Length Biased Burr- III Distribution for Modeling the Glass and Carbon Fiber Strength

Abstract

This paper introduces the length-biased Burr III (LBBIII) distribution, derived by applying a length-biased weighting scheme to the standard two-parameter Burr III distribution. The model is particularly suitable for lifetime and strength data, where the probability of observing an event is proportional to its magnitude. Most of its statistical and reliability properties were derived. Four estimation methods are examined and a simulation study evaluates the finite-sample performance of these estimators. The practical utility of the LBBIII distribution is demonstrated using two real datasets on glass and carbon fiber strength. The proposed distribution consistently outperforms the standard Burr-III and some well-known distributions. These findings establish the LBBIII distribution as a flexible and superior model for length-biased reliability and strength data.

Keywords: Burr III distribution; Length Biased; Survival Function; Order Statistics; Maximum Likelihood Method.

1. Introduction

The Burr III distribution, introduced by Burr (1942) as part of a system of cumulative frequency functions, represents a flexible family of distributions widely employed in reliability analysis, survival analysis, hydrology, and finance [1]. The distribution's probability density function (PDF) and cumulative distribution function (CDF) for a two-parameter Burr III specification are given by:

$$f(x) = \alpha\beta x^{-(\alpha+1)}(1+x^{-\alpha})^{-(\beta+1)}, \qquad x,\ \alpha,\ \beta > 0$$

$$F(x) = (1+x^{-\alpha})^{-\beta}$$

where α and β are shape parameters. It's known that, if a random variable X follows a Burr III distribution, then $1/X$ follows a Burr XII distribution, allowing results to be transferred between these formulations.

The flexibility of the Burr III distribution in accommodating various hazard rate shapes—including decreasing, increasing, and unimodal forms—has motivated extensive research into its generalizations. This survey synthesizes the major extensions of the Burr III distribution, organizing them by the methodological approach used to generate the extension.

Shao et al. [2] proposed an extension of the three-parameter Burr III distribution motivated by low-flow frequency analysis in water resources research. Their extension added an extra parameter to improve flexibility in modeling hydrological extremes. The authors evaluated three estimation methods – method of moments, probability-weighted moments (L-moments), and maximum likelihood estimation – and provided guidance on computational implementation. The extended distribution fit Australian low-flow data better than the standard formulation.

Cifarelli and colleagues [3] introduced the semi-Burr distribution as a generalization of the semi-Pareto distribution. Their work characterized Pareto type III distributions and developed random coefficient autoregressive models with semi-Burr marginal distributions. This framework connects Burr III distributions to time series applications, especially for minification processes.

Gomes et al. [4] developed the Beta Burr III distribution, applying the beta generator framework to the Burr III baseline. This approach, inspired by Eugene et al.'s [5] beta-normal distribution, adds two shape parameters that control the tails and overall flexibility. The resulting five-parameter model includes the standard Burr III as a special case and offers enhanced ability to model lifetime data with complex hazard behaviors.

Cordeiro et al. [6] introduced an extended Burr III model specifically designed for survival analysis and reliability applications. This extension can be expressed as a linear combination of Burr III distributions and offers tractable properties for ordinary and incomplete moments, generating functions, quantile functions, mean deviations, and reliability measures. The authors derived the density of order statistics as an infinite linear combination of Burr III densities and provided maximum likelihood estimation procedures with the observed information matrix. Real-data applications demonstrated the model's practical potential for reliability problems.

Ali et al. [7] proposed a modified Burr-III distribution, investigating its properties and applications. This modification focused on improving flexibility for modeling failure time data while keeping the parameterization simple.

Ali and Ahmad [8] introduced the transmuted modified Burr-III distribution, applying the quadratic rank transmutation map to add a skewness parameter. This extension allows the distribution to handle both positive and negative skewness beyond the basic Burr III specification, making it more useful in financial and environmental contexts where asymmetry is common.

Al-Saiari et al. [9] applied the Marshall-Olkin transformation to create the Marshall-Olkin extended Burr III distribution. This approach adds a tilt parameter that gives more flexibility in tail behavior and hazard rate shapes. Later, Haq and colleagues further developed the Marshall-Olkin modified Burr-III distribution, combining the Marshall-Olkin framework with modified Burr-III specifications.

Behairy et al. [10] proposed the Kumaraswamy-Burr type III distribution, applying the Kumaraswamy generator to the Burr III baseline. The Kumaraswamy distribution offers a simpler closed-form CDF and quantile function than the beta distribution. This extension inherits the flexibility of the Kumaraswamy family while remaining computationally tractable.

Jamal et al. [11] introduced the odd Burr-III family of distributions, a general framework where any baseline distribution can be transformed using the Burr III CDF as a generator. This family produces distributions with very flexible hazard rate shapes, including bathtub-shaped hazards. Alizadeh et al. [12] further developed the odd Burr generalized family, establishing various mathematical properties and showing real-data applications.

Handique et al. [13] proposed a three-parameter extended Burr-III distribution, studying its distributional properties, reliability characteristics, and parameter estimation. They used Monte Carlo simulations to see how well the estimation worked for different sample sizes and parameter values. The model was tested against other Burr III extensions using two real datasets, and it showed superior performance based on goodness-of-fit tests and model selection criteria.

Chakraborty et al. [14] developed a four-parameter extended Burr-III distribution, which they called a "simple extension" because of its parsimonious generation approach. Their results showed that this four-parameter extension outperformed all other Burr III extensions they compared it against when applied to a failure time dataset.

Most recently, Haq et al. [15] introduced a three-parameter unit probability distribution as a generalization of the Burr III distribution, specifically designed for data on the unit interval (like proportions, rates, or percentages). The unit modified Burr III is more flexible than existing unit distributions and can handle various PDF and hazard function shapes.

The reminder of this paper is organized as follows. The proposed distribution, namely the length biased BurrIII (LBBIII), was defined in Section 2. The statistical properties of LBBIII distribution are provided in Section 3. In Section 4, the order statistics are provided. The parameter estimation by using four estimation methods is considered in Section 5. Two simulation schemes are performed in Section 6. Two real datasets are used to examine the flexibility of LBBIII distribution in section 7.

2. Building the LBBIII Distribution

Let X be a random variable distributed follows the Burr III distribution, with parameters α and β, the density function (PDF) as follows,

$$g(x;\alpha,\beta) = \alpha\beta\, x^{-1-\alpha}(1+x^{-\alpha})^{-1-\beta} \tag{1}$$

where both α and β are shape parameters. The distribution function of X is:

$$G(x;\alpha,\beta) = (1+x^{-\alpha})^{-\beta}$$

The weighted distribution concept is:

$$f^{w}(x) = \frac{w(x)g(x)}{w}, \quad for\ x > 0 \tag{2}$$

where $w = \int_{-\infty}^{\infty} w\,(x)g(x)\,dy$ and $w(x) = x\ or\ x^{a}$ which called length biased.

Let X, denote the random variable with PDF (1) and let we choose the weighted function as $w(x) = x$. According to (2), the PDF of the Length Biased Burr -III distribution is

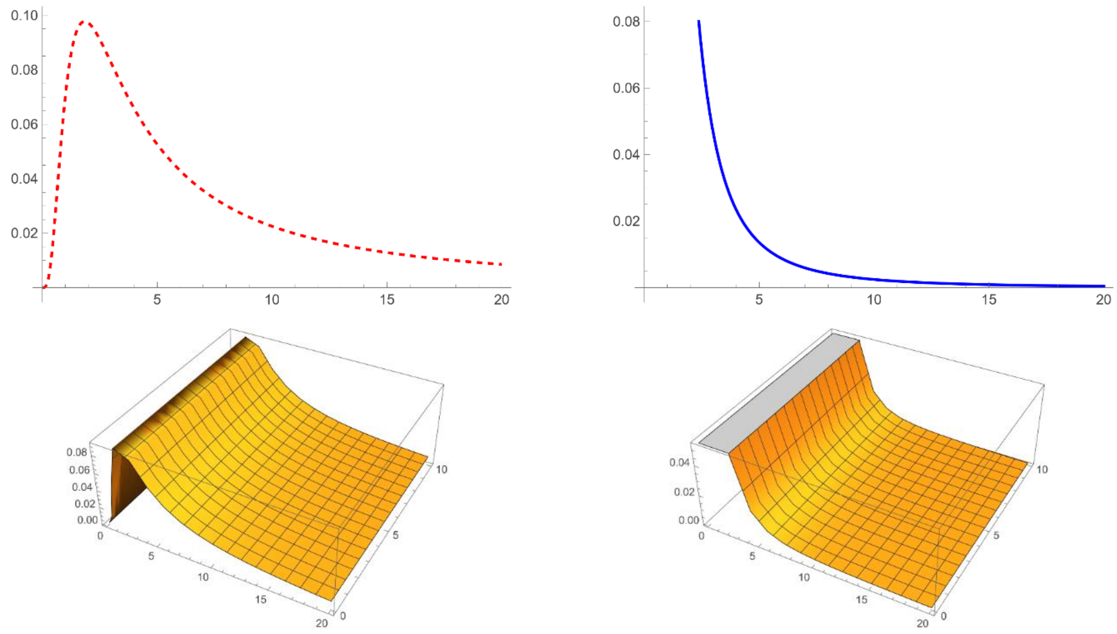
$$f(x;\alpha,\beta) = \frac{\Gamma[\beta]}{\Gamma\left[\frac{-1+\alpha}{\alpha}\right]\,\Gamma\left[\frac{1}{\alpha}+\beta\right]}\frac{(1+x^{-\alpha})^{-\beta}\alpha\beta}{(1+x^{\alpha})}; \qquad x,\beta > 0, \alpha > 1 \tag{3}$$

In addition, the CDF is:

$$F(x;\alpha,\beta) = 1 - \frac{x^{1-\alpha}\beta\,\Gamma[\beta]\ HG2F1R\left[\frac{-1+\alpha}{\alpha}, 1+\beta, 2-\frac{1}{\alpha}, -x^{-\alpha}\right]}{\Gamma\left[\frac{1}{\alpha}+\beta\right]}; \quad x > 0 \tag{4}$$

where, $HG2F1R$ is the Regularized Hypergeometric 2F1 function.

Figure 1 illustrates the behavior of the PDF of LBBIII distribution at different parameter values.



$\alpha = 1.5, \beta = 2.5$ $\qquad$ $\alpha = 2.5, \beta = 0.01$

Figure 1: The density function of LBBXII

3. Properties of the LBBIII distribution

In this section some properties of the LBBIII distribution will be derived.

3.1 The r-th Moment

Generally, the *r-th* moment of a continuous random variable X is given by:

$$E(x^r) = \int_0^{\infty} x^r \frac{\text{Gamma}[\beta]}{\text{Gamma}[\frac{-1+\alpha}{\alpha}]\text{Gamma}[\frac{1}{\alpha}+\beta]} \frac{(1+x^{-\alpha})^{-\beta}\alpha\beta}{(1+x^{\alpha})} dx$$

$$= \frac{\beta \,\text{Gamma}\left[\frac{-1-r+\alpha}{\alpha}\right]\text{Gamma}[\beta]\,\text{Gamma}[\frac{1+r+\alpha\beta}{\alpha}]}{\text{Gamma}\left[\frac{-1+\alpha}{\alpha}\right]\text{Gamma}[1+\beta]\,\text{Gamma}[\frac{1}{\alpha}+\beta]} \tag{5}$$

Hence, the first four moments can be derived easily as:

$$E(x) = \frac{4^{-1/\alpha}\,\Gamma[\frac{1}{2}-\frac{1}{\alpha}]\,\Gamma[\frac{2}{\alpha}+\beta]}{\sqrt{\pi}\,\Gamma[\frac{1}{\alpha}+\beta]};$$

$$E(x^2) = \frac{\Gamma[\frac{-3+\alpha}{\alpha}]\,\Gamma[\frac{3}{\alpha}+\beta]}{\Gamma[\frac{-1+\alpha}{\alpha}]\,\Gamma[\frac{1}{\alpha}+\beta]};$$

$$E(x^3) = \frac{\Gamma[\frac{-4+\alpha}{\alpha}]\,\Gamma[\frac{4}{\alpha}+\beta]}{\Gamma[\frac{-1+\alpha}{\alpha}]\,\Gamma[\frac{1}{\alpha}+\beta]};$$

$$and\;\; E(x^4) = \frac{\Gamma\left[\frac{-5+\alpha}{\alpha}\right]\Gamma[\frac{5}{\alpha}+\beta]}{\Gamma[\frac{-1+\alpha}{\alpha}]\,\Gamma[\frac{1}{\alpha}+\beta]}$$

3.2 The Quantile function

The quantile function is the mathematical inverse of the CDF. While the CDF gives the probability that a random variable will be less than or equal to a certain value x, the quantile function does the opposite: it gives the value x corresponds to a specific cumulative probability q. For a random variable with a CDF $F(x)$, the quantile function $Q(q)$ is defined for a probability q between 0 and 1 as:

$$Q(q) = F^{-1}(q)$$

When X follows the LBBIII distribution, its quantile function can be written as:

$$Q(q) = \left(\frac{I_q^{-1}\left(\frac{1}{\alpha}+\beta, 1-\frac{1}{\alpha}\right)}{1-I_q^{-1}\left(\frac{1}{\alpha}+\beta, 1-\frac{1}{\alpha}\right)}\right)^{\frac{1}{\alpha}}$$

where, $I_q^{-1}(a,b)$ is the quantile function of the $Beta(a,b)$ distribution.

3.3 The hazard function of LBBIII distribution

Generally, the survival function of a random variable X can be given by

$$S(x) = \frac{x^{1-\alpha}\beta\,\Gamma[\beta]\ \ HG2F1R\left[\frac{-1+\alpha}{\alpha},1+\beta,2-\frac{1}{\alpha},-x^{-\alpha}\right]}{\Gamma\left[\frac{1}{\alpha}+\beta\right]} \tag{7}$$

Then, the hazard function can be derived as:

$$h(x) = \frac{x^{-1+\alpha}\,(1+x^{-\alpha})^{-\beta}\alpha}{(1+x^{\alpha})\,\Gamma\left[\frac{-1+\alpha}{\alpha}\right]\,HG2F1R[\frac{-1+\alpha}{\alpha},1+\beta,2-\frac{1}{\alpha},-x^{-\alpha}]} \tag{8}$$

where, $HG2F1R$ is the Regularized Hypergeometric 2F1 function.

4. Order statistics of LBBIII distribution

Let $x_1, x_2, \ldots, x_n$ be a random sample drawn from LBBIII. As these are independent and identically distributed continuous random variables, the probability that any two or more of them are exactly equal is zero. Thus, the sample observations can be arranged in a unique order from smallest to largest.

Let $x_{(1:n)}, x_{(2:n)}, \ldots, x_{(n:n)}$ be the order statistics. Then the PDF of $x_{(i:n)}, 1 \le i \le n,$ denoted by $f_{i:n}(x)$ is given by

$$f_{i:n}(x) = C_{i:n}[F(x)]^{i-1}[S(x)]^{n-i} f(x)\,, \tag{9}$$

where, $C_{i:n} = n!/(i-1)!\,(n-i)!$.

The smallest observation in the sample denoted by $x_{(1:n)} = min(x_1, \ldots, x_n)$, the largest observation in the sample is $x_{(n:n)} = max(x_1, \ldots, x_n)$ and the median order as $x_{(m+1:n)}$, if $n = 2m+1$, thus the pdf of the smallest, largest and the median are given by

$$f_{1:n}(x;\alpha,\beta) = \frac{n(1+x^{-\alpha})^{-\beta}\alpha\beta\,\Gamma[\beta]\left(\frac{x^{1-\alpha}\beta\,\Gamma[\beta]HG2F1R\,[\frac{-1+\alpha}{\alpha},1+\beta,2-\frac{1}{\alpha},-x^{-\alpha}]}{\Gamma[\frac{1}{\alpha}+\beta]}\right)^{-1+n}}{(1+x^{\alpha})\,\Gamma[\frac{-1+\alpha}{\alpha}]\,\Gamma[\frac{1}{\alpha}+\beta]};$$

$$f_{n:n}(x;\alpha,\beta) = \frac{(1+x^{-\alpha})^{-\beta}\alpha\beta n!\,\Gamma[\beta]\left(1-\frac{x^{1-\alpha}\beta\,\Gamma[\beta]HG2F1R\,[\frac{-1+\alpha}{\alpha},1+\beta,2-\frac{1}{\alpha},-x^{-\alpha}]}{\Gamma[\frac{1}{\alpha}+\beta]}\right)^{-1+n}}{(1+x^{\alpha})\,\Gamma[n]\,\Gamma[\frac{-1+\alpha}{\alpha}]\,\Gamma[\frac{1}{\alpha}+\beta]};$$

and $F_{m+1:n}(x;\alpha,\beta) =$

$$\frac{\left((1+x^{-\alpha})^{-\beta}\alpha\beta n!\ \Gamma[\beta]\left(\frac{x^{1-\alpha}\beta G\ \Gamma[\beta]HG2F1R\ [\frac{-1+\alpha}{\alpha},1+\beta,2-\frac{1}{\alpha},-x^{-\alpha}]}{\Gamma[\frac{1}{\alpha}+\beta]}\right)^{-1-m+n}\left(1-\frac{x^{1-\alpha}\beta\ \Gamma[\beta]HG2F1R\ [\frac{-1+\alpha}{\alpha},1+\beta,2-\frac{1}{\alpha},-x^{-\alpha}]}{\Gamma[\frac{1}{\alpha}+\beta]}\right)^{m}\right)}{\left((1+x^{\alpha})\,\Gamma[1+m]\,\Gamma[-m+n]\,\Gamma[\frac{-1+\alpha}{\alpha}]\,\Gamma[\frac{1}{\alpha}+\beta]\right)}$$

5. Estimation methods

Once a parametric probability distribution is proposed, their parameters governing its shape, scale and location. Hence, these parameters should be estimated from the available sample. Estimation methods transform sample information into numerical values of unknown parameters. In this paper, four estimation methods were used to estimate the two parameters α

and β of the LBBIII distribution, namely, maximum likelihood estimation, least squares, Cramér-Von Mises and weighted least squares methods.

5.1 Maximum Likelihood Estimation (MLE)

Let $x_1, x_2, \dots, x_n$ be a random sample of size n from the LBBIII distribution, the log-likelihood function can be written as follows:

$$\ell(x;\alpha,\beta) = n\,ln(\Gamma[\beta]) + n\,ln(\alpha) + n\ \ ln(\beta) - n\,ln\left(\Gamma\left[\frac{-1+\alpha}{\alpha}\right]\right) - n\,ln\left(\Gamma\left[\frac{1}{\alpha}+\beta\right]\right)$$
$$+ \sum_{i=1}^{n} ln(1+x^{-\alpha})^{-\beta} - \sum_{i=1}^{n} ln(1+x^{\alpha}) \qquad (10)$$

Differentiating the log-likelihood function with respect to α and β we have:

$$\frac{\partial\ell}{\partial\alpha} = \frac{n}{\alpha^2}\left(-H[-\tfrac{1}{\alpha}] + H[-1+\tfrac{1}{\alpha}+\beta] + \frac{\alpha\left(1+x^{\alpha}+\alpha\ln(x)\left(-x^{\alpha}+\beta\,ln[1+x^{-\alpha}]^{-1-\beta}\right)\right)}{1+x^{\alpha}}\right) \qquad (11)$$

where $H[n]$ is the $n-th$ harmonic number, defined as the sum of the reciprocals of the first positive integers, that is $H[n] = \sum_{i=1}^{n}\frac{1}{k}$.

$$\frac{\partial\ell}{\partial\beta} = n\left(\tfrac{1}{\beta} - ln[1+x^{-\alpha}]^{-\beta}\,ln[ln[1+x^{-\alpha}]] + \psi[0,\beta] - \psi[0,\tfrac{1}{\alpha}+\beta]\right) \qquad (12)$$

where $\psi[z]$ is the digamma function which is defined as the logarithmic derivative of the Gamma function.

5.2 least Squares Estimation (LSE)

While the standard "Ordinary Least Squares" is used to find the relationship between two variables, the least squares method used to find the relationship between the ordered sample data and their theoretical positions. Given an ordered sample $x_{1:n} < x_{2:n} < \cdots < x_{n:n}$, the least squares estimator is obtained by minimizing the sum of squared differences between the empirical and theoretical distribution functions as follows:

$$V(x_i) = \sum_{i=1}^{n}\left(F(x_{i:n}) - \frac{i}{n+1}\right)^2$$

where $F(x_{i:n})$ is the CDF of the LBBIII distribution.

5.3 Cramér-Von Mises (CVM)

Cramér-Von Mises method seeks to minimize the distance between the Empirical Distribution Function (EDF) and the CDF of the model. Given an ordered sample $x_{1:n} < x_{2:n} < \cdots < x_{n:n}$, the Cramér-von Mises estimator is obtained by minimizing the following objective function with respect to the parameters:

$$W^2(x_i) = \frac{1}{12n} + \sum_{i=1}^{n}\left(F(x_{i:n}) - \frac{2i-1}{2n}\right)^2,$$

where $\frac{2i-1}{2n}$ represents the "middle" probability point of the i-th observation in an ideal distribution. The estimates are the values of α and β that give the smallest possible value for W^2.

5.4 Weighted Least Squares (WLS)

The method of weighted least squares (WLS) is an extension of the ordinary least squares (OLS) method. It solves the main limitation of OLS: the assumption that all data points are equally reliable. In reality, some observations provide more information than others. When estimating a distribution, observations near the center of the sample are usually more stable than those in the extreme tails. WLS addresses this by giving each observation a weight that reflects its reliability. In the WLS we need to minimize:

$$W(x_i) = \sum_{i=1}^{n} w_i\left[F(x_{i:n}) - \frac{i}{n+1}\right]^2$$

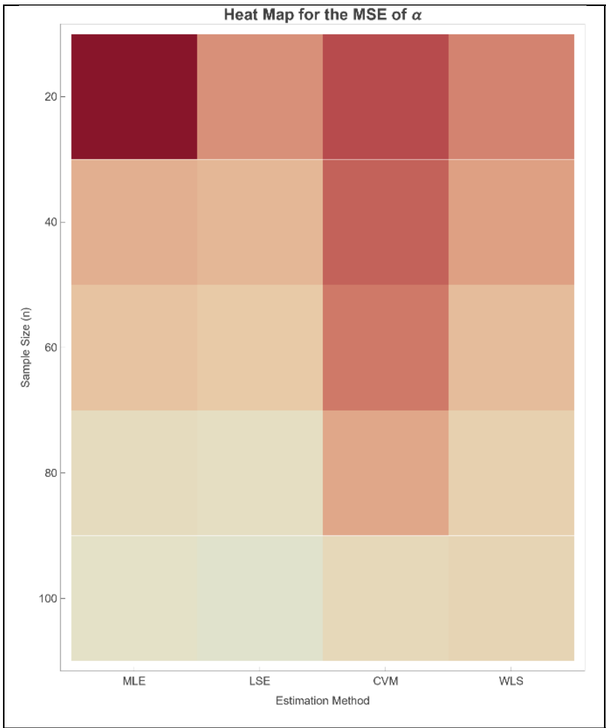
where, w_i is the weight assigned to the i-th ordered observation. Choosing what the weights should be is the main point in this method. The common strategy is to choose $w_i = \frac{(n+1)^2(n+2)}{i(n-i+1)}$ based on the variance of empirical distribution. In this case, observations with low variance (usually those in the middle of the sample as they close to the mean) get higher weights.

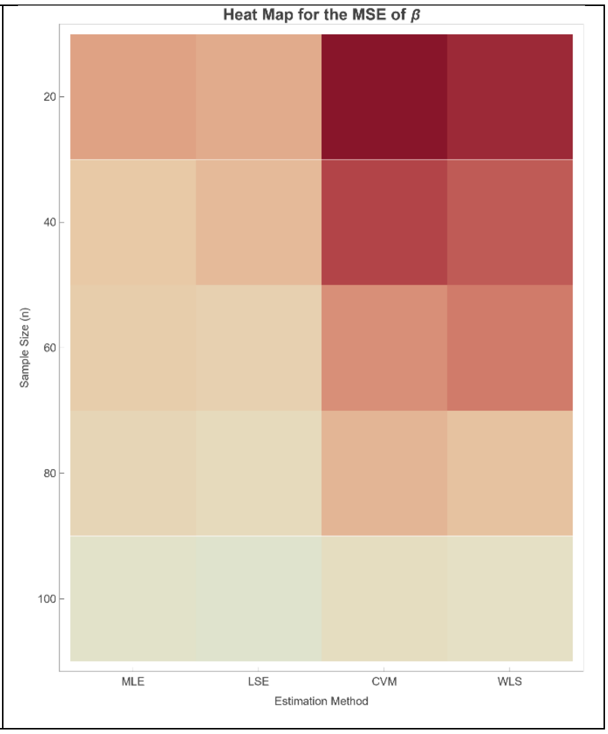
6. simulation study

The simulation study presented in Tables 1 and 2 was designed to evaluate the finite-sample performance of four competing estimators—maximum likelihood (MLE), least squares (LSE), Cramér–von Mises (CVM), and weighted least squares (WLS)—for estimating the two shape parameters, α and β, of the proposed length-biased Burr III distribution. Two distinct parameter configurations were considered: the first scheme set α = 1.5 and β = 0.5, while the second increased both parameters to α = 2.5 and β = 1.5. Sample sizes ranged from as small as 20 up to 100 observations. Performance was assessed using mean squared error (MSE) and average bias, with lower values on both metrics indicating better estimation quality.

Table 1: MSE and Average bias of the estimates at $\alpha = 1.5$ and $\beta = 0.5$

	MSE α				MSE β			
n	**MLE**	**LSE**	**CVM**	**WLS**	**MLE**	**LSE**	**CVM**	**WLS**
20	0.0336	0.0100	0.0198	0.0116	0.1543	0.1333	0.7097	0.6166
40	0.0066	0.0059	0.0162	0.0079	0.0725	0.0986	0.4919	0.4126
60	0.0052	0.0048	0.0125	0.0058	0.0638	0.0635	0.2166	0.2866
80	0.0025	0.0024	0.0074	0.0032	0.0490	0.0305	0.1004	0.0834
100	0.0021	0.0012	0.0026	0.0029	0.0150	0.0141	0.0291	0.0283
	Bias α				**Bias β**			
n	**MLE**	**LSE**	**CVM**	**WLS**	**MLE**	**LSE**	**CVM**	**WLS**
20	0.1046	0.0256	0.0735	0.0874	0.2973	0.1156	0.4366	0.2412
40	0.0335	0.0214	0.0499	0.0382	0.1374	0.0431	0.2143	0.1297
60	0.0287	0.0194	0.0321	0.0145	0.0514	0.0225	0.1926	0.0999
80	0.0055	0.0170	0.0123	0.0092	0.0412	0.0167	0.0934	0.0758
100	0.0052	0.0125	0.0058	0.0031	0.0156	0.0126	0.0324	0.0482





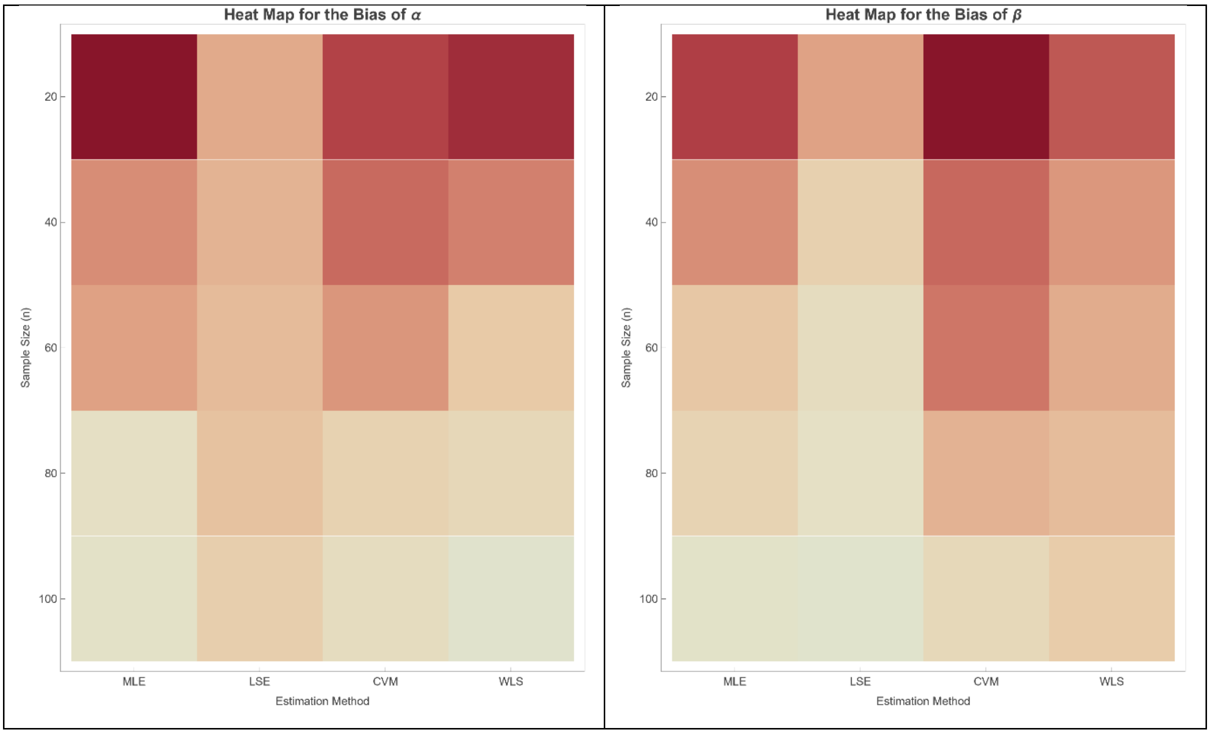


Figure 2: Graphical representations of the MSE and the Average Bias in Table 1

In Table 1, where the true parameters are relatively small, the results for estimating α show that the maximum likelihood estimator consistently delivers the lowest MSE across all sample sizes. At n = 20, the MSE for MLE is 0.0336, and this drops steadily to just 0.0021 at n = 100. This behavior aligns perfectly with the expected asymptotic efficiency of MLE. The least squares estimator performs nearly as well, especially once the sample size reaches 40 or more; at n = 40, the MSE for LSE is 0.0059 compared to MLE's 0.0066. In contrast, the Cramér–von Mises estimator shows noticeably higher MSE for small samples, with a value of 0.0198 at n = 20, although it improves considerably as n increases, reaching 0.0026 at n = 100. Weighted least squares sit somewhere between LSE and CVM but never outperform MLE or LSE. Overall, for α, MLE proves to be the most efficient, followed closely by LSE, while CVM requires larger samples to become truly competitive.

For β, MLE again dominates in terms of MSE, dropping from 0.1543 at $n = 20$ to 0.0150 at $n = 100$. The least squares estimator is also quite competitive, particularly at the larger sample sizes of 80 and 100. However, both CVM and WLS perform poorly for small to moderate samples; for example, the MSE for CVM at $n = 20$ is a striking 0.7097, indicating enormous variability. Even at $n = 100$, CVM and WLS still lag noticeably behind MLE and LSE. This suggests that β is inherently more difficult to estimate than α, especially for method-of-moments-type estimators, and that CVM and WLS should be used with great caution, if at all, when the sample size is limited.

Examining bias in Table 1 reveals another interesting pattern. For α, the least squares estimator exhibits the smallest bias overall, with a value of 0.0256 at $n = 20$ and only 0.0125 at $n = 100$. Maximum likelihood bias is somewhat larger at small sample sizes 0.1046 at $n = 20$ but decays rapidly as n increases, reaching just 0.0052 at $n = 100$. CVM and WLS show moderate bias that becomes acceptable only at the largest sample sizes. For β, the story is similar but more dramatic: LSE again achieves the lowest bias across the board, while MLE bias is larger but still decays nicely. The CVM estimator, however, displays unacceptably high bias for small samples, reaching 0.4366 at $n = 20$, which indicates a systematic tendency to overestimate β. Taken together, these results suggest that LSE provides the most unbiased estimates for both parameters, whereas MLE offers the best overall trade-off between bias and variance.

Table 2: MSE and Average bias of the estimates at $\alpha = 2.5$ and $\beta = 1.5$

	MSE α				**MSE β**			
n	**MLE**	**LSE**	**CVM**	**WLS**	**MLE**	**LSE**	**CVM**	**WLS**
20	0.0412	0.0125	0.0211	0.0134	0.1685	0.1452	0.7254	0.6341
40	0.0084	0.0068	0.0185	0.0091	0.0813	0.1024	0.512	0.4328
60	0.0063	0.0051	0.0142	0.0064	0.0721	0.0711	0.2355	0.2982
80	0.0031	0.0029	0.0089	0.0042	0.0544	0.0382	0.1214	0.0921
100	0.0025	0.0016	0.0031	0.0035	0.0182	0.0175	0.0346	0.0315
	Bias α				**Bias β**			
n	**MLE**	**LSE**	**CVM**	**WLS**	**MLE**	**LSE**	**CVM**	**WLS**
20	0.1124	0.0284	0.0782	0.0912	0.3125	0.1248	0.4582	0.2634
40	0.0382	0.0231	0.0524	0.0415	0.1456	0.0512	0.2314	0.1425
60	0.0315	0.0205	0.0361	0.0182	0.0624	0.0284	0.2085	0.1124
80	0.0072	0.0182	0.0154	0.0114	0.0482	0.0201	0.1042	0.0821
100	0.0068	0.0141	0.0071	0.0048	0.0195	0.0154	0.0415	0.0521

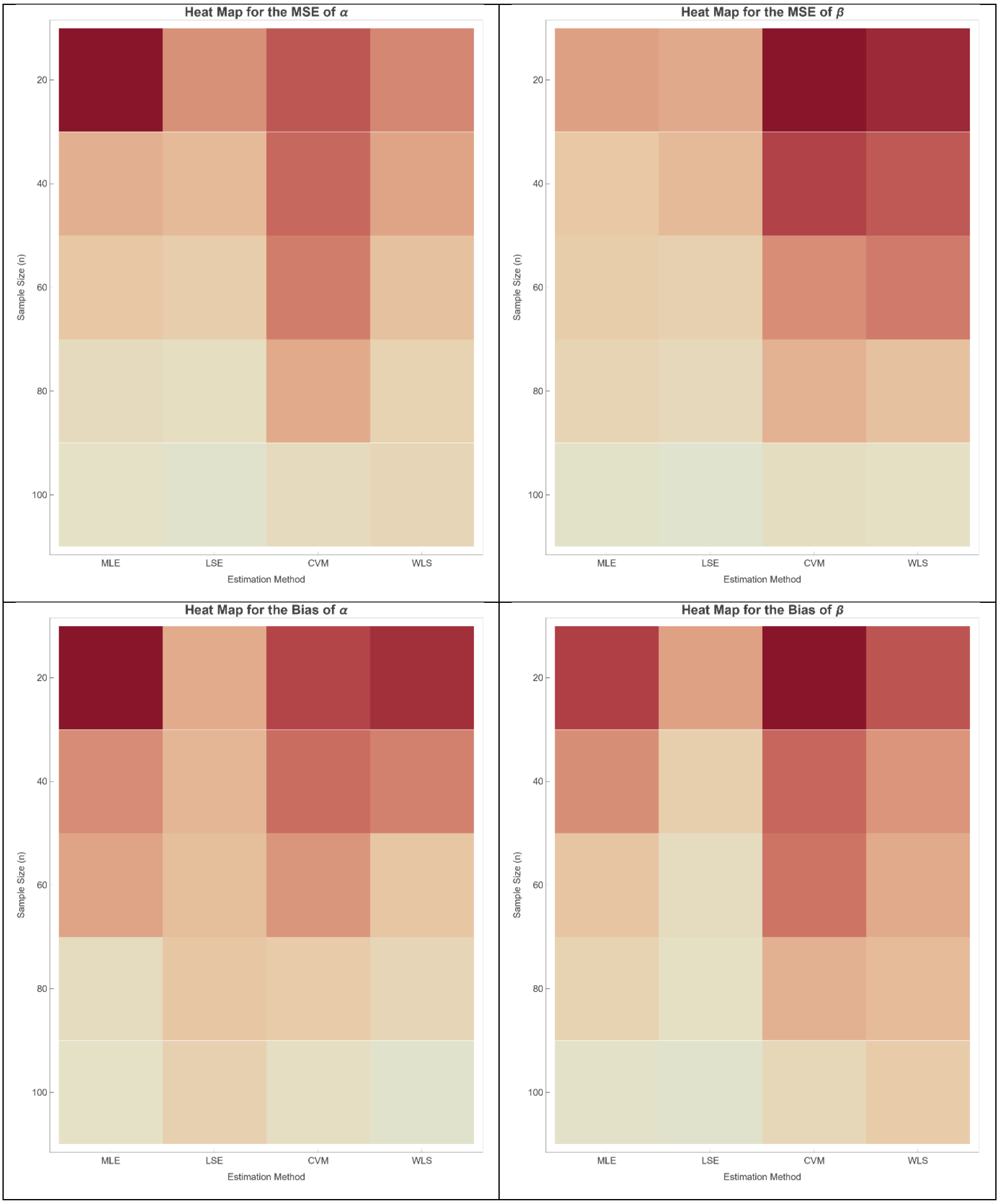


Figure 3: Graphical representations of the MSE and the Average Bias in Table 2

In Table 2, where the true parameters are larger ($\alpha = 2.5$ and $\beta = 1.5$), the overall rankings of the estimators remain unchanged, which is a reassuring sign of robustness. However, the numerical values of MSE and bias are generally slightly higher across all estimators compared to Table 1. For example, the MSE of MLE for α at $n = 20$ is 0.0412 in Table 2 versus 0.0336 in Table 1. This increase is not surprising from a theoretical perspective, because larger true parameter values often lead to higher sampling variability. Similarly, bias magnitudes are also a bit larger; the MLE bias for β at $n = 20$ is 0.3125 in Table 2 compared to 0.2973 previously. The CVM estimator again struggles with β in small samples, showing a bias of 0.4582 at $n = 20$. These findings confirm that while estimation becomes marginally more challenging when the parameters are larger, the relative strengths and weaknesses of each method remain consistent.

The graphical representations referenced in Figures 2 and 3 would almost certainly reinforce these conclusions. Darker cells, indicating lower MSE and bias, would be expected to appear concentrated in the MLE and LSE columns, while CVM and WLS would appear lighter, especially for small sample sizes and particularly for β. As sample size increases, progressive darkening would be observed across all estimators, confirming their consistency. In short, the figures would visually communicate what the tables show numerically: MLE and LSE are the clear winners, with CVM and WLS only becoming acceptable when samples are large.

Based on these findings, if the primary goal is to minimize mean squared error—for instance, when constructing prediction intervals—then maximum likelihood estimation is the clear choice, as it consistently delivers the lowest MSE for both parameters across all sample sizes. If instead the main concern is to minimize bias, for example in hypothesis testing or when unbiasedness is a regulatory requirement, then the least squares estimator is preferable, particularly for small to moderate samples. For large samples of 80 or more observations, both MLE and LSE perform excellently, and even CVM and WLS become acceptable, though they never surpass the top two. For small samples of 40 or fewer, however, CVM and WLS should be avoided when estimating β, as their MSE and bias can be unacceptably high. Finally, if computational simplicity is a priority, LSE has the advantage of requiring no iterative numerical optimization, unlike MLE.

In conclusion, this simulation study convincingly demonstrates that for the length-biased Burr III distribution, the maximum likelihood estimator is the most efficient in terms of MSE, while the least squares estimator is the most unbiased. Both perform reliably across sample sizes, whereas the Cramér–von Mises and weighted least squares estimators are not recommended for small samples, especially for estimating β. The results are robust to changes in the true parameter values, as seen by comparing the two tables. These findings should provide clear and actionable guidance for anyone modeling length-biased lifetime data using the LBBIII distribution.

7. Applications

In this section, two real data sets illustrate that the LBBIII distribution might fit better than a model based on the B-III distribution and other known distributions .

Dataset 1.

The first dataset shows the strength of 1.5 cm glass fibers, measured at the National Physical Laboratory in England.

list 1- Dataset showing better performance of LBBIII distribution model

0.55	0.93	1.25	1.36	1.49	1.52	1.58	1.61	1.64	1.68	1.73	1.81	2
0.74	1.04	1.27	1.39	1.49	1.53	1.59	1.61	1.66	1.68	1.76	1.82	2.01
0.77	1.11	1.28	1.42	1.5	1.54	1.6	1.62	1.66	1.69	1.76	1.84	2.24
0.81	1.13	1.29	1.48	1.5	1.55	1.61	1.62	1.66	1.7	1.77	1.84	1.84
1.24	1.3	1.48	1.51	1.55	1.61	1.63	1.67	1.7	1.78	1.89		

The real-world data from Smith and Naylor [16] is used to assess how well the LBBIII distribution performs empirically. Its adaptability is examined by comparing it with four other models, including the standard Burr-III distribution. Several statistical measures are applied in the analysis: the Log-likelihood $(-\hat{\ell})$ to evaluate overall fit, information criteria (AIC, BIC, AICC, CAIC) to adjust for model complexity, and goodness-of-fit tests (Kolmogorov-Smirnov, Anderson-Darling, Cramér-von Mises and Watson) to assess distributional accuracy. Detailed findings are presented in Tables 1 and 2, where better performance is indicated by the smallest values across these metrics.

Table 3: Information criteria for dataset 1.

Distribution	$\hat{\alpha}$	$\hat{\beta}$	**AIC**	**BIC**	**AICC**	**HQIC**	**CAIC**
LBBIII	4.628	2.713	81.178	85.464	81.378	82.863	81.378
BIII	3.613	1.479	110.399	114.685	110.599	112.084	110.599
Exponential	1.045	0.550	130.860	135.146	131.060	132.546	131.060
Pareto	0.550	1.022	175.328	179.614	175.528	177.013	175.528
Fréchet	3.712	1.253	111.719	116.006	111.919	113.405	111.919

Table 4: Goodness-of-fit tests for dataset 1.

Distribution	KS	AD	CVM	W	$-\hat{\ell}$
LBBIII	0.252	5.512	1.034	0.953	38.589
BIII	0.439	19.248	4.166	1.730	53.199
Exponential	0.371	13.610	2.753	2.013	63.430
Pareto	0.421	17.028	3.581	2.720	85.664
Fréchet	0.298	7.419	1.397	1.091	53.860

The goodness-of-fit outcomes shown in Tables 3 and 4 indicate that the LBBIII distribution consistently yields the lowest goodness-of-fit values among all models tested, meaning it most closely matches the empirical data distribution. This finding establishes the LBBIII's enhanced fit quality compared to the baseline Burr-III model as well as other alternative distributions, providing robust statistical support for its use in analogous contexts.

Dataset 2.

The tensile strength of 69 carbon fibers, measured in gigapascals (GPa), is provided in the following dataset. These fibers were tested under tension at a gauge length of 20 mm, as reported in [17].

list 2- Dataset showing superior performance of LBBIII distribution model

1.312	1.479	1.479	1.552	1.700	1.803	1.861	1.865	1.944	1.958	1.966	1.997
2.006	2.027	2.027	2.055	2.063	2.098	2.140	2.179	2.224	2.240	2.253	2.270
2.272	2.301	2.301	2.301	2.359	2.382	2.382	2.426	2.434	2.435	2.478	2.490
2.511	2.535	2.535	2.554	2.566	2.570	2.586	2.629	2.633	2.642	2.648	2.684
2.697	2.770	2.770	2.773	2.800	2.809	2.818	2.821	2.848	2.880	2.954	3.012
3.067	3.084	3.090	3.096	3.128	3.233	3.433	3.585	3.858			

The dataset's performance is evaluated through comparisons with four alternative models, including the standard Burr-III distribution. The analysis applies several statistical measures: the Log-likelihood $(-\hat{\ell})$ for assessing overall fit; information criteria (AIC, BIC, AICC, CAIC) to account for model complexity; and goodness-of-fit tests (Kolmogorov–Smirnov, Anderson–Darling, Cramér–von Mises, and Watson) to evaluate distributional accuracy. Detailed results are shown in Tables 3 and 4, where superior performance is indicated by the lowest values.

Table 5: Information criteria for dataset 2.

Distribution	$\hat{\alpha}$	$\hat{\beta}$	AIC	BIC	AICC	HQIC	CAIC
LBBIII	4.940	33.716	129.144	133.612	129.326	130.917	129.326
BIII	3.613	14.512	134.402	138.870	134.584	136.175	134.584
Exponential	0.872	1.312	161.179	165.648	161.361	162.952	161.361
Pareto	1.312	1.648	194.320	198.788	194.502	196.093	194.502
Fréchet	5.423	2.158	138.570	143.038	138.751	140.342	138.751

Table 6: Goodness-of-fit tests for dataset 2.

Distribution	KS	AD	CVM	W	$-\hat{\ell}$
LBBIII	0.131	2.505	0.387	0.364	62.572
BIII	0.168	4.123	0.726	0.642	65.201
Exponential	0.308	10.180	1.988	1.342	78.590
Pareto	0.361	13.450	2.773	1.969	95.160
Fréchet	0.139	2.604	0.389	0.246	67.285

As shown in Tables 5 and 6, the LBBIII distribution consistently produces the smallest goodness-of-fit values among all tested models, indicating that it aligns most closely with the observed data distribution. This result confirms that the LBBIII offers a superior fit compared to both the standard Burr-III model and the other alternative distributions, offering strong statistical justification for its application in similar settings.

Conclusion

This paper introduced the length-biased Burr III (LBBIII) distribution and derived its main statistical properties. A simulation study compared four estimation methods: maximum likelihood, least squares, Cramér–von Mises and weighted least squares. The results show that maximum likelihood estimation provides the lowest mean squared error, while least squares estimation yields the smallest bias. The practical value of the LBBIII distribution was confirmed using two real datasets on glass and carbon fiber strength. In both cases, the LBBIII distribution fitted the data better than the standard Burr III, exponential, Pareto, and Fréchet distributions. These findings support the LBBIII distribution as a flexible and effective model for length-biased reliability and strength data.

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