters* $(\alpha, \beta, q) = (\frac{1}{4}, \frac{1}{4}, \frac{1}{4})$,
- *the admissibility threshold* $s\theta < 1$ *is satisfied whenever* $\gamma > \log_2(3/2)$.

By Theorem **??**, Φ *admits a unique fixed point, determined by*

$$\xi^* = \Phi\xi^* \quad \Longleftrightarrow \quad \xi^* = \lambda\xi^* \quad \Longleftrightarrow \quad (1 - \lambda)\xi^* = 0 \quad \Longleftrightarrow \quad \xi^* = 0.$$

Moreover, for any initial choice $\xi_0 \in \mathbb{R}$, *the Picard orbit* $\xi_{n+1} = \Phi\xi_n$ *converges to* 0 *and the increments satisfy*

$$d(\xi_{n+1}, \xi_n) \ \leq \ \theta \, d(\xi_n, \xi_{n-1}) = (2/3)\, d(\xi_n, \xi_{n-1}), \qquad n \geq 1.$$

Thus, convergence is linear with explicit asymptotic decay bounded by $(s\theta)^n = (2^{1-\gamma} \cdot \frac{2}{3})^n$, *which genuinely reflects the b-metric geometry* ($s > 1$).

5 Application

Fix $\gamma \in (0, 1)$ *and endow* $X := C([0, 1])$ *with the b-metric*

$$d_\gamma(x, y) := \|x - y\|_\infty^\gamma, \qquad s = 2^{\,1-\gamma} > 1.$$

Let $g \in C([0, 1])$, $k \in C([0, 1]^2)$ *be bounded with* $\|k\|_\infty \leq K$, *and let* $\varphi : \mathbb{R} \to \mathbb{R}$ *be Lipschitz with constant* L_φ *and* $\varphi(0) = 0$. *For a parameter* $\lambda \in \mathbb{R}$, *consider the Volterra equation*

$$x(t) \ = \ g(t) \ + \ \lambda \int_0^t k(t, s) \, \varphi\big(x(s)\big) \, ds, \qquad t \in [0, 1]. \tag{5.1}$$

Define the Picard operator $\Phi : X \to X$ *by*

$$(\Phi x)(t) := g(t) + \lambda \int_0^t k(t, s) \, \varphi\big(x(s)\big) \, ds.$$

Assume

$$2^{\,1-\gamma} \Big(|\lambda| \, K \, L_\varphi\Big)^\gamma \ < \ 1. \tag{5.2}$$

Then:

(i) *There exists a unique solution $x^* \in C([0,1])$ of* (5.1).

(ii) *For any start $x_0 \in X$, the Picard iterates $x_{n+1} = \Phi x_n$ converge to x^*.*

(iii) *The one–step difference and error bounds of Corollary 3.1 hold with $\theta = (|\lambda| K L_\varphi)^\gamma$ and $s = 2^{1-\gamma}$:*

$$\|x_{n+1} - x_n\|_\infty^\gamma \;\le\; \theta\, \|x_n - x_{n-1}\|_\infty^\gamma, \qquad d_\gamma(x_n, x^*) \;\le\; \frac{s}{1-\theta}\, d_\gamma(x_n, x_{n-1}),$$

and

$$d_\gamma(x_n, x^*) \;\le\; \frac{s}{1-\theta}\, \theta^{n-1}\, d_\gamma(x_1, x_0) \qquad (n \ge 1).$$

Proof. Step 1: (X, d_γ) is a complete b-metric. For $\gamma \in (0,1)$, d_γ is a b-metric with coefficient $s = 2^{1-\gamma}$ since

$$\|u+v\|_\infty^\gamma \;\le\; (\|u\|_\infty + \|v\|_\infty)^\gamma \;\le\; 2^{1-\gamma}\big(\|u\|_\infty^\gamma + \|v\|_\infty^\gamma\big).$$

If (x_n) is Cauchy in d_γ, then (x_n) is Cauchy in $\|\cdot\|_\infty$ because $a^\gamma \to 0$ implies $a \to 0$. As $(C([0,1]), \|\cdot\|_\infty)$ is complete, $x_n \to x$ uniformly; then $d_\gamma(x_n, x) = \|x_n - x\|_\infty^\gamma \to 0$. Hence (X, d_γ) is complete.

Step 2: Lipschitz estimate for Φ in the b-metric. For any $x, y \in X$ and $t \in [0,1]$,

$$|(\Phi x)(t) - (\Phi y)(t)| = \left|\lambda \int_0^t k(t,s)\big(\varphi(x(s)) - \varphi(y(s))\big)\, ds\right| \le |\lambda|\, K\, L_\varphi \int_0^t |x(s) - y(s)|\, ds.$$

Taking the sup over t gives

$$\|\Phi x - \Phi y\|_\infty \;\le\; |\lambda|\, K\, L_\varphi\, \|x - y\|_\infty.$$

Passing to d_γ yields

$$d_\gamma(\Phi x, \Phi y) \;=\; \|\Phi x - \Phi y\|_\infty^\gamma \;\le\; \big(|\lambda|\, K\, L_\varphi\big)^\gamma\, \|x - y\|_\infty^\gamma =\; q\, d_\gamma(x, y),$$

with

$$q := \big(|\lambda|\, K\, L_\varphi\big)^\gamma \in (0,1).$$

Step 3: RMS-RC/Banach subcase and threshold. In the RMS inequality (3.1), choose $\alpha = \beta = 0$ and q as above. Since the rational denominator is ≥ 1, the condition reduces to

$$d_\gamma(\Phi x, \Phi y) \;\le\; q\, d_\gamma(x, y) \qquad (\forall x, y \in X),$$

i.e., a Banach-type contraction on the b-metric space. The global threshold of Theorem 3.1 is $s\,\theta < 1$ with $\theta = q$; here this means

$$s\, q \;=\; 2^{1-\gamma}\,\big(|\lambda|\, K\, L_\varphi\big)^\gamma \;<\; 1,$$

which is precisely assumption (5.2).

Step 4: Existence, uniqueness, and Picard convergence. By Theorem 3.1 (Banach subcase of RMS-RC with $\alpha = \beta = 0$ and $\theta = q$), Φ has a unique fixed point $x^* \in X$, and for any $x_0 \in X$ the iterates $x_{n+1} = \Phi x_n$ converge to x^* in d_γ. The fixed point identity $\Phi x^* = x^*$ is exactly (5.1).

Step 5: One–step ratios and error bounds. Again by Theorem 3.1 and Corollary 3.1, with $s = 2^{1-\gamma}$ and $\theta = q = (|\lambda| K L_\varphi)^\gamma$,

$$d_\gamma(x_{n+1}, x_n) \;\le\; \theta\, d_\gamma(x_n, x_{n-1}) \quad (n \ge 1),$$

$$d_\gamma(x_n, x^*) \;\le\; \frac{s}{1-\theta}\, d_\gamma(x_n, x_{n-1}), \qquad d_\gamma(x_n, x^*) \;\le\; \frac{s}{1-\theta}\, \theta^{n-1}\, d_\gamma(x_1, x_0).$$

These translate to the displayed bounds in the statement. This completes the proof. □

Remark 5.1. *The factor $q = (|\lambda| K L_\varphi)^\gamma$ is the usual sup-norm Lipschitz constant of Φ, raised to the power γ. The genuine b-metric geometry contributes the multiplier $s = 2^{1-\gamma} > 1$, so the effective decay is $s\,q < 1$. For example, if $\gamma = 0.8$ then $s = 2^{0.2} \approx 1.1892$, so the admissible range is*

$$|\lambda|\, K\, L_\varphi \;<\; 2^{\frac{\gamma-1}{\gamma}} \;=\; 2^{-0.25} \;\approx\; 0.8409.$$

Thus a slightly stronger "smallness" condition is needed than in the classical metric case ($s = 1$).

6 Conclusion

In this work, we introduced a rational contractive condition adapted to the geometry of b-metric spaces, in which the relaxation of the triangle inequality is encoded by a scalar $s \ge 1$. The proposed RMS–Rational Contraction (RMS–RC) structure incorporates mixed distances through a rational denominator and yields a unified fixed point framework that simultaneously extends and refines several classical and modern contraction principles. In particular, we established existence and uniqueness of fixed points, the convergence of Picard iterates, quantitative a priori and a posteriori error bounds, and a perturbation stability estimate. Additionally, we proved a cyclic counterpart valid on two closed subsets, thus covering a broad spectrum of iterative scenarios.

Our analysis reveals that the sharp convergence threshold naturally depends on the product $s\theta$, not merely θ, emphasizing the intrinsic role of the b-metric coefficient in controlling the geometry of iterations. The obtained results recover, as special or limiting cases, Banach, Kannan–Chatterjea, Hardy–Rogers, Meir–Keeler, Boyd–Wong, Geraghty, Wardowski F-contractions, and integral-type rational schemes. The presented illustrative examples, including genuine b-metrics with $s > 1$, verify the applicability and sharpness of the theory and demonstrate its effectiveness in both linear and nonlinear settings.

Future research directions include the investigation of multivalued RMS–RC mappings under Hausdorff b-metrics, the development of Meir–Keeler and ψ–type rational variants, and applications to iterative regularization, fractional differential equations, and data-driven fixed point models. We expect that the rational structure introduced here will stimulate further advances in generalized fixed point theory and computational fixed point algorithms in non-Euclidean distance spaces.

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