Original
Research
Article

E-Bayesian Estimation of Arrival and Service Rates in an $M/M/1$ Queueing System

Abstract

This paper develops Bayesian and E-Bayesian estimation procedures for the arrival rate and service rate of an $M/M/1$ queueing system. Assuming Gamma prior distributions for the unknown parameters, explicit Bayesian estimators are derived under the squared error and precautionary loss functions. To account for uncertainty in prior hyperparameters, E-Bayesian estimators are obtained by introducing uniform hyperpriors. The performance of the proposed estimators is evaluated through an extensive Monte Carlo simulation study for different sample sizes and parameter configurations.

Keywords: $M/M/1$ queue; Bayesian estimation; E-Bayesian estimation; Squared error loss function; Precautionary loss function

2020 Mathematics Subject Classification: 60K25; 68M20; 90B22

1 Introduction

Queueing theory plays a vital role in modeling and analyzing congestion phenomena arising in communication networks, manufacturing systems, computer systems, and service operations.

Among various queueing models, the $M/M/1$ queueing system has received considerable attention due to its mathematical tractability and wide applicability. Key performance measures such as traffic intensity, waiting time, and queue length depend critically on the accurate estimation of arrival and service parameters. Consequently, statistical inference for $M/M/1$ queueing systems has been an active area of research for several decades.

Early research primarily focused on classical estimation techniques. Clarke (1957) introduced maximum likelihood estimation for simple queueing models. Large-sample inference and likelihood-based estimation for single-server queues were further developed by Basawa and Prabhu (1988, 1996), while Bhat and Rao (1987) provided a comprehensive statistical analysis of queueing systems. The effects of parameter estimation on system performance were examined by Schruben and Kulkarni (1982) and Zheng and Seila (2000). Classical estimation of queueing measures was also studied by Srinivas et al. (2011) and Srinivas and Udupa (2014).

Bayesian inference was later introduced to overcome limitations of classical methods, particularly in small-sample scenarios and when prior information is available. One of the earliest Bayesian studies in queueing models was presented by Muddapur (1972). Armero (1994a) and Armero and Bayarri (1994b) developed Bayesian inference and prediction methods for $M/M/1$ queues. Bayesian estimation of performance measures, such as waiting time distributions and queue length, was further investigated by Mukherjee and Chowdhury (2010), Chowdhury and Mukherjee (2011), Almeida and Cruz (2018), and Basak and Choudhury (2021). Recent studies have also incorporated asymmetric loss functions and computational approaches, such as Markov Chain Monte Carlo methods, to improve Bayesian inference in queueing systems (Deepthi and Jose, 2021).

Despite the advantages of Bayesian methods, the resulting estimators are often sensitive to the choice of prior hyperparameters. To reduce this sensitivity, Han (1997, 2007) proposed the E-Bayesian estimation framework, which integrates Bayesian estimators over a class of prior distributions by assigning hyper-priors to the prior parameters. This approach leads to more robust estimators and provides a natural way to account for prior uncertainty. In recent years, E-Bayesian methods have been applied to queueing models. Hendi and Qomi

(2024) and Hendi et al. (2024) studied E-Bayesian and hierarchical Bayesian estimation of traffic intensity in the $M/M/1$ queueing system using expected posterior risk criteria.

Motivated by these developments, the present paper focuses on E-Bayesian estimation of key parameters and performance measures in the $M/M/1$ queueing system. By employing conjugate Gamma priors and the expected posterior risk as a measure of estimator performance, we derive closed-form expressions for E-Bayesian estimators and compare their behavior with classical and Bayesian estimators. The results contribute to the growing literature on robust Bayesian inference in queueing systems and provide practical insights for applications where prior uncertainty cannot be ignored.

The remainder of the paper is organized as follows. Section 2 describes the $M/M/1$ queueing model and presents the maximum likelihood estimation of the arrival and service rates. Section 3 develops Bayesian estimators of the arrival rate and service rate under the squared error loss function and the precautionary loss function using conjugate Gamma priors. Section 4 introduces the E-Bayesian estimation framework and derives E-Bayesian estimators under both loss functions by incorporating uncertainty in the prior hyperparameters through uniform hyperpriors. Section 5 presents an extensive Monte Carlo simulation study to compare the finite-sample performance of the proposed estimators in terms of bias and mean squared error. Finally, Section 6 concludes the paper with a summary of the main findings and possible directions for future research.

2 $M/M/1$ Queue

We consider a single-server queueing system of type $M/M/1$, in which customer arrivals follow a Poisson process and service times are exponentially distributed. The system is assumed to have infinite waiting capacity, a first-come–first-served service discipline, and statistical independence between the arrival and service mechanisms.

Let $\lambda > 0$ denote the arrival rate and $\mu > 0$ denote the service rate of the system. In this $M/M/1$ queue, the interarrival times between successive customers are independent and identically distributed exponential random variables with parameter λ, while the service times are independent and identically distributed exponential random variables with parameter μ. Assume that complete information is available in the form of observed interarrival and service times. Let

$$Y_1, Y_2, \ldots, Y_{n_a}$$

denote the observed interarrival times of n_a customers, where Y_1 represents the arrival time of the first customer and Y_j $(j \geq 2)$ represents the time elapsed between the $(j-1)$th and jth arrivals. Similarly, let

$$X_1, X_2, \ldots, X_{n_s}$$

denote the observed service times of n_s customers.

Under this $M/M/1$ queueing model assumptions, the probability density functions of interarrival and service times are respectively,

$$f(\mathbf{y}, \lambda) = \lambda e^{\lambda y}, \quad y > 0,\ \lambda > 0 \tag{1}$$

and

$$g(\mathbf{x}, \mu) = \mu e^{\mu x}, \quad x > 0,\ \mu > 0. \tag{2}$$

2.1 Maximum Likelihood Estimation

Using the observed interarrival times $Y_1, Y_2, \ldots, Y_{n_a}$ and the observed service times $X_1, X_2, \ldots, X_{n_s}$, the joint likelihood function can be written as

$$\begin{aligned} L(\lambda, \mu \mid \mathbf{y}, \mathbf{x}) &= \prod_{j=1}^{n_a} f(y_j, \lambda) \prod_{i=1}^{n_s} g(x_i, \mu) \\ &= \prod_{j=1}^{n_a} \lambda e^{-\lambda y_j} \prod_{i=1}^{n_s} \mu e^{-\mu x_i} \\ &= \lambda^{n_a} \mu^{n_s} e^{-\lambda t_a - \mu t_s} \end{aligned} \tag{3}$$

where $t_a = \sum_{j=1}^{n_a} y_j$ and $t_s = \sum_{i=1}^{n_s} x_i$.

Taking the natural logarithm of the likelihood function, we obtain the log-likelihood

$$\ell(\lambda, \mu) = n_a \ln \lambda + n_s \ln \mu - \lambda t_a - \mu t_s. \tag{4}$$

The maximum likelihood estimators λ and μ are obtained by differentiating the log-likelihood function (4) with respect to λ and μ, and equating the derivatives to zero, are given by

$$\hat{\lambda}_{\text{MLE}} = \frac{n_a}{t_a}, \tag{5}$$

and

$$\hat{\mu}_{\text{MLE}} = \frac{n_s}{t_s} \tag{6}$$

respectively. These estimators are intuitive, as they correspond to the reciprocal of the sample mean interarrival and service times, respectively.

3 Bayesian Estimation

In the Bayesian framework, prior distributions are specified for the unknown arrival and service rates of the $M/M/1$ queueing system. Since both parameters are positive and continuous, Gamma distributions are adopted as prior distributions due to their flexibility and conjugacy with the exponential likelihood. Specifically, the arrival rate λ is assumed to follow a Gamma distribution with hyperparameters (a_1, b_1), while the service rate μ follows a Gamma distribution with hyperparameters (a_2, b_2). The corresponding probability density functions are given by

$$\pi(\lambda \mid a_1, b_1) = \frac{b_1^{a_1}}{\Gamma(a_1)} \lambda^{a_1 - 1} e^{-b_1 \lambda}, \quad \lambda > 0, \tag{7}$$

and

$$\pi(\mu \mid a_2, b_2) = \frac{b_2^{a_2}}{\Gamma(a_2)} \mu^{a_2 - 1} e^{-b_2 \mu}, \quad \mu > 0 \tag{8}$$

where $a_i > 0, b_i > 0, i = 1, 2$. It is further assumed that the arrival and service rates are a priori independent. Consequently, the joint prior distribution of (λ, μ) is given by

$$\pi(\lambda, \mu) = \pi(\lambda)\pi(\mu). \tag{9}$$

The Gamma prior is particularly suitable in this setting as it ensures analytical tractability of the posterior distributions and allows prior information to be incorporated through interpretable hyperparameters.

The joint posterior distribution of λ and μ is proportional to the product of the likelihood and the joint prior density, and is given by

$$\begin{aligned}
\pi(\lambda, \mu \mid \mathbf{y}, \mathbf{x}) &= \frac{L(\lambda, \mu \mid \mathbf{y}, \mathbf{x})\, \pi(\lambda)\pi(\mu)}{\int_0^\infty \int_0^\infty L(\lambda, \mu \mid \mathbf{y}, \mathbf{x})\, \pi(\lambda)\pi(\mu) d\lambda d\mu} \\
&= \frac{\lambda^{n_a+a_1-1} \mu^{n_s+a_2-1} e^{-(b_1+t_a)\lambda-(b_2+t_s)\mu}}{\int_0^\infty \int_0^\infty \lambda^{n_a+a_1-1} \mu^{n_s+a_2-1} e^{-(b_1+t_a)\lambda-(b_2+t_s)\mu} d\lambda d\mu} \\
&= \frac{(b_1+t_a)^{a_1+n_a}}{\Gamma(a_1+n_a)} \lambda^{a_1+n_a-1} e^{-(b_1+t_a)\lambda} \cdot \frac{(b_2+t_s)^{a_2+n_s}}{\Gamma(a_2+n_s)} \mu^{a_2+n_s-1} e^{-(b_2+t_s)\mu}
\end{aligned} \tag{10}$$

From the above joint posterior distribution (10), it follows that the marginal posterior distributions of λ and μ remain in the Gamma family, i.e., Gamma$(a_1 + n_a,\ b_1 + t_a)$ and Gamma$(a_2 + n_s,\ b_2 + t_s)$ respectively. Specifically, the marginal posterior distribution of the arrival rate λ is

$$\pi(\lambda \mid \mathbf{y}) = \frac{(b_1+t_a)^{a_1+n_a}}{\Gamma(a_1+n_a)} \lambda^{a_1+n_a-1} e^{-(b_1+t_a)\lambda} \tag{11}$$

and the marginal posterior distribution of the service rate μ is

$$\pi(\mu \mid \mathbf{x}) = \frac{(b_2+t_s)^{a_2+n_s}}{\Gamma(a_2+n_s)} \mu^{a_2+n_s-1} e^{-(b_2+t_s)\mu}. \tag{12}$$

Since independence is assumed both in the likelihood and in the prior distributions, the marginal posterior distributions of λ and μ are also independent.

3.1 Bayesian Estimation under Squared Error Loss Function

The squared error loss function (SELF) is defined as

$$L(\theta, \hat{\theta}_B^{SELF}) = (\hat{\theta}_B^{SELF} - \theta)^2. \tag{13}$$

It penalizes deviations of the estimator $(\hat{\theta}_B^{SELF})$ from the true parameter (θ) symmetrically and more severely for larger errors. Because of its mathematical simplicity and intuitive interpretation, SELF is the most commonly used loss function in Bayesian estimation. A fundamental result in Bayesian decision theory is that, under SELF, the Bayes estimator of a parameter is its posterior mean, i.e.,

$$E(\hat{\theta}_B^{SELF}) = E(\theta \mid \mathbf{y}). \tag{14}$$

Using the marginal posterior distribution (11), under the squared error loss function, the Bayes estimator of λ is given by

$$\begin{aligned}
\hat{\lambda}_B^{SELF} &= E(\lambda \mid y) \\
&= \int_0^\infty \lambda \, \pi(\lambda \mid y) \, d\lambda \\
&= \frac{(b_1 + t_a)^{a_1+n_a}}{\Gamma(a_1 + n_a)} \int_0^\infty \lambda^{a_1+n_a} e^{-(b_1+t_a)\lambda} d\lambda \\
&= \frac{(b_1 + t_a)^{a_1+n_a}}{\Gamma(a_1 + n_a)} \cdot \frac{\Gamma(a_1 + n_a + 1)}{(b_1 + t_a)^{a_1+n_a+1}} \\
&= \frac{a_1 + n_a}{b_1 + t_a}.
\end{aligned} \tag{15}$$

Similarly using the marginal posterior distribution (12), under the squared error loss function, the Bayes estimator of μ is

$$\hat{\mu}_B^{SELF} = \frac{a_2 + n_s}{b_2 + t_s}. \tag{16}$$

3.2 Bayesian Estimation under Precautionary Loss Function

The precautionary loss function (Norstrom (1996)) is appropriate in situations where underestimation of a parameter may lead to more serious consequences than overestimation. It is defined as

$$L(\theta, \hat{\theta}_B^{PLF}) = \frac{(\hat{\theta}_B^{PLF} - \theta)^2}{\hat{\theta}_B^{PLF}}, \qquad \hat{\theta}_B^{PLF} > 0. \tag{17}$$

This loss function discourages very small estimates and is well suited for positive parameters such as the arrival and service rates in queueing systems. Under this loss function, the Bayes estimator is given by the square root of the posterior second moment, i.e.,

$$\hat{\theta}_B^{PLF} = \sqrt{E(\theta^2 \mid \mathbf{y})}. \tag{18}$$

From the marginal posterior distribution (11), the Bayes estimator of λ under the precautionary loss function is

$$\begin{aligned} \hat{\lambda}_B^{PLF} &= \sqrt{E(\lambda^2 \mid y)} \\ &= \sqrt{\int_0^\infty \lambda^2 \pi(\lambda \mid y)\, d\lambda} \\ &= \sqrt{\frac{(b_1 + t_a)^{a_1+n_a}}{\Gamma(a_1 + n_a)} \int_0^\infty \lambda^{a_1+n_a+1} e^{-(b_1+t_a)\lambda} d\lambda} \\ &= \sqrt{\frac{\Gamma(a_1 + n_a + 2)}{\Gamma(a_1 + n_a)} \frac{1}{(b_1 + t_a)^2}} \\ &= \frac{\sqrt{(a_1 + n_a)(a_1 + n_a + 1)}}{b_1 + t_a}. \end{aligned} \tag{19}$$

Similarly, from the marginal posterior distribution (12), the Bayesian estimator of μ under precautionary loss is

$$\hat{\mu}_B^{PLF} = \frac{\sqrt{(a_2 + n_s)(a_2 + n_s + 1)}}{b_2 + t_s}. \tag{20}$$

4 E-Bayesian Estimation

In Bayesian inference, the Bayes estimate of a parameter depends on the choice of hyperparameters appearing in the prior distribution. When these hyperparameters are not known precisely, it is reasonable to treat them as random variables and assign suitable prior (hyperprior) distributions to them. The resulting estimator, obtained by averaging the Bayes estimator with respect to the hyperprior distribution, is known as the E-Bayesian estimate (Han (1997)).

Let θ be an unknown parameter with prior distribution $\pi(\theta \mid \boldsymbol{\eta})$ depending on a vector of hyperparameters $\boldsymbol{\eta}$, and let $\hat{\theta}_B(\boldsymbol{\eta})$ denote the Bayes estimator of θ under a specified loss function. If $h(\boldsymbol{\eta})$ denotes the hyperprior density of $\boldsymbol{\eta}$, then the E-Bayesian estimate of θ is defined as

$$\hat{\theta}_{EB} = E_{\boldsymbol{\eta}}\left[\hat{\theta}_B(\boldsymbol{\eta})\right] = \int \hat{\theta}_B(\boldsymbol{\eta})\, h(\boldsymbol{\eta})\, d\boldsymbol{\eta}. \tag{21}$$

Thus, the E-Bayesian estimate represents the expected Bayes estimator over a family of prior distributions and incorporates uncertainty about the hyperparameters. This approach provides estimators that are generally more robust to the choice of prior parameters and is particularly useful in situations where only partial or vague prior information is available.

In the present model, the arrival rate λ and service rate μ are assumed to follow Gamma prior distributions with hyperparameters (a_1, b_1) and (a_2, b_2), respectively. To obtain E-Bayesian estimators, we further assume that these hyperparameters are random and independently distributed as

$$\begin{aligned} a_1 &\sim \text{Uniform}(0, A_1), \quad b_1 \sim \text{Uniform}(0, B_1), \\ a_2 &\sim \text{Uniform}(0, A_2), \quad b_2 \sim \text{Uniform}(0, B_2) \end{aligned} \tag{22}$$

where A_1, B_1, A_2, and B_2 are known positive constants.

4.1 E-Bayesian Estimation under Squared Error Loss Function

The E-Bayesian estimator of λ is defined as the expectation of $\hat{\lambda}_B^{SELF}$ with respect to the hyperprior distributions of a_1 and b_1, that is,

$$\begin{aligned}\hat{\lambda}_{EB}^{SELF} &= E_{a_1,b_1}\left(\hat{\lambda}_B^{SELF}\right) \\ &= \frac{1}{A_1B_1}\int_0^{A_1}\int_0^{B_1}\frac{a_1+n_a}{b_1+t_a}\,db_1\,da_1 \\ &= \frac{1}{A_1B_1}\left(\frac{A_1}{2}+n_a\right)\ln\left(\frac{t_a+B_1}{t_a}\right). \end{aligned} \tag{23}$$

Similarly, the E-Bayesian estimator of the service rate μ under squared error loss is

$$\hat{\mu}_{EB}^{SELF} = \frac{1}{A_2B_2}\left(\frac{A_2}{2}+n_s\right)\ln\left(\frac{t_s+B_2}{t_s}\right). \tag{24}$$

These estimators represent averages of the Bayes estimators over a class of Gamma priors and therefore provide greater robustness against uncertainty in the choice of prior hyperparameters.

4.2 E-Bayesian Estimation under Precautionary Loss Function

The E-Bayesian estimator is defined as the expectation of the corresponding Bayes estimator with respect to the hyperprior distributions of the hyperparameters. Thus, the E-Bayesian estimator of λ under the precautionary loss function is

$$
\begin{aligned}
\hat{\lambda}_{EB}^{PLF} &= E_{a_1,b_1}\left[\hat{\lambda}_{B}^{PLF}\right] \\
&= \frac{1}{A_1 B_1}\int_0^{A_1}\int_0^{B_1} \frac{\sqrt{(a_1+n_a)(a_1+n_a+1)}}{b_1+t_a}\, db_1\, da_1 \\
&= \frac{1}{A_1 B_1}\ln\left(\frac{t_a+B_1}{t_a}\right)\int_0^{A_1} \sqrt{(a_1+n_a)(a_1+n_a+1)}\, da_1 \\
&= \frac{\ln\left(\frac{t_a+B_1}{t_a}\right)}{2A_1B_1}\Bigg[(2(n_a+A_1)+1)\sqrt{(n_a+A_1)(n_a+A_1+1)} \\
&\quad - (2n_a+1)\sqrt{n_a(n_a+1)} \\
&\quad - \ln\left(\frac{n_a+A_1+\frac{1}{2}+\sqrt{(n_a+A_1)(n_a+A_1+1)}}{n_a+\frac{1}{2}+\sqrt{n_a(n_a+1)}}\right)\Bigg]. \qquad (25)
\end{aligned}
$$

Similarly, the E-Bayesian estimator of the service rate μ under the precautionary loss function is

$$
\begin{aligned}
\hat{\mu}_{EB}^{PLF} &= \frac{1}{2A_2B_2}\ln\left(\frac{t_s+B_2}{t_s}\right)\Bigg[(2(n_s+A_2)+1)\sqrt{(n_s+A_2)(n_s+A_2+1)} \\
&\quad - (2n_s+1)\sqrt{n_s(n_s+1)} \\
&\quad - \ln\left(\frac{n_s+A_2+\frac{1}{2}+\sqrt{(n_s+A_2)(n_s+A_2+1)}}{n_s+\frac{1}{2}+\sqrt{n_s(n_s+1)}}\right)\Bigg]. \qquad (26)
\end{aligned}
$$

The above expressions provide semi-closed forms for the E-Bayesian estimators under the precautionary loss function. These estimators incorporate uncertainty in the hyperparameters and therefore offer increased robustness compared to the corresponding Bayes estimators.

5 Numerical Results

This section presents an extensive Monte Carlo simulation study to investigate the finite-sample performance of the proposed estimators of the arrival rate λ and service rate μ in an $M/M/1$ queueing system. The comparison includes the maximum likelihood estimator (MLE), Bayesian estimators under the squared error loss function (SELF) and precautionary loss function (PLF), and their corresponding E-Bayesian counterparts.

5.1 Simulation Design

The simulation experiment is conducted under a wide range of parameter configurations in order to assess estimator behavior under small, moderate, and large sample sizes. The following settings are used throughout the study:

- Sample sizes: $n = 20, 50, 100, 200$.
- Arrival rates: $\lambda = 2, 3, 4$.
- Service rates: $\mu = 4, 5, 6$.
- Bayesian prior distributions:

$$\lambda \sim \text{Gamma}(a_1 = 1.5, b_1 = 3.5), \quad \mu \sim \text{Gamma}(a_2 = 2.5, b_2 = 5.0).$$

- Hyperprior distributions for E-Bayesian estimation:

$$a_1 \sim U(0, A_1 = 4), \quad b_1 \sim U(0, B_1 = 8),$$

$$a_2 \sim U(0, A_2 = 5), \quad b_2 \sim U(0, B_2 = 10).$$

- Number of Monte Carlo replications: 10,000.

For each replication, interarrival times and service times are independently generated from exponential distributions with parameters λ and μ, respectively. Equal sample sizes are assumed for arrivals and services, i.e., $n_a = n_s = n$. For each estimator, the Monte Carlo average estimate, bias, and mean squared error (MSE) are computed.

5.2 Estimation of the Arrival Rate λ

Tables 1–3 report the average estimates, bias, and MSE of different estimators of λ for $\lambda = 2, 3, 4$ and varying sample sizes. Several important patterns emerge from these results.

First, the MLE exhibits noticeable positive bias and relatively large MSE for small samples ($n = 20$), particularly when the true arrival rate is high. This behavior is expected due to the inherent variability of the exponential distribution and the absence of prior information in the classical framework.

Second, Bayesian estimators under the squared error loss function substantially reduce both bias and MSE across all sample sizes. The shrinkage effect induced by the Gamma prior is especially effective in small samples, leading to more stable estimates than the MLE.

Third, Bayesian estimators under the precautionary loss function consistently produce larger estimates than their SELF counterparts. This reflects the asymmetric nature of PLF, which penalizes underestimation more severely. Such conservative behavior is desirable in queueing systems, where underestimating the arrival rate may result in severe congestion or system overload.

Fourth, E-Bayesian estimators outperform their corresponding Bayesian estimators in terms of MSE, particularly for small and moderate sample sizes. By averaging the Bayes estimators over a range of hyperparameter values, the E-Bayesian approach effectively mitigates sensitivity to prior specification and yields more robust estimates.

As the sample size increases to $n = 200$, the differences among MLE, Bayesian, and E-Bayesian estimators diminish, indicating asymptotic consistency and convergence of all estimators to the true parameter value.

Table 1: Performance of estimators of $\lambda = 2$

n	Estimator	Average	Bias	MSE
20	MLE	2.1183	0.1183	0.1453
	Bayes (SELF)	2.0621	0.0621	0.0946
	Bayes (PLF)	2.1093	0.1093	0.1321
	E-Bayes (SELF)	2.047	0.047	0.082
	E-Bayes (PLF)	2.0841	0.0841	0.1182
50	MLE	2.0711	0.0711	0.0812
	Bayes (SELF)	2.0313	0.0313	0.0527
	Bayes (PLF)	2.0587	0.0587	0.0712
	E-Bayes (SELF)	2.0242	0.0242	0.0461
	E-Bayes (PLF)	2.0463	0.0463	0.0632
100	MLE	2.0397	0.0397	0.0411
	Bayes (SELF)	2.0181	0.0181	0.0273
	Bayes (PLF)	2.0342	0.0342	0.0365
	E-Bayes (SELF)	2.0145	0.0145	0.0231
	E-Bayes (PLF)	2.0281	0.0281	0.0314
200	MLE	2.0214	0.0214	0.0216
	Bayes (SELF)	2.0101	0.0101	0.0147
	Bayes (PLF)	2.0183	0.0183	0.0187
	E-Bayes (SELF)	2.0082	0.0082	0.0128
	E-Bayes (PLF)	2.0158	0.0158	0.0169

Table 2: Performance of estimators of $\lambda = 3$

n	Estimator	Average	Bias	MSE
20	MLE	3.1411	0.1411	0.1845
	Bayes (SELF)	3.0783	0.0783	0.1214
	Bayes (PLF)	3.1528	0.1528	0.1690
	E-Bayes (SELF)	3.0617	0.0617	0.1054
	E-Bayes (PLF)	3.1134	0.1134	0.1487
50	MLE	3.0830	0.0830	0.1026
	Bayes (SELF)	3.0411	0.0411	0.0689
	Bayes (PLF)	3.0892	0.0892	0.0897
	E-Bayes (SELF)	3.0340	0.0340	0.0601
	E-Bayes (PLF)	3.0711	0.0711	0.0787
100	MLE	3.0461	0.0461	0.0532
	Bayes (SELF)	3.0237	0.0237	0.0351
	Bayes (PLF)	3.0611	0.0611	0.0467
	E-Bayes (SELF)	3.0190	0.0190	0.0311
	E-Bayes (PLF)	3.0484	0.0484	0.0417
200	MLE	3.0251	0.0251	0.0288
	Bayes (SELF)	3.0127	0.0127	0.0195
	Bayes (PLF)	3.0415	0.0415	0.0257
	E-Bayes (SELF)	3.010	0.010	0.017
	E-Bayes (PLF)	3.0337	0.0337	0.0221

Table 3: Performance of estimators of $\lambda = 4$

n	Estimator	Average	Bias	MSE
20	MLE	4.1623	0.1623	0.2129
	Bayes (SELF)	4.0916	0.0916	0.1481
	Bayes (PLF)	4.1949	0.1949	0.1976
	E-Bayes (SELF)	4.0737	0.0737	0.1296
	E-Bayes (PLF)	4.1528	0.1528	0.1738
50	MLE	4.0935	0.0935	0.1162
	Bayes (SELF)	4.0446	0.0446	0.0767
	Bayes (PLF)	4.1112	0.1112	0.1015
	E-Bayes (SELF)	4.0367	0.0367	0.0681
	E-Bayes (PLF)	4.0871	0.0871	0.0891
100	MLE	4.0523	0.0523	0.0613
	Bayes (SELF)	4.0251	0.0251	0.0402
	Bayes (PLF)	4.0732	0.0732	0.0538
	E-Bayes (SELF)	4.0208	0.0208	0.0369
	E-Bayes (PLF)	4.0570	0.0570	0.0480
200	MLE	4.0280	0.0280	0.0329
	Bayes (SELF)	4.0136	0.0136	0.0217
	Bayes (PLF)	4.0523	0.0523	0.0288
	E-Bayes (SELF)	4.0112	0.0112	0.0191
	E-Bayes (PLF)	4.0414	0.0414	0.0254

5.3 Estimation of the Service Rate μ

Tables 4–6 summarize the simulation results for the estimation of μ corresponding to $\mu = 4, 5, 6$. The overall behavior of the estimators closely mirrors that observed for the arrival rate.

For small samples, the MLE again shows relatively high bias and MSE. Bayesian estimators under SELF achieve substantial efficiency gains by incorporating prior information, while PLF estimators provide conservative estimates that guard against underestimation of service capacity.

E-Bayesian estimators consistently yield the smallest MSE across almost all configurations, demonstrating their robustness to uncertainty in hyperparameter selection. This advantage is particularly pronounced when the service rate is large and the sample size is small, a scenario frequently encountered in practical service systems.

As the sample size increases, all estimators become nearly indistinguishable, confirming their asymptotic equivalence.

Table 4: Performance of estimators of $\mu = 4$

n	Estimator	Average	Bias	MSE
20	MLE	4.1423	0.1423	0.1980
	Bayes (SELF)	4.0817	0.0817	0.1367
	Bayes (PLF)	4.1586	0.1586	0.1813
	E-Bayes (SELF)	4.0633	0.0633	0.1188
	E-Bayes (PLF)	4.1299	0.1299	0.1622
50	MLE	4.0841	0.0841	0.1095
	Bayes (SELF)	4.0395	0.0395	0.0729
	Bayes (PLF)	4.0915	0.0915	0.0960
	E-Bayes (SELF)	4.0315	0.0315	0.0647
	E-Bayes (PLF)	4.0739	0.0739	0.0854
100	MLE	4.0469	0.0469	0.0567
	Bayes (SELF)	4.0227	0.0227	0.0370
	Bayes (PLF)	4.0592	0.0592	0.0491
	E-Bayes (SELF)	4.0184	0.0184	0.0334
	E-Bayes (PLF)	4.0475	0.0475	0.0436
200	MLE	4.0247	0.0247	0.0295
	Bayes (SELF)	4.0123	0.0123	0.0195
	Bayes (PLF)	4.0396	0.0396	0.0257
	E-Bayes (SELF)	4.0106	0.0106	0.0172
	E-Bayes (PLF)	4.0314	0.0314	0.0222

Table 5: Performance of estimators of $\mu = 5$

n	Estimator	Average	Bias	MSE
20	MLE	5.1589	0.1589	0.2248
	Bayes (SELF)	5.0960	0.0960	0.1543
	Bayes (PLF)	5.1915	0.1915	0.2069
	E-Bayes (SELF)	5.0799	0.0799	0.1379
	E-Bayes (PLF)	5.1525	0.1525	0.1869
50	MLE	5.0914	0.0914	0.1243
	Bayes (SELF)	5.0446	0.0446	0.0810
	Bayes (PLF)	5.1129	0.1129	0.1076
	E-Bayes (SELF)	5.0369	0.0369	0.0739
	E-Bayes (PLF)	5.0873	0.0873	0.0953
100	MLE	5.0516	0.0516	0.0643
	Bayes (SELF)	5.0252	0.0252	0.0429
	Bayes (PLF)	5.0730	0.0730	0.0569
	E-Bayes (SELF)	5.0215	0.0215	0.0381
	E-Bayes (PLF)	5.0578	0.0578	0.0502
200	MLE	5.0283	0.0283	0.0337
	Bayes (SELF)	5.0149	0.0149	0.0229
	Bayes (PLF)	5.0524	0.0524	0.0296
	E-Bayes (SELF)	5.0127	0.0127	0.0208
	E-Bayes (PLF)	5.0418	0.0418	0.0261

Table 6: Performance of estimators of $\mu = 6$

n	Estimator	Average	Bias	MSE
20	MLE	6.1919	0.1919	0.2649
	Bayes (SELF)	6.1122	0.1122	0.1861
	Bayes (PLF)	6.2221	0.2221	0.2410
	E-Bayes (SELF)	6.0948	0.0948	0.1670
	E-Bayes (PLF)	6.1783	0.1783	0.2194
50	MLE	6.1091	0.1091	0.1435
	Bayes (SELF)	6.0536	0.0536	0.0962
	Bayes (PLF)	6.1410	0.1410	0.1231
	E-Bayes (SELF)	6.0457	0.0457	0.0874
	E-Bayes (PLF)	6.1122	0.1122	0.1090
100	MLE	6.0615	0.0615	0.0752
	Bayes (SELF)	6.0293	0.0293	0.0513
	Bayes (PLF)	6.0899	0.0899	0.0670
	E-Bayes (SELF)	6.0251	0.0251	0.046 6
	E-Bayes (PLF)	6.0704	0.0704	0.0600
200	MLE	6.0339	0.0339	0.0394
	Bayes (SELF)	6.0167	0.0167	0.0269
	Bayes (PLF)	6.0715	0.0715	0.0357
	E-Bayes (SELF)	6.0149	0.0149	0.0242
	E-Bayes (PLF)	6.0561	0.0561	0.0316

6 Conclusion

This paper investigated Bayesian and E-Bayesian estimation of the arrival rate λ and service rate μ in an $M/M/1$ queueing system. Explicit estimators were derived under squared error and precautionary loss functions using conjugate Gamma priors, and uncertainty in prior hyperparameters was addressed through the E-Bayesian framework with uniform hyperpriors. From the Monte Carlo simulations we observed that Bayesian estimators perform better than the maximum likelihood estimator in terms of mean squared error, particularly for small and moderate samples. Estimators under the precautionary loss function were conservative, reducing the risk of underestimation in congestion-prone systems. E-Bayesian estimators consistently achieved the lowest mean squared error, demonstrating robustness to prior misspecification. As sample size increased, all estimators exhibited asymptotic consistency.

References

[1] Almeida, M. A. C. and Cruz, F. R. B. (2018). A note on Bayesian estimation of traffic intensity in single-server Markovian queues. Communications in Statistics - Simulation and Computation,47(9), 2577–2586.

[2] Armero, C. (1994a). Bayesian inference in Markovian queues. Queueing Systems 15, (1-4), 419–426.

[3] Armero, C. and Bayarri, M. J. (1994b). A Bayesian prediction in $M/M/1$ queues. Queueing system , 15, 401–417.

[4] Basak, A. and Choudhury, A. (2021). Bayesian inference and prediction in single server $M/M/1$ queuing model based on queue length. Communications in Statistics - Simulation and Computation, 50(6), 1576–1588.

[5] Basawa, I. V., and Prabhu, N. U.(1988). Large sample inference in single server queues. Queueing system , 3(4), 289–304.

[6] Basawa, I. V., and Prabhu, N. U.(1996). Maximum likelihood estimation for single server queues from waiting time data. Queueing Systems, 24(1-4), 155–167.

[7] Bhat, u. N., and Rao, S. S.(1987). Statistical analysis of queueing systems. Queueing Systems, 3(1), 217–247.

[8] Bura, G. S. and Vaish, Y. (2025). Classical and Bayesian estimates of traffic intensity for $M/M/1$ queue with state-dependent service. Communications in Statistics-Simulation and Computation, 54(12), 5322–5348.

[9] Chowdhury, S. and Mukherjee, S. P. (2011). Estimation of waiting time distribution in an $M/M/1$ queue. Opsearch, 48(4), 306–317.

[10] Clarke, A. B. (1957). Maximum likelihood estimates in a simple queue. The Annals of Mathematical Statistics, 28(4), 1036-1040.

[11] Deepthi, V., and Jose, J. K. (2021). Bayesian inference on $M/M/1$ queue under asymmetric loss function using Markov Chain Monte Carlo method. American Journal of Mathematical and Management Sciences, 24(5), 1003–1023.

[12] Gross, D., Shortle, J. F., Thompson, J. M., and Harris, C. M. (2009). Fundamentals of queueing theory. Wiley-Interscience, New York, NY, 4th edition.

[13] Han, M. (1997). The structure of hierachical prior distribution and its applications. Chinese Operations Research and Management Science, 6, 31–40.

[14] Han, M. (2007). E-Bayesian estimation of failure probability and its application. Mathematical and Computer Modeling, 45(9-10), 1272–79.

[15] Hendi, M. K., and Qomi, M. N. (2024). E-Bayesian estimation of traffic intensity in a $M/M/1$ system using expected posterior risk criteria. Electronic Journal of Applied Statistical Analysis, 17(03), 609–620.

[16] Hendi, M. K.,Qomi, M. N., and Singh, S. K. (2024). E-Bayesian and Hierarchical Bayesian Estimation of Traffic Intensity in the $M/M/1$ Queueing System. American journal of mathematical and management sciences, 43(3), 163–172.

[17] Kiapour, A. (2022). Bayesian estimation of the expected queue length of a system $M/M/1$ with certain and uncertain priors. Communications in statistics-Theory and Methods, 51(15), 5310–5317.

[18] Muddapur, M. V. (1972). Bayesian estimates of parameters in some queueing models. Annals of the Institute of Statistical Mathematics, 24(1), 327–331.

[19] Mukherjee, S. P., and Chowdhury, S. (2010). Bayes estimation of measures of effectiveness in an $M/M/1$ queueing model. Calcutta Statistical Association Bulletin, 62(1-2), 97–108.

[20] Norstrom, J. G. (1996). The use of precautionary loss functions in risk analysis. IEEE Transactions on Reliability, 45(3), 400–403.

[21] Schruben, L., and Kulkarni, R. (1982). Some consequences of estimating parameters for the $M/M/1$ queue. Operations Research Letters, 1(2), 75–78.

[22] Singh, S. K. and Acharya, S. K. (2019). Equivalence between Bayes and the maximum likelihood estimator in $M/M/1$ queue. Communications in Statistics - Theory and Methods, 48(19), 4780–4793.

[23] Singh, S. K., Cruz, F. R. B., Gomes, E. S. and Banik, A. D. (2024) Classical and Bayesian estimations of performance measures in a single server Markovian queueing system based on arrivals during service times. Communications in Statistics - Theory and Methods , 53(10), 3517–3546.

[24] Srinivas, V., and Udupa, H. J. (2014). Best unbiased estimation and can property in the stable $M/M/1$ queue. Communications in Statistics - Theory and Methods, 43(2), 321–327.

[25] Srinivas, V., Rao, S. S., and Kale, B. K. (2011). Estimation of measures in $M/M/1$ queue. Communication in Statistics - Theory and Methods, 40(18), 3327–3336.

[26] Zheng, S., and Seila, A. F. (2000). Some well-behaved estimators for the $M/M/1$ queue. Operations Research Letters, 26(5), 231–235.