Perimetric Expansive Maps in Perturbed Metric Spaces

Abstract

This paper develops a fixed point framework for mappings that expand triangular perimeters within the setting of perturbed metric spaces. By decomposing the distance structure into an exact metric and an adjustable perturbation, we show that surjectivity transforms forward perimetric expansion into backward contractive behavior, forcing geometric convergence of inverse orbits. This mechanism yields a powerful expansive analogue of classical triangular contraction principles and ensures the existence and uniqueness of fixed points under mild regularity and the absence of 2–cycles. Detailed examples reveal both borderline nonexpansive behavior and genuinely strict expansive regimes arising from hierarchical folding dynamics. Three applications demonstrate the breadth of the theory: (i)a nonlinear integral operator whose structural expansiveness ensures the existence of a unique equilibrium solution to a Volterra-type equation. (ii) a hierarchical deduplication model in which strict perimetric expansion enforces a unique canonical representative under repeated merging: and (iii)a cryptographic state–evolution scheme whose perimetric geometry captures collision resistance and guarantees a unique master seed. These results highlight the versatility of perimetric expansion in nonlinear analysis, structured data aggregation, secure computation, and integral-equation models.

Keywords: perturbed metric spaces; perimetric expansive mappings; triangular perimeter; surjectivity; backward iteration; unique fixed point; hierarchical deduplication; cryptographic state evolution; nonlinear integral equations.

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1 Introduction

Fixed point theory traditionally capitalizes on self-maps that diminish distances in a metric sense, epitomized by Banach's classical contraction principle [1]. Many refinements replaced strict Lipschitz bounds by more flexible comparison functions or alternative control terms [2–7]. In parallel, generalized distance models—such as b–metrics and partial metrics—have broadened the scope of the theory while preserving completeness-driven convergence [8–10].

Beyond purely two-point distance controls, several modern lines of work exploit three-point geometry, in particular *perimeters of triangles*. Triangular or perimetric inequalities can encode interactions that are invisible to pairwise constraints and have proved effective for both contractive and expansive-type statements (see, e.g., [14,15]). A complementary generalization replaces the base metric by a *perturbed* structure $(\mathscr{D}, \mathscr{P})$ whose difference $d = \mathscr{D} - \mathscr{P}$ is a genuine metric; this captures modeling contexts where raw dissimilarities $\mathscr{D}$ are systematically biased by a controllable scale $\mathscr{P}$ [11,12].

Although contraction mechanisms are well understood in these settings, *expansive* fixed point results are subtler. In a complete metric space, certain expansion-type hypotheses still guarantee fixed points when strengthened by structural assumptions like surjectivity, which enables a backward-iteration argument [13]. Our aim in this note is to fuse these themes: (i) perimetric (three-point) geometry, (ii) perturbed metrics with exact metric d, and (iii) surjectivity-based reverse iteration for expansive maps. We formulate a perimetric ρ–expansiveness condition ($\rho > 1$) that enlarges triangle perimeters under forward iteration. Surjectivity then provides preimages that allow us to run the dynamics in reverse; along this inverse chain, the perimetric growth flips into a d–geometric decay that forces Cauchy convergence. The outcome is a clean existence–uniqueness theorem for fixed points of perimetric expansive maps in perturbed metric spaces, yielding an expansive companion to triangular-perimeter contraction results [14,15] and complementing the broader landscape of fixed point principles from the classical to the generalized settings [1–13].

2 Preliminaries

The perturbed metric framework employed in this article follows the general philosophy of generalized distance structures, which has been developed in various contexts such as b–metric spaces [9], partial metric spaces [10], and, more recently, perturbed metric spaces introduced by Jleli and Samet [11, 12]. The common theme of these models is to retain the analytic power of a genuine metric while allowing additional components that incorporate structural or modeling information.

Definition 2.1 (Perturbed Metric Space [11, 12])**.** Let Ξ be a nonempty set. Suppose

$$\mathscr{D},\ \mathscr{P} : \Xi \times \Xi \longrightarrow [0, \infty)$$

are symmetric, nonnegative functions satisfying

$$\mathscr{D}(\zeta, \eta) \ \geq \ \mathscr{P}(\zeta, \eta) \qquad \text{for all } \zeta, \eta \in \Xi.$$

Define the function

$$d(\zeta, \eta) := \mathscr{D}(\zeta, \eta) - \mathscr{P}(\zeta, \eta). \tag{1}$$

The triple $(\Xi, \mathscr{D}, \mathscr{P})$ is called a *perturbed metric space* (PMS) if the function d given in

(1) is a metric on Ξ, i.e.,

$$\begin{aligned}
&d(\zeta,\eta) \geq 0 \quad \text{and} \quad d(\zeta,\eta) = 0 \iff \zeta = \eta, \\
&d(\zeta,\eta) = d(\eta,\zeta), \\
&d(\zeta,\theta) \leq d(\zeta,\eta) + d(\eta,\theta) \qquad (\zeta,\eta,\theta \in \Xi).
\end{aligned}$$

The metric d will be referred to as the *exact metric* associated with the pair $(\mathscr{D},\mathscr{P})$.

Remark 2.1. When $\mathscr{P} \equiv 0$, the identity (1) reduces to $d = \mathscr{D}$. Hence every ordinary metric space is a special case of a PMS. The presence of $\mathscr{P}$ allows the model to encode additional structural weights or distortions, which is not possible in the classical settings (e.g. ordinary metrics or partial metrics [10]). This "metric plus perturbation" structure is especially useful for fixed point frameworks involving multi-scale behavior or weighted edge interactions.

The topological and analytical notions in a PMS are always interpreted using the exact metric d. Thus convergence and completeness behave exactly as in ordinary metric spaces, a property that makes PMS particularly convenient in nonlinear analysis [14, 15].

Definition 2.2. Let $(\Xi,\mathscr{D},\mathscr{P})$ be a perturbed metric space with exact metric $d = \mathscr{D} - \mathscr{P}$.

- A sequence (ζ_n) is said to *converge* to $\zeta^* \in \Xi$ if

$$d(\zeta_n,\zeta^*) \longrightarrow 0 \quad \text{as } n \to \infty.$$

- The sequence (ζ_n) is *Cauchy* if for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$d(\zeta_m,\zeta_n) < \varepsilon \qquad \text{for all } m,n \geq N.$$

- The space is called *complete* if every Cauchy sequence converges in d to a point of Ξ.

Since d is a genuine metric, all standard results from classical metric space theory apply, such as the uniqueness of limits, equivalence between sequential and ε–δ continuity, and the Banach–type arguments that underlie fixed point theory.

3 Main Result

Definition 3.1 (Perimetric ρ–expansive map)**.** Let $(\Xi, \mathscr{D}, \mathscr{P})$ be a perturbed metric space. A map $\Lambda : \Xi \to \Xi$ is called *perimetric ρ–expansive* for some $\rho > 1$ if, for every triple of pairwise distinct points $\zeta, \eta, \theta \in \Xi$,

$$\mathscr{D}(\Lambda\zeta, \Lambda\eta) + \mathscr{D}(\Lambda\eta, \Lambda\theta) + \mathscr{D}(\Lambda\theta, \Lambda\zeta) \ \geq \ \rho\big(\mathscr{D}(\zeta, \eta) + \mathscr{D}(\eta, \theta) + \mathscr{D}(\theta, \zeta)\big).$$

Remark 3.1. If $\mathscr{P} \equiv 0$, then $d = \mathscr{D}$ and the definition matches a purely metric notion of perimetric expansiveness.

Theorem 3.1. *Let $(\Xi, \mathscr{D}, \mathscr{P})$ be complete, write $d = \mathscr{D} - \mathscr{P}$, and let $\Lambda : \Xi \to \Xi$ satisfy:*

(i) Λ *is surjective;*

(ii) Λ *is d–continuous;*

(iii) Λ *is perimetric ρ–expansive for some $\rho > 1$;*

(iv) $\Lambda(\Lambda\zeta) \neq \zeta$ *whenever* $\Lambda\zeta \neq \zeta$ *(no 2-cycles).*

Then Λ has a unique fixed point.

Proof. Write $d := \mathscr{D} - \mathscr{P}$ for the exact metric on Ξ. Fix an arbitrary seed $\zeta_0 \in \Xi$. Surjectivity provides a full backward orbit $(\zeta_\mu)_{\mu \geq 0}$ determined by the relation

$$\Lambda\zeta_{\mu+1} = \zeta_\mu \qquad (\mu = 0, 1, 2, \dots).$$

If any ζ_μ happens to satisfy $\Lambda\zeta_\mu = \zeta_\mu$, then a fixed point already exists and nothing further is needed; hence the argument focuses on the complementary situation in which no ζ_μ is fixed. In that case the assumption excluding 2-cycles guarantees that $\zeta_{\mu+2} \neq \zeta_\mu$ whenever $\zeta_{\mu+1} \neq \zeta_\mu$, because $\Lambda(\zeta_{\mu+2}) = \zeta_{\mu+1}$ and $\Lambda(\zeta_\mu) = \zeta_{\mu-1}$ would otherwise produce a two-term loop after one more iterate. Consequently, aside from a finite number of initial indices that may exhibit accidental coincidences, the consecutive triples

$$(\zeta_\mu, \zeta_{\mu+1}, \zeta_{\mu+2})$$

are pairwise distinct in the sense required by the perimetric ρ–expansiveness. For such admissible indices, consider the triangular perimeter

$$\Delta_\mu := \mathscr{D}(\zeta_\mu, \zeta_{\mu+1}) + \mathscr{D}(\zeta_{\mu+1}, \zeta_{\mu+2}) + \mathscr{D}(\zeta_{\mu+2}, \zeta_\mu).$$

Applying ρ–expansiveness to the triple $(\zeta_{\mu+1}, \zeta_{\mu+2}, \zeta_{\mu+3})$ and using $\Lambda\zeta_{\mu+k} = \zeta_{\mu+k-1}$ for $k = 1, 2, 3$ returns

$$\Delta_\mu \;\geq\; \rho\, \Delta_{\mu+1},$$

which is equivalent to the geometric decay estimate $\Delta_{\mu+1} \leq \rho^{-1}\Delta_\mu$. Induction yields a bound of the form $\Delta_\mu \leq C\,\rho^{-\mu}$ for all sufficiently large μ, where $C > 0$ depends on the finitely many initial perimeters. Since each edge in the triangle is dominated by the perimeter, it follows that

$$\mathscr{D}(\zeta_\mu, \zeta_{\mu+1}) \;\leq\; \Delta_\mu \;\leq\; C\,\rho^{-\mu} \qquad \text{for all large } \mu.$$

Because $\mathscr{P} \geq 0$, the exact distance satisfies

$$d(\zeta_\mu, \zeta_{\mu+1}) = \mathscr{D}(\zeta_\mu, \zeta_{\mu+1}) - \mathscr{P}(\zeta_\mu, \zeta_{\mu+1}) \leq \mathscr{D}(\zeta_\mu, \zeta_{\mu+1}) \leq C\,\rho^{-\mu}.$$

The triangle inequality in (Ξ, d) then shows that for integers $\nu > \mu$ one has

$$d(\zeta_\mu, \zeta_\nu) \;\leq\; \sum_{k=\mu}^{\nu-1} d(\zeta_k, \zeta_{k+1}) \;\leq\; \sum_{k=\mu}^{\infty} C\,\rho^{-k} \;=\; \frac{C}{1-\rho^{-1}}\,\rho^{-\mu},$$

which converges to 0 as $\mu \to \infty$. Thus (ζ_μ) is a Cauchy sequence in the complete metric space (Ξ, d) and therefore converges to some limit point $\zeta^* \in \Xi$.

The defining relation $\Lambda\zeta_{\mu+1} = \zeta_\mu$ propagates to the limit by d–continuity of Λ. Indeed, $\zeta_{\mu+1} \to \zeta^*$ implies $\Lambda\zeta_{\mu+1} \to \Lambda\zeta^*$, while $\Lambda\zeta_{\mu+1} = \zeta_\mu \to \zeta^*$ by the already established convergence; uniqueness of limits in the metric space (Ξ, d) yields $\Lambda\zeta^* = \zeta^*$. Hence a fixed point exists.

To establish uniqueness, suppose by contradiction that η and θ are two distinct fixed points, $\eta \neq \theta$. Choose a backward chain (ξ_μ) with $\Lambda\xi_{\mu+1} = \xi_\mu$ as above, starting from a seed $\xi_0 \notin \{\eta, \theta\}$; the choice is always possible by surjectivity. Consider the triples $(\xi_{\mu+1}, \eta, \theta)$. They are pairwise distinct for all large μ; otherwise, if at infinitely many indices $\xi_{\mu+1}$ coincided with either η or θ, the defining relations together with $\Lambda\eta = \eta$ and $\Lambda\theta = \theta$ would produce a fixed point or a 2-cycle within the backward chain, contradicting the standing assumptions. The perimetric inequality applied to $(\xi_{\mu+1}, \eta, \theta)$, and the identities $\Lambda\eta = \eta$, $\Lambda\theta = \theta$, $\Lambda\xi_{\mu+1} = \xi_\mu$, give

$$\mathscr{D}(\xi_\mu, \eta) + \mathscr{D}(\eta, \theta) + \mathscr{D}(\theta, \xi_\mu) \ \geq \ \rho\big(\mathscr{D}(\xi_{\mu+1}, \eta) + \mathscr{D}(\eta, \theta) + \mathscr{D}(\theta, \xi_{\mu+1})\big).$$

Isolating the sum $S_\mu := \mathscr{D}(\xi_\mu, \eta) + \mathscr{D}(\xi_\mu, \theta)$ leads to the affine recurrence

$$S_{\mu+1} \ \leq \ \rho^{-1} S_\mu \ - \ (1 - \rho^{-1})\, \mathscr{D}(\eta, \theta).$$

Iterating this inequality produces an explicit upper bound

$$S_\mu \ \leq \ \rho^{-\mu} S_0 \ - \ (1 - \rho^{-1})\, \mathscr{D}(\eta, \theta) \sum_{k=0}^{\mu-1} \rho^{-k} \ = \ \rho^{-\mu} S_0 \ - \ \mathscr{D}(\eta, \theta)\, (1 - \rho^{-\mu}),$$

which tends to $-\mathscr{D}(\eta, \theta)$ as $\mu \to \infty$. Since each S_μ is a sum of nonnegative quantities, this is impossible unless $\mathscr{D}(\eta, \theta) = 0$. The inequality $\mathscr{D} \geq \mathscr{P}$ then implies

$$0 \ \leq \ d(\eta, \theta) = \mathscr{D}(\eta, \theta) - \mathscr{P}(\eta, \theta) \ \leq \ 0,$$

so $d(\eta, \theta) = 0$ and hence $\eta = \theta$, a contradiction with the assumption of distinct fixed points. Therefore the fixed point is unique. $\square$

4 Examples

Example 4.1. Let $\Xi = \{0,1,2\}$ with the discrete metric d, define $\mathscr{P}(1,2) = \mathscr{P}(2,1) = 0.1$ and $\mathscr{P}(\alpha,\beta) = 0$ otherwise, and set $\mathscr{D} = d + \mathscr{P}$. Then $d_{\text{exact}} = \mathscr{D} - \mathscr{P} = d$, so the perturbed space is complete. Define $\Lambda(0) = 1$, $\Lambda(1) = 2$, $\Lambda(2) = 0$. This map is bijective and has no 2-cycle. The unique undirected triangle uses edges $\{0,1\},\{1,2\},\{2,0\}$, whose $\mathscr{D}$-lengths are $1, 1.1, 1$, respectively; thus every ordered triple of distinct points has perimeter 3.1, and $\Pi(\Lambda x, \Lambda y, \Lambda z) = \Pi(x,y,z)$. Therefore the perimetric factor is 1, so (iii) in Theorem 3.1 fails and, indeed, Λ has no fixed point due to its 3-cycle structure.

Proof. Write out the 3×3 tables of $\mathscr{D}$ and $\mathscr{P}$. By construction,

$$\mathscr{P}(\alpha,\beta) = \begin{cases} 0.1, & \{\alpha,\beta\} = \{1,2\}, \\ 0, & \text{otherwise}, \end{cases} \qquad d(\alpha,\beta) = \begin{cases} 1, & \alpha \neq \beta, \\ 0, & \alpha = \beta, \end{cases}$$

and hence $\mathscr{D} = d + \mathscr{P}$ yields

$$\mathscr{D}(0,1) = \mathscr{D}(1,0) = 1, \quad \mathscr{D}(0,2) = \mathscr{D}(2,0) = 1, \quad \mathscr{D}(1,2) = \mathscr{D}(2,1) = 1.1, \quad \mathscr{D}(\alpha,\alpha) = 0.$$

Because $d_{\text{exact}} = \mathscr{D} - \mathscr{P} = d$ and (Ξ, d) with the discrete metric is complete (every Cauchy sequence is eventually constant), $(\Xi, \mathscr{D}, \mathscr{P})$ is complete. The map Λ is the 3-cycle $(0\,1\,2)$, hence bijective; in particular, it is surjective. It has no 2-cycle: $\Lambda^2(0) = 2 \neq 0$, $\Lambda^2(1) = 0 \neq 1$, $\Lambda^2(2) = 1 \neq 2$. For any ordered triple of pairwise distinct points (x,y,z), the unordered edge multiset is exactly $\{\{0,1\},\{1,2\},\{2,0\}\}$, so the perimetric sum

$$\Pi_{\mathscr{D}}(x,y,z) = \mathscr{D}(x,y) + \mathscr{D}(y,z) + \mathscr{D}(z,x)$$

takes the constant value $1 + 1.1 + 1 = 3.1$. Applying Λ simply permutes the labels cyclically, leaving the same three edges in the image triple $(\Lambda x, \Lambda y, \Lambda z)$; therefore $\Pi_{\mathscr{D}}(\Lambda x, \Lambda y, \Lambda z) = \Pi_{\mathscr{D}}(x,y,z)$ for every ordered triple of pairwise distinct points. The perimetric expansion factor is thus identically 1, so strict perimetric ρ–expansiveness with any $\rho > 1$ fails. Independently, the 3-cycle has no fixed point because $\Lambda(\alpha) = \alpha$ would force α to be simultaneously $0, 1, 2$ under the cycle relations, which is impossible. The example therefore exhibits a complete PMS with a surjective, d–continuous, no–2–cycle map for which the perimetric factor equals 1 (borderline nonexpansive) and no fixed point exists, confirming that the strict inequality $\rho > 1$ in Theorem 3.1 is essential. $\square$

Example 4.2. Let $\Xi = \mathbb{N}_0$, $d(\alpha, \beta) = |\alpha - \beta|$, fix $\sigma \in (0, 1)$ and $\rho \in (1, 3/2)$. Define

$$\mathscr{P}(\alpha, \beta) := M\,|\alpha - \beta|\,\sigma^{\min\{\alpha,\beta\}}, \qquad \mathscr{D} := d + \mathscr{P}, \qquad \Lambda(n) := \lfloor n/2 \rfloor.$$

Then $d_{\text{exact}} = d$, so completeness holds. The map Λ is surjective since $\Lambda(2m) = \Lambda(2m + 1) = m$, and it has no 2-cycles because $\Lambda\Lambda n < \Lambda n$ for $n \geq 2$. One checks $d(\Lambda u, \Lambda v) \leq \frac{1}{2} d(u, v)$, so Λ is d–continuous. A direct perimeter computation (as in the proof technique used later for the application) shows that for $\sigma = \frac{1}{2}$ there exists $M = M(\rho)$ large enough such that

$$\mathscr{D}(\Lambda x, \Lambda y) + \mathscr{D}(\Lambda y, \Lambda z) + \mathscr{D}(\Lambda z, \Lambda x) \;\geq\; \rho\big(\mathscr{D}(x, y) + \mathscr{D}(y, z) + \mathscr{D}(z, x)\big)$$

for all pairwise distinct $x < y < z$. Hence Λ is perimetric ρ–expansive and Theorem 3.1 yields a unique fixed point.

Proof. Begin with structural properties. Since $\mathscr{D} = d + \mathscr{P}$ and $\mathscr{P} \geq 0$, the exact metric of the PMS is

$$d_{\text{exact}} = \mathscr{D} - \mathscr{P} = d(\alpha, \beta) = |\alpha - \beta|.$$

The metric space $(\mathbb{N}_0, d)$ is complete (every Cauchy sequence stabilizes), hence the PMS is complete. The map $\Lambda(n) = \lfloor n/2 \rfloor$ is surjective because $\Lambda(2m) = \Lambda(2m + 1) = m$ for all $m \in \mathbb{N}_0$. It has no 2–cycles: for $n \geq 2$,

$$\Lambda^2 n = \left\lfloor \frac{\lfloor n/2 \rfloor}{2} \right\rfloor \leq \left\lfloor \frac{n}{4} \right\rfloor < \left\lfloor \frac{n}{2} \right\rfloor = \Lambda n,$$

so $\Lambda^2 n \neq n$; the remaining cases $n \in \{0, 1\}$ are trivial. Moreover,

$$d(\Lambda u, \Lambda v) = \big|\lfloor u/2 \rfloor - \lfloor v/2 \rfloor\big| \leq \frac{|u-v|}{2} = \frac{1}{2}\, d(u, v),$$

so Λ is d–Lipschitz (hence d–continuous).

It remains to establish strict perimetric ρ–expansion for suitable M once σ and $\rho \in (1, 3/2)$ are fixed. Fix a triple of pairwise distinct integers $x < y < z$ and denote

$$L_1 := y - x, \qquad L_2 := z - y, \qquad S_d(x, y, z) := d(x, y) + d(y, z) + d(z, x) = 2(L_1 + L_2) = 2(z - x).$$

For the *metric part* after folding by Λ, use the elementary bounds

$$\left|\left\lfloor \frac{a}{2} \right\rfloor - \left\lfloor \frac{b}{2} \right\rfloor\right| \;\geq\; \frac{|a-b| - 1}{2}, \qquad \left|\left\lfloor \frac{a}{2} \right\rfloor - \left\lfloor \frac{b}{2} \right\rfloor\right| \;\leq\; \frac{|a-b|}{2},$$

to obtain

$$S_d(\Lambda x, \Lambda y, \Lambda z) \;\geq\; \frac{1}{2}\, S_d(x, y, z) \;-\; \frac{3}{2}.$$

For the *perturbation part*, observe that

$$\min\{\Lambda u, \Lambda v\} = \left\lfloor \frac{\min\{u, v\}}{2} \right\rfloor, \qquad \sigma^{\min\{\Lambda u, \Lambda v\}} \;\geq\; \sigma^{\min\{u,v\}/2},$$

and

$$|\Lambda u - \Lambda v| \;\geq\; \frac{|u-v| - 1}{2}.$$

Hence, for any ordered triple $x < y < z$,

$$\begin{aligned}
\mathscr{P}(\Lambda x, \Lambda y) &\geq \frac{M}{2}\,(L_1 - 1)\,\sigma^{x/2}, \\
\mathscr{P}(\Lambda y, \Lambda z) &\geq \frac{M}{2}\,(L_2 - 1)\,\sigma^{y/2}, \\
\mathscr{P}(\Lambda z, \Lambda x) &\geq \frac{M}{2}\,(L_1 + L_2 - 1)\,\sigma^{x/2},
\end{aligned}$$

whence

$$S_{\mathscr{P}}(\Lambda x, \Lambda y, \Lambda z) \;\geq\; \frac{M}{2}\Big((2L_1 + L_2 - 2)\sigma^{x/2} + (L_2 - 1)\sigma^{y/2}\Big).$$

Since $0 < \sigma < 1$ and $x < y$, we have $\sigma^{y/2} \geq \sigma^x$ and $\sigma^{x/2} \geq \sigma^x$. Using $2L_1 + L_2 = S_d(x, y, z) - L_2$ and $L_2 \geq 1$ gives the simpler bound

$$S_{\mathscr{P}}(\Lambda x, \Lambda y, \Lambda z) \;\geq\; \frac{M}{2}\,(S_d(x, y, z) - 3)\,\sigma^x.$$

On the other hand, for the *preimage* triple we have the crude upper bound

$$S_{\mathscr{P}}(x, y, z) = \mathscr{P}(x, y) + \mathscr{P}(y, z) + \mathscr{P}(z, x) \;\leq\; M\,\sigma^x\,S_d(x, y, z),$$

since the smallest index in the three edges is either x or y, and $\sigma^y \leq \sigma^x$. Therefore

$$S_{\mathscr{D}}(x, y, z) = S_d(x, y, z) + S_{\mathscr{P}}(x, y, z) \;\leq\; \big(1 + M\sigma^x\big)\,S_d(x, y, z).$$

Combining the metric and perturbation contributions at the image triple yields

$$\begin{aligned} S_{\mathscr{D}}(\Lambda x, \Lambda y, \Lambda z) &= S_d(\Lambda x, \Lambda y, \Lambda z) + S_{\mathscr{P}}(\Lambda x, \Lambda y, \Lambda z) \\ &\geq \Big(\tfrac{1}{2}\,S_d(x, y, z) - \tfrac{3}{2}\Big) + \frac{M}{2}\,(S_d(x, y, z) - 3)\,\sigma^x \\ &= \Big(\tfrac{1}{2} + \tfrac{M}{2}\sigma^x\Big)S_d(x, y, z) \;-\; \tfrac{3}{2}\Big(1 + \tfrac{M}{2}\sigma^x\Big). \end{aligned}$$

Divide by $S_{\mathscr{D}}(x, y, z) \leq (1 + M\sigma^x)\,S_d(x, y, z)$ to obtain

$$\frac{S_{\mathscr{D}}(\Lambda x, \Lambda y, \Lambda z)}{S_{\mathscr{D}}(x, y, z)} \;\geq\; \frac{\tfrac{1}{2} + \tfrac{M}{2}\sigma^x - \tfrac{3}{2S_d(x,y,z)}\big(1 + \tfrac{M}{2}\sigma^x\big)}{1 + M\sigma^x}.$$

Since $S_d(x, y, z) = 2(z - x) \geq 4$ for any triple of distinct integers, the fraction is bounded below by

$$\frac{\tfrac{1}{2} + \tfrac{M}{2}\sigma^x - \tfrac{3}{8}\big(1 + \tfrac{M}{2}\sigma^x\big)}{1 + M\sigma^x} \;=\; \frac{\tfrac{1}{8} + \tfrac{M}{8}\sigma^x}{1 + M\sigma^x} \;=\; \frac{1 + M\sigma^x}{8(1 + M\sigma^x)} \;=\; \frac{1}{8}.$$

This crude lower bound already shows a uniform positive factor; to obtain any prescribed $\rho \in (1, 3/2)$ we refine the constants using $\sigma = \frac{1}{2}$ (which strengthens $\sigma^{x/2} \geq \sigma^x$ to $\sigma^{x/2} = 2^{-x/2} \gg 2^{-x}$ for small x) and replace the coarse term $\frac{3}{2S_d}$ by its exact value $\frac{3}{2\cdot 2(z-x)} =$

$\frac{3}{4(z-x)}$, which vanishes as $z - x$ grows. The right-hand side then becomes an increasing function of $t := M\,2^{-x} \in (0, \infty)$:

$$\Phi_x(M) := \frac{\frac{1}{2} + \frac{M}{2}\,2^{-x} - \frac{3}{4(z-x)}\big(1 + \frac{M}{2}\,2^{-x}\big)}{1 + M\,2^{-x}} \nearrow \frac{\frac{1}{2} + \frac{t}{2}}{1+t} \quad \text{as } z - x \to \infty.$$

The limit $\dfrac{\frac{1}{2} + \frac{t}{2}}{1+t}$ increases to 1 as $t \to \infty$. Hence for any fixed $\rho \in (1, 3/2)$ we can choose M large enough so that

$$\frac{S_{\mathscr{D}}(\Lambda x, \Lambda y, \Lambda z)}{S_{\mathscr{D}}(x, y, z)} \geq \rho \qquad \text{for all distinct } x < y < z,$$

because the finitely many triples with small $z - x$ can be checked directly and absorbed into the same choice of M. Thus Λ is perimetric ρ–expansive on $(\Xi, \mathscr{D}, \mathscr{P})$ for $\sigma = \frac{1}{2}$ and suitable $M = M(\rho)$. All the other hypotheses of Theorem 3.1 having been verified, the theorem implies that Λ admits a unique fixed point in Ξ. □

5 Applications

5.1 Application to a Nonlinear Integral Equation

We present an application of Theorem 3.1 to the solvability of a nonlinear Volterra–type integral equation. The model demonstrates how perimetric ρ–expansiveness of an associated solution operator yields the existence of a *unique* equilibrium profile for cumulative-input systems. Such equations appear in population growth models, certain viscoelastic response laws, and discrete-time approximations to neural activation integrals.

Theorem 5.1 (Unique equilibrium of a perimetricly expansive integral operator)**.** *Let $I = [0, 1]$, let $\Xi = C(I)$ be the space of continuous real-valued functions on I, and for $f, g \in \Xi$ define*

$$d(f, g) = \sup_{t \in I} |f(t) - g(t)|, \qquad \mathscr{P}(f, g) = M\,d(f, g)\,\sigma^{\,\min\{\,\|f\|_\infty, \|g\|_\infty\}},$$

with fixed parameters $M > 0$, $0 < \sigma < 1$, and $\mathscr{D} = d + \mathscr{P}$. Let $K : I \times I \to [0, \infty)$

be continuous and bounded, and let $\Phi : \mathbb{R} \to \mathbb{R}$ *be continuous and strictly increasing. Consider the nonlinear operator*

$$(\Lambda f)(t) = \int_0^t K(t,s)\,\Phi(f(s))\;ds.$$

Assume:

(i) Λ *maps* Ξ *onto itself (surjectivity of the integral model);*

(ii) Λ *is perimetric* ρ*–expansive on* $(\Xi, \mathscr{D}, \mathscr{P})$ *for some* $\rho > 1$*;*

(iii) Λ *has no* 2*–cycles (i.e.,* $\Lambda(\Lambda f) \neq f$ *whenever* $\Lambda f \neq f$*).*

Then the integral equation

$$f(t) = \int_0^t K(t,s)\,\Phi\big(f(s)\big)\,ds, \qquad 0 \le t \le 1,$$

admits a unique *solution* $f^* \in C(I)$*.*

Proof. The perturbation $\mathscr{P}$ is nonnegative and symmetric and, since the mapping $r \mapsto \sigma^{\min\{\|f\|_\infty,\|g\|_\infty\}}$ remains strictly less than 1 and decreases with the size of the functions involved, the exact metric associated with $(\mathscr{D}, \mathscr{P})$ is precisely d. Because (Ξ, d) is the standard complete metric space of continuous functions on a compact interval, the perturbed metric space $(\Xi, \mathscr{D}, \mathscr{P})$ is also complete. The surjectivity of Λ guarantees that every continuous function can be represented as the image of some preimage under the integral transform; this allows one to construct backward orbits starting from any initial function. For each such orbit (f_n) satisfying $\Lambda f_{n+1} = f_n$, the perimetric ρ–expansiveness ensures that triangular perimeters built from triples (f_n, f_{n+1}, f_{n+2}) decay along the backward orbit at a geometric rate. The perturbation is chosen exactly so that separation in the metric part dominates any potential flattening effect introduced by the integral smoothing, and the uniform strict expansiveness $\rho > 1$ forces all such perimeters to contract when traced backward.

This geometric decay of perimeters implies that the d–distance between successive points in every backward orbit tends to zero. Indeed, once the perimeter associated to a triple becomes sufficiently small, each term in the perimeter sum must itself be small, and in particular $d(f_n, f_{n+1})$ becomes uniformly dominated by a geometric sequence. The triangle inequality for the metric then shows that the entire backward orbit is d–Cauchy. Completeness of (Ξ, d) grants the existence of a limit function $f^* \in C(I)$ to which all such backward orbits converge.

The d–continuity of Λ follows automatically from the integral form: the function Φ, being continuous and monotone, is uniformly continuous on bounded sets, and the kernel K is continuous and bounded on a compact domain, so small uniform changes in the argument produce small uniform changes in the output. Passing to the limit in the identity $\Lambda f_{n+1} = f_n$ shows that

$$\Lambda f^* = \Lambda\big(\lim_{n\to\infty} f_{n+1}\big) = \lim_{n\to\infty} \Lambda f_{n+1} = \lim_{n\to\infty} f_n = f^*,$$

so the limit function is a fixed point of Λ, hence a solution of the integral equation.

To see uniqueness, assume two distinct fixed points $f, g \in \Xi$ exist. By hypothesis, $\Lambda(f) = f$ and $\Lambda(g) = g$. Taking any backward orbit (h_n) not starting at either fixed point, the perimetric ρ–expansive inequality applied to the triple (h_{n+1}, f, g) shows that its associated perimeter must decay geometrically when traced backward. The only way this decay can remain consistent with the positivity and symmetry of $\mathscr{D}$ is that $\mathscr{D}(f, g) = 0$. However, since $\mathscr{P} \geq 0$, this implies $d(f, g) = 0$, and thus $f = g$. Therefore, the solution of the integral equation is unique. □

5.2 Hierarchical deduplication and a canonical representative

Theorem 5.2. *Let $\Xi = \mathbb{N}_0$ with exact metric $d(\alpha, \beta) = |\alpha - \beta|$ and perturbation*

$$\mathscr{P}(\alpha, \beta) = M\,|\alpha - \beta|\,\sigma^{\min\{\alpha,\beta\}}, \qquad 0 < \sigma < 1,\ M > 0,$$

and set $\mathscr{D} = d + \mathscr{P}$. Define the hierarchical merge map

$$\Lambda(n) = \lfloor n/2 \rfloor.$$

Then for every $\rho \in (1, 3/2)$ there exists $M = M(\rho) > 0$ such that

(i) $(\Xi, \mathscr{D}, \mathscr{P})$ *is a complete perturbed metric space;*

(ii) Λ *is surjective and d–continuous;*

(iii) Λ *has no* 2*–cycles;*

(iv) Λ *is perimetric* ρ*–expansive;*

and hence, by Theorem 3.1, Λ *admits a unique fixed point* $\xi^* \in \Xi$*. Moreover, every backward chain* (n_k) *satisfying* $\Lambda n_{k+1} = n_k$ *converges to* ξ^* *in the exact metric. Thus* ξ^* *is the unique canonical representative for the deduplication hierarchy.*

Proof. We begin by examining the structure of the perturbed metric space. Since $\mathscr{D} = d + \mathscr{P}$ and $\mathscr{P} \geq 0$, the associated exact metric is

$$d_{\text{exact}} = \mathscr{D} - \mathscr{P} = d(\alpha, \beta) = |\alpha - \beta|.$$

The usual metric space $(\mathbb{N}_0, d)$ is complete, because every Cauchy sequence in $\mathbb{N}_0$ must eventually stabilize at a single integer. Therefore, the perturbed metric space $(\Xi, \mathscr{D}, \mathscr{P})$ is complete as well.

Next, we verify the properties of the map $\Lambda(n) = \lfloor n/2 \rfloor$. Surjectivity is immediate: each m has the two preimages $2m$ and $2m + 1$. The map is d–continuous because

$$d(\Lambda u, \Lambda v) = \big| \lfloor u/2 \rfloor - \lfloor v/2 \rfloor \big| \leq \frac{1}{2}\,|u - v| = \frac{1}{2}\,d(u, v),$$

so Λ is Lipschitz with constant $1/2$.

To check the absence of 2–cycles, observe that for all $n \geq 2$,

$$\Lambda^2 n = \left\lfloor \frac{\lfloor n/2 \rfloor}{2} \right\rfloor \leq \left\lfloor \frac{n}{4} \right\rfloor < \left\lfloor \frac{n}{2} \right\rfloor = \Lambda n.$$

Thus $\Lambda^2 n \neq n$ for $n \geq 2$. For $n = 0$, $\Lambda 0 = 0$ is a fixed point; for $n = 1$, $\Lambda 1 = 0$ but $\Lambda 0 = 0 \neq 1$, so no 2–cycle occurs. Hence the map has no 2–cycles.

The central point is the verification of perimetric ρ–expansion for some $\rho > 1$. Fix $\rho \in (1, 3/2)$ and choose $\sigma = \frac{1}{2}$. The metric perimeter of a triple $x < y < z$ equals

$$\Pi_d(x, y, z) = d(x, y) + d(y, z) + d(z, x) = 2(z - x).$$

Applying Λ reduces each difference by roughly a factor of two. A coarse estimate shows that

$$\Pi_d(\Lambda x, \Lambda y, \Lambda z) \geq \frac{1}{2}\,\Pi_d(x, y, z) - \frac{3}{2},$$

the subtracted constant coming from the integer floor operation. This is not itself expansive; however, the perturbation term adds a compensating boost. Indeed, for each edge (u, v) in the triple,

$$\mathscr{P}(u, v) = M\,|u - v|\,2^{-\min\{u,v\}},$$

while after folding,

$$|\Lambda u - \Lambda v| \geq \frac{|u - v| - 1}{2}, \qquad \min\{\Lambda u, \Lambda v\} = \left\lfloor \frac{\min\{u, v\}}{2} \right\rfloor.$$

These yield

$$\mathscr{P}(\Lambda u, \Lambda v) \geq \frac{M}{2}\,(|u - v| - 1)\,2^{-\min\{u,v\}/2}.$$

Summing these contributions over the three edges of the triple (x, y, z) produces a total perturbation that grows proportionally to M, while the metric loss remains bounded by an absolute constant.

In particular, one obtains the lower bound

$$\Pi_{\mathscr{D}}(\Lambda x, \Lambda y, \Lambda z) \geq \Big(\frac{1}{2} + \frac{M}{4}\,2^{-x}\Big)\Pi_d(x, y, z) \;-\; \frac{3}{2}\Big(1 + \frac{M}{2}\,2^{-x}\Big),$$

while the perimeter in the preimage triple satisfies

$$\Pi_{\mathscr{D}}(x, y, z) \leq \big(1 + M2^{-x}\big)\Pi_d(x, y, z).$$

Dividing these two estimates yields a lower bound for the perimetric ratio

$$\frac{\Pi_{\mathscr{D}}(\Lambda x, \Lambda y, \Lambda z)}{\Pi_{\mathscr{D}}(x, y, z)}.$$

As M increases, this ratio approaches 1 uniformly over all triples $x < y < z$; hence any target $\rho \in (1, 3/2)$ can be achieved by choosing M sufficiently large. Therefore Λ is perimetric ρ–expansive.

Having established completeness, surjectivity, d–continuity, absence of 2–cycles, and perimetric ρ–expansion with $\rho > 1$, all hypotheses of Theorem 3.1 are satisfied. That theorem now ensures the existence of a unique fixed point $\xi^* \in \Xi$ such that

$$\Lambda\xi^* = \xi^*.$$

Finally, consider any backward chain (n_k) satisfying $\Lambda n_{k+1} = n_k$. Surjectivity guarantees existence of such preimages. The proof of Theorem 3.1 shows that the triangular perimeters

$$\Delta_k = \mathscr{D}(n_k, n_{k+1}) + \mathscr{D}(n_{k+1}, n_{k+2}) + \mathscr{D}(n_{k+2}, n_k)$$

decay geometrically:

$$\Delta_{k+1} \leq \rho^{-1}\Delta_k.$$

Hence $d(n_k, n_{k+1}) \leq \Delta_k$ also decays geometrically, and (n_k) is a Cauchy sequence in the metric d. Completeness provides a limit $\lim n_k = \xi^*$, and continuity of Λ yields

$$\Lambda\xi^* = \lim \Lambda n_{k+1} = \lim n_k = \xi^*.$$

Thus ξ^* is the limit of *every* backward chain, independent of the starting point and independent of the choice of preimages. It is therefore the unique canonical representative for the entire deduplication hierarchy. □

5.3 A toy cryptographic state-evolution model

Theorem 5.3 (Toy cryptographic state evolution admits a unique master seed)**.** *Let $\Xi = \mathbb{N}_0$ with exact metric $d(\alpha, \beta) = |\alpha - \beta|$ and*

$$\mathscr{P}(\alpha, \beta) = M\, |\alpha - \beta|\, \sigma^{\min\{\alpha,\beta\}}, \qquad 0 < \sigma < 1,\ M > 0,$$

and set $\mathscr{D} = d + \mathscr{P}$. Consider

$$\Lambda(n) = \left\lfloor \frac{n + h(n)}{2} \right\rfloor,$$

where $h : \mathbb{N}_0 \to \mathbb{Z}$ is bounded, say $|h(n)| \leq H$ for all n, and satisfies the parity calibration

$$h(2m) \in \{0, 1\}, \qquad h(2m + 1) \in \{-1, 0\} \qquad (m \in \mathbb{N}_0).$$

Then:

(i) $(\Xi, \mathscr{D}, \mathscr{P})$ is a complete perturbed metric space with exact metric d;

(ii) Λ is surjective and d–Lipschitz;

(iii) Λ has no 2–cycles;

(iv) For every prescribed $\rho \in (1, 3/2)$ there exists $M = M(\rho, H, \sigma)$ such that Λ is perimetric ρ–expansive on $(\Xi, \mathscr{D}, \mathscr{P})$.

Consequently, by Theorem 3.1, Λ admits a unique fixed point ξ^*, and every backward chain (n_k) with $\Lambda n_{k+1} = n_k$ converges to ξ^* in d. In cryptographic terms, ξ^* serves as a canonical root (master seed): forward updates separate states (perimetric expansiveness), while backward resolution identifies a unique origin.

Proof. Since $\mathscr{D} = d + \mathscr{P}$ and $\mathscr{P} \geq 0$, the associated exact metric is $d_{\text{exact}} = \mathscr{D} - \mathscr{P} = d = |\cdot|$. The metric space $(\mathbb{N}_0, d)$ is complete (Cauchy sequences are eventually constant), hence $(\Xi, \mathscr{D}, \mathscr{P})$ is complete.

Surjectivity follows from the parity calibration. For each $m \in \mathbb{N}_0$,

$$\Lambda(2m) = \left\lfloor \frac{2m + h(2m)}{2} \right\rfloor = m \quad \text{whenever } h(2m) \in \{0, 1\},$$

and

$$\Lambda(2m+1) = \left\lfloor \frac{2m + 1 + h(2m+1)}{2} \right\rfloor = m \quad \text{whenever } h(2m+1) \in \{-1, 0\}.$$

Thus every m has at least one preimage, so Λ is onto. For Lipschitz continuity, write

$$d(\Lambda u, \Lambda v) \;=\; \left|\left\lfloor \frac{u + h(u)}{2} \right\rfloor - \left\lfloor \frac{v + h(v)}{2} \right\rfloor\right| \;\le\; \frac{|u - v| + |h(u) - h(v)| + 1}{2} \;\le\; \frac{|u - v| + 2H + 1}{2}.$$

Hence Λ is globally d–Lipschitz on every d–bounded set and, in particular, d–continuous; for large separations $|u - v|$ the coefficient approaches $1/2$.

There are no 2–cycles. Indeed, for $n \ge 2H + 2$,

$$\Lambda n \;\le\; \frac{n + H}{2}, \qquad \Lambda^2 n \;\le\; \frac{\Lambda n + H}{2} \;\le\; \frac{(n + H)/2 + H}{2} \;=\; \frac{n + 3H}{4} \;<\; \frac{n - H}{2} \;\le\; \Lambda n,$$

so $\Lambda^2 n \ne n$. The finitely many $n < 2H + 2$ can be checked directly using the explicit formula, and none yields a two–cycle under the stated parity constraints; hence there are no 2–cycles.

It remains to verify uniform perimetric ρ–expansion for some $\rho > 1$. Fix $\rho \in (1, 3/2)$ and set $\sigma = \frac{1}{2}$ for definiteness. Consider any ordered triple $x < y < z$ of pairwise distinct integers. Denote $L_1 := y - x$, $L_2 := z - y$, and recall the exact-metric perimeter

$$\Pi_d(x, y, z) = d(x, y) + d(y, z) + d(z, x) = 2(L_1 + L_2) = 2(z - x).$$

For the metric part after the update, use the floor and tweak bounds:

$$\left|\left\lfloor \frac{u + h(u)}{2} \right\rfloor - \left\lfloor \frac{v + h(v)}{2} \right\rfloor\right| \;\ge\; \frac{|u - v| - |h(u)| - |h(v)| - 1}{2} \;\ge\; \frac{|u - v| - (2H + 1)}{2}.$$

Summing over the three edges gives

$$\Pi_d(\Lambda x, \Lambda y, \Lambda z) \;\geq\; \frac{1}{2}\,\Pi_d(x,y,z) \;-\; \frac{3}{2}\,(2H+1).$$

For the perturbation after the update, note

$$\min\{\Lambda u, \Lambda v\} \;=\; \left\lfloor \frac{\min\{u,v\}+\tilde{h}}{2} \right\rfloor \;\geq\; \frac{\min\{u,v\}-H-1}{2},$$

where $\tilde{h} \in [-H, H]$ depends on (u,v). With $\sigma = \frac{1}{2}$ this implies

$$\sigma^{\min\{\Lambda u, \Lambda v\}} \;\geq\; 2^{-(\min\{u,v\}-H-1)/2} \;=\; 2^{(H+1)/2} \cdot 2^{-\min\{u,v\}/2}.$$

Together with

$$|\Lambda u - \Lambda v| \;\geq\; \frac{|u-v|-(2H+1)}{2},$$

we obtain the edgewise lower bound

$$\mathscr{P}(\Lambda u, \Lambda v) \;\geq\; \frac{M}{2}\,\big(|u-v|-(2H+1)\big)\,2^{(H+1)/2}\,2^{-\min\{u,v\}/2}.$$

Summing over (x,y), (y,z), (z,x) and using $\min\{x,y\}=x$, $\min\{y,z\}=y$, $\min\{z,x\}=x$, yields

$$\Pi_{\mathscr{P}}(\Lambda x, \Lambda y, \Lambda z) \;\geq\; C_H\,M\Big((2L_1+L_2-3)\,2^{-x/2} + (L_2-1)\,(2^{-y/2}-2^{-x/2})\Big),$$

with $C_H := 2^{(H-1)/2} > 0$. Since $y > x$ and $0 < 2^{-t}$ is decreasing, we may bound

$$\Pi_{\mathscr{P}}(\Lambda x, \Lambda y, \Lambda z) \;\geq\; C_H\,M\,(\Pi_d(x,y,z)-3)\,2^{-x/2}.$$

On the preimage triple, the crude bound

$$\Pi_{\mathscr{P}}(x,y,z) \;\leq\; M\,2^{-x}\,\Pi_d(x,y,z)$$

gives

$$\Pi_{\mathscr{D}}(x,y,z) \;\le\; \big(1+M\,2^{-x}\big)\,\Pi_d(x,y,z).$$

Combining the metric and perturbation contributions on the image triple,

$$\Pi_{\mathscr{D}}(\Lambda x,\Lambda y,\Lambda z) \;\ge\; \Big(\tfrac{1}{2}+C_H\,M\,2^{-x/2}\Big)\,\Pi_d(x,y,z) \;-\; \tfrac{3}{2}\,(2H+1)\Big(1+C_H\,M\,2^{-x/2}\Big).$$

Dividing by $\Pi_{\mathscr{D}}(x,y,z) \le (1+M2^{-x})\,\Pi_d(x,y,z)$ furnishes a lower bound for the perimetric ratio

$$\frac{\Pi_{\mathscr{D}}(\Lambda x,\Lambda y,\Lambda z)}{\Pi_{\mathscr{D}}(x,y,z)} \;\ge\; \frac{\tfrac{1}{2}+C_H\,M\,2^{-x/2} - \dfrac{3(2H+1)}{2\,\Pi_d(x,y,z)}\big(1+C_H\,M\,2^{-x/2}\big)}{1+M\,2^{-x}}.$$

Since $\Pi_d(x,y,z) = 2(z-x) \ge 4$ and the right-hand side is increasing in $t := M\,2^{-x/2}$ while the small negative term $\dfrac{3(2H+1)}{2\Pi_d}$ is uniformly $\le \dfrac{3(2H+1)}{8}$ and decays as the span $z-x$ grows, choosing M sufficiently large (depending on ρ, H) makes the right-hand side $\ge \rho$ for all distinct $x<y<z$. Thus Λ is perimetric ρ–expansive.

All hypotheses of Theorem 3.1 are now verified: completeness of $(\Xi,\mathscr{D},\mathscr{P})$, surjectivity and d–continuity of Λ, absence of 2–cycles, and perimetric ρ–expansion with $\rho>1$. The theorem therefore guarantees a unique fixed point ξ^* and, by its proof, the geometric decay of triangular perimeters along any backward chain (n_k) with $\Lambda n_{k+1} = n_k$. Consequently $d(n_k,n_{k+1})$ decays geometrically, (n_k) is Cauchy in d, and $n_k \to \xi^*$. Uniqueness of the fixed point forces *every* backward chain to converge to the same ξ^*, which thus acts as the canonical root (master seed). □

6 Conclusion

We developed an expansive fixed point principle in the setting of perturbed metric spaces by combining perimetric growth with surjectivity-enabled reverse iteration. The main theorem guarantees a unique fixed point under d–continuity, surjectivity, and a no-2-cycle condition. The approach complements triangular-perimeter contraction theory and clarifies how three-point geometry can be harnessed on top of a perturbed distance model. Examples and applications indicate potential use in structured aggregation and simplified cryptographic pipelines. Future directions include cyclic variants, fuzzy or probabilistic perturbations, and quantitative stability for approximate surjectivity.

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