

A Mathematical and Computational Study of Melodic Structures in North Indian Classical Music using Raga-Restricted Operations

Abstract

We present a structured mathematical and computational framework for analyzing melodic structures in North Indian classical music by introducing a novel concept termed the *raga-restricted operation*, which models rule-based note transitions under raga grammar constraints. The framework integrates algebraic structures, graph-theoretic representation, and stochastic modeling via Markov chains to provide a unified approach to melodic analysis. It is applied to two representative ragas, Raga Yaman and Raga Bhupali, where generated melodic sequences, directed graphs, transition probability matrices, and stationary distributions are examined to capture both local transition dynamics and long-term structural behavior. The results demonstrate that the proposed model preserves the intrinsic grammatical rules of ragas while yielding a mathematically rigorous and computationally interpretable description of melodic evolution. Consequently, the framework establishes a systematic link between music theory and mathematical modelling, offering new insights into probabilistic structure, note dependency, and long-term behavior in raga-based systems.

Keywords: Raga-restricted operation, Indian classical music, Markov chains, transition probability matrix, stationary distribution, graph theory, computational musicology.

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1 Introduction

Mathematics and music have long been interconnected through the presence of patterns, structures, and formal relationships inherent in musical compositions. Early investigations revealed that musical sounds can be described using numerical ratios and

harmonic relationships, while later developments introduced algebraic and structural frameworks to analyze musical systems [9–11]. In addition, probabilistic and signal-processing approaches have been widely employed to study tonal structure and sequential behavior in music [7,12].

In Western music theory, well-defined constructs such as symmetry, transformations, and ordered tonal systems provide a structured foundation for mathematical analysis [10,12]. These frameworks enable the application of algebraic methods and computational techniques for modelling musical compositions. Furthermore, modern research has incorporated statistical and computational models, including Markov chain approaches, to analyze note transitions and dependency structures in musical sequences [2,14,15].

In contrast, North Indian classical music exhibits a fundamentally different structure. It is governed by the concept of a *raga*, which defines a rule-based yet flexible system of permissible notes, characteristic phrases, and constrained transitions. The melodic progression depends not only on the set of allowed notes but also on their ordering, ascending (*aroha*) and descending (*avaroha*) patterns, and contextual usage within performance [13]. Due to these layered constraints, conventional algebraic or statistical models are often insufficient to fully capture the structural richness and contextual dependencies inherent in raga-based systems [3,6].

To address these challenges, several recent works have explored computational and probabilistic approaches to modelling raga structures. In particular, Markov chain models have been employed to represent note transitions and capture statistical dependencies in melodic sequences [1, 2, 5]. Graph-based representations and transition probability matrices have also been used to analyze structural properties and transition dynamics in raga systems [4,6]. Moreover, category-theoretic and abstract frameworks have been explored to provide deeper structural insights into musical systems [8].

Despite these advances, existing approaches are primarily statistical or computational in nature and often lack a unified algebraic formulation that explicitly incorporates the grammatical constraints of ragas. In particular, the absence of a mathematically defined operation governing permissible transitions limits the ability to systematically integrate algebraic, graphical, and probabilistic perspectives within a single framework.

Motivated by these observations, the present study proposes a comprehensive mathematical framework that integrates algebraic structures, graph theory, and stochastic processes to analyze melodic systems in North Indian classical music. A novel concept, termed the *raga-restricted operation*, is introduced to model permissible note transitions while explicitly excluding forbidden movements. This operation provides an algebraic foundation for representing raga grammar. Furthermore, melody sequence generation, directed graph representation of note transitions, and transition probability analysis using Markov models are employed to investigate both local transition behavior and global structural properties.

The proposed framework is applied to two representative ragas, Raga Yaman and Raga Bhupali, which differ significantly in tonal structure and complexity. Through this analysis, the study aims to provide a clear, systematic, and mathematically rigorous understanding of melodic behavior in raga-based systems, thereby establishing a meaningful connection between music theory, probability, and modern mathematical modeling.

2 Preliminaries

In this section, we recall the fundamental mathematical and musical concepts required for the development of the proposed framework.

Definition 2.1 (Set). *A set is a well-defined collection of distinct objects, called elements. Mathematically, a set is represented as*

$$S = \{x \mid x \text{ satisfies a given property}\}.$$

Definition 2.2 (Binary Operation). *Let S be a non-empty set. A binary operation on S is a mapping*

$$* : S \times S \rightarrow S.$$

Definition 2.3 (Group [9,10]). *A group is an ordered pair $(G, *)$ where G is a non-empty set and $*$ is a binary operation satisfying:*

- *Closure: $a * b \in G$*
- *Associativity: $(a * b) * c = a * (b * c)$*
- *Identity: $\exists e \in G$ such that $a * e = e * a = a$*
- *Inverse: $\forall a \in G, \exists a^{-1}$ such that $a * a^{-1} = e$*

Definition 2.4 (Semigroup and Monoid). *A semigroup is a non-empty set $(G, *)$ satisfying closure and associativity. A monoid is a semigroup with an identity element.*

Definition 2.5 (Transformation). *Let S be a set. A transformation on S is a function*

$$T : S \rightarrow S.$$

A transformation is called a symmetry if it is bijective and preserves structure.

Definition 2.6 (Function). *A function f from a set A to a set B is a mapping*

$$f : A \rightarrow B$$

such that each element of A is assigned exactly one element of B .

Definition 2.7 (Relation). *A relation between two sets A and B is a subset of their Cartesian product:*

$$R \subseteq A \times B.$$

If $(a, b) \in R$, then a is related to b .

Definition 2.8 (Graph [4]). *A graph is an ordered pair*

$$G = (V, E),$$

where V is the set of vertices and $E \subseteq V \times V$ is the set of edges.

Definition 2.9 (Probability [12]). Probability measures the likelihood of occurrence of an event A and is defined as

$$P(A) = \frac{\text{Number of favourable outcomes}}{\text{Total number of outcomes}}, \quad 0 \leq P(A) \leq 1.$$

For mutually exclusive events A and B ,

$$P(A \cup B) = P(A) + P(B).$$

Definition 2.10 (Swar (Musical Note) [13]). A swar is the fundamental unit of sound in Indian classical music. The basic set of swars is

$$S = \{Sa, Re, Ga, Ma, Pa, Dha, Ni\}.$$

Definition 2.11 (Raga [13]). A raga is a structured and rule-based subset of swars used for musical expression. Mathematically,

$$R \subseteq S.$$

Definition 2.12 (Aroha–Avroha). Aroha represents the ascending sequence of swars and Avroha represents the descending sequence:

$$A = (s_1, s_2, \dots, s_n), \quad D = (s_n, \dots, s_2, s_1).$$

Definition 2.13 (Note Dependency). In a raga, the occurrence of a note depends on the previous note. This dependency can be expressed as:

$$.P(s_{i+1} | s_j)$$

Definition 2.14 (Permissible Notes). Let S be the set of all notes. A subset $R \subseteq S$ represents permissible notes of a raga.

Definition 2.15 (Forbidden Notes (Varjit Swars)). Let $R \subseteq S$. The set of forbidden notes is defined as

$$F = S \setminus R.$$

Definition 2.16 (Improvisation in Raga). A melodic system can be represented as

$$M = (R, T),$$

where $T \subseteq R \times R$ is the set of permissible transitions. Improvisation must satisfy

$$(s_i, s_j) \in T.$$

Definition 2.17 (Raga Grammar). A raga can be represented as a structured system:

$$R = (R, A, D, T),$$

where

- $R \subseteq S$ is the set of permissible notes,
- A is the ascending sequence (Aroha),
- D is the descending sequence (Avroha),
- $T \subseteq R \times R$ is the set of permissible transitions.

Definition 2.18 (Markov Chain [2,3]). A Markov chain is a stochastic process in which the probability of the next state depends only on the current state:

$$P(s_{n+1} | s_n, s_{n-1}, \dots) = P(s_{n+1} | s_n)$$

The transition probabilities are represented by a matrix

$$P = [P_{ij}]$$

Where $p_{ij} = P(s_j | s_i)$

Definition 2.19 (Transition Probability Matrix [1]). The transition probability matrix (TPM) is defined as

$$p_{ij} \geq 0, \sum_j p_{ij} = 1 \text{ for all } i.$$

Thus, P is a stochastic matrix representing transition behavior.

3 Methodology

This study develops a mathematical framework for modelling melodic structures in North Indian classical music by integrating algebraic operations, graph theory, and stochastic processes. The objective is to investigate whether musical systems, particularly raga-based systems, can be represented through consistent algebraic and probabilistic structures.

Let the musical note space be denoted by

$$S = \{s_1, s_2, \dots, s_n\}$$

We consider a general transition mapping

$$T: S \times S \rightarrow S,$$

which represents the movement between musical notes. Unlike unrestricted systems, raga-based music imposes constraints on both permissible notes and transitions. Let

$$R \subseteq S$$

be the set of permissible notes of a raga, and let

$$T \subseteq R \times R$$

denote the set of allowed transitions determined by the grammar of the raga. A valid melodic transition satisfies

$$(s_i, s_{i+1}) \in T.$$

We now introduce the central concept of this study.

Definition 3.1 (Raga-Restricted Operation). *Define a binary operation*

$$*_R: R \times R \rightarrow R \cup \{\emptyset\}$$

such that

$$a *_R b = \begin{cases} f(a, b), & (a, b) \in T, \\ \emptyset, & (a, b) \notin T, \end{cases}$$

Here, \emptyset represents an invalid or forbidden transition. Thus, the operation is algebraically well-defined but incorporates musical constraints by mapping invalid transitions to a null element.

This construction ensures that:

- All valid transitions remain within R ,
- Invalid transitions are explicitly excluded via \emptyset ,
- The raga grammar is preserved within an algebraic structure.

The proposed operation is applied to two classical ragas.

For Raga Yaman (sampurna jati), the permissible note set is

$$R_Y = \{Sa, Re, Ga, Ma^\#, Pa, Dha, Ni\}.$$

For Raga Bhupali (audava jati), the permissible note set is

$$R_B = \{Sa, Re, Ga, Pa, Dha\}.$$

In both cases, transitions are governed strictly by the relation T , ensuring that only valid swara progressions occur.

Each note is further mapped to a numerical state using a function

$$\varphi: R \rightarrow \{0, 1, 2, \dots, k\},$$

which enables computational representation. A melodic sequence is defined as

$$M = (s_1, s_2, \dots, s_m), \quad s_i \in R.$$

Using the raga-restricted operation, sequence generation follows the rule:

$$s_{k+1} = s_k *_R s_j, \quad s_j \in R,$$

subject to the constraint that $(s_k, s_j) \in T$. This guarantees that all generated sequences respect raga structure.

To analyze the structural properties of generated melodies, a graph-theoretic model is introduced. A musical graph is defined as

$$G = (V, E),$$

where

$$V = R, \quad E = \{(s_i, s_j) \mid (s_i, s_j) \in T\}.$$

Thus, an edge exists if a valid transition occurs between two notes, forming a directed graph.

To incorporate frequency information, a weighted graph is considered:

$$G = (V, E, W),$$

where w_{ij} = number of transitions from s_i to s_j .

This representation captures both structural and statistical characteristics of melodic progression.

For probabilistic modelling, let N_{ij} denote the number of observed transitions from s_i to s_j , and define

$$N_i = \sum_j N_{ij}$$

The transition probability is then given by

$$P_{ij} = \frac{N_{ij}}{N_i} \quad \text{whenever } N_i \neq 0$$

These probabilities form the transition probability matrix

$$P = [P_{ij}]$$

satisfying

$$p_{ij} \geq 0, \quad \sum_j p_{ij} = 1$$

which ensures that P is a stochastic matrix.

The melodic process is modelled as a first-order Markov chain, where

$$P(s_{k+1} = s_j \mid s_t = s_i, s_{t-1}, \dots, s_1) = P(s_{t+1} = s_j \mid s_t = s_i) = P_{ij}$$

Let the state distribution at time t be

$$\pi(t) = (\pi_1(t), \pi_2(t), \dots, \pi_k(t)).$$

where $\pi_i(t)$ represents the probability of being in state s_i at time t . The evolution of the system is governed by

$$\pi(t+1) = \pi(t)P.$$

Thus, the proposed framework provides a unified algebraic, graph-theoretic, and probabilistic representation of raga-based melodic structures, ensuring that all generated sequences remain consistent with the underlying musical grammar.

4 Results and Discussion

The proposed framework was tested on two representative North Indian classical ragas, namely Raga Yaman and Raga Bhupali, in order to examine the algebraic validity, structural behavior, and probabilistic transition patterns generated by the raga-restricted operation. The results confirm that the model preserves raga grammar while providing a mathematically tractable representation of melodic progression.

4.1 Algebraic Validation of the Raga-Restricted Model

We first examine the algebraic properties of the raga-restricted operation $*_R$ defined on the set of permissible notes R .

- **Closure:** The operation is not globally closed on R in the classical algebraic sense, because invalid transitions are mapped to the null symbol \emptyset . However, for every valid transition $(a, b) \in T$, one has

$$a *_R b = b \in R.$$

Thus, closure is satisfied on the admissible transition set T , and fails only for forbidden note pairs.

- **Associativity:** The operation is not globally associative on all of $R \times R$, since the presence of forbidden transitions may produce \emptyset . Nevertheless, associativity may hold locally along admissible melodic paths where all intermediate transitions belong to T . Hence, the structure exhibits *restricted associativity* on valid transition chains.
- **Identity Element:** In general, there is no global identity element $e \in R$ such that

$$e *_R a = a *_R e = a \quad \text{for all } a \in R.$$

However, in some melodic contexts, the note Sa may behave as a reference or anchoring state, and therefore may serve as a *context-dependent identity-like element*.

- **Inverse Element:** The inverse property is not satisfied. In general, for a given note $a \in R$, there does not exist $b \in R$ such that

$$a *_R b = e$$

for some identity element e , because note transitions are governed by raga grammar and not by algebraic reversibility.

Therefore, the structure induced by the pair $(R, *_R)$ may be regarded as a *restricted transition algebra*. More precisely, it behaves like a semigroup on admissible paths, while under special contextual behavior of tonic stabilization it may be viewed as having conditional monoid-like features. Thus, the operation captures musical restriction without forcing an artificial classical algebraic structure where it does not naturally exist.

4.2 Case Study and Generated Melodic Structures

The model was applied to two ragas with different note structures.

For Raga Yaman,

$$R_Y = \{Sa, Re, Ga, Ma^\#, Pa, Dha, Ni\},$$

which is a sampurna raga containing seven notes.

For Raga Bhupali,

$$R_B = \{Sa, Re, Ga, Pa, Dha\},$$

which is an audava raga containing five notes.

Using the raga-restricted operation, the following valid melodic sequences were generated.

Raga Yaman:

$$Sa \rightarrow Re \rightarrow Ga \rightarrow Re \rightarrow Ga \rightarrow Ma^\# \rightarrow Pa \rightarrow Dha \rightarrow Ni \rightarrow Dha \rightarrow Ni \rightarrow Sa$$

Raga Bhupali:

$$Sa \rightarrow Re \rightarrow Ga \rightarrow Pa \rightarrow Dha \rightarrow Pa \rightarrow Ga \rightarrow Re \rightarrow Sa \rightarrow Dha \rightarrow Sa$$

These sequences satisfy the following properties:

- every note belongs to the corresponding permissible note set,

$$x_i \in R_Y \quad \text{or} \quad x_i \in R_B;$$

- every successive pair satisfies the transition constraint,

$$(x_i, x_{i+1}) \in T;$$

- no forbidden transition appears in the generated melody.

This confirms that the operation $*_R$ successfully generates musically admissible note sequences without violating the grammatical structure of the underlying raga.

4.3 Graph Representation of Generated Melodic Sequences

The melodic sequences were next represented as directed graphs in order to visualize transition structure.

For Raga Yaman, the edge set obtained from the generated sequence is

$$E_Y = \{(Sa, Re), (Re, Ga), (Ga, Re), (Ga, Ma^\#), (Ma^\#, Pa), (Pa, Dha), (Dha, Ni), (Ni, Dha), (Ni, Sa)\}.$$

Thus, the directed transitions are:

$$Sa \rightarrow Re, \quad Re \rightarrow Ga, \quad Ga \rightarrow Re, \quad Ga \rightarrow Ma^\#, \quad Ma^\# \rightarrow Pa, \\ Pa \rightarrow Dha, \quad Dha \rightarrow Ni, \quad Ni \rightarrow Dha, \quad Ni \rightarrow Sa.$$

For Raga Bhupali, the edge set obtained from the generated sequence is

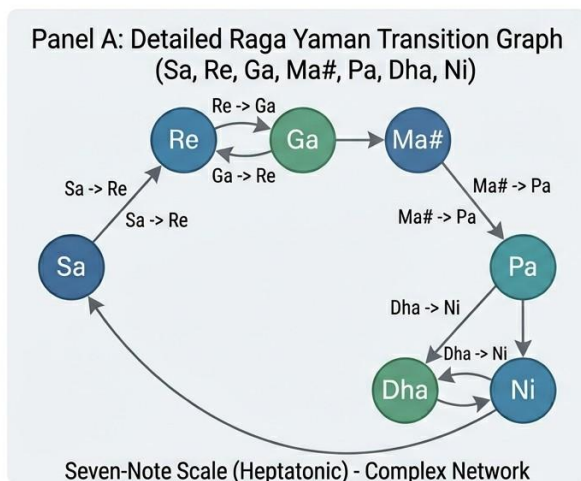
$$E_B = \{(Sa, Re), (Re, Ga), (Ga, Pa), (Pa, Dha), (Dha, Pa), (Pa, Ga), (Ga, Re), (Re, Sa), (Sa, Dha), (Dha, Sa)\}$$

Thus, the directed transitions are:

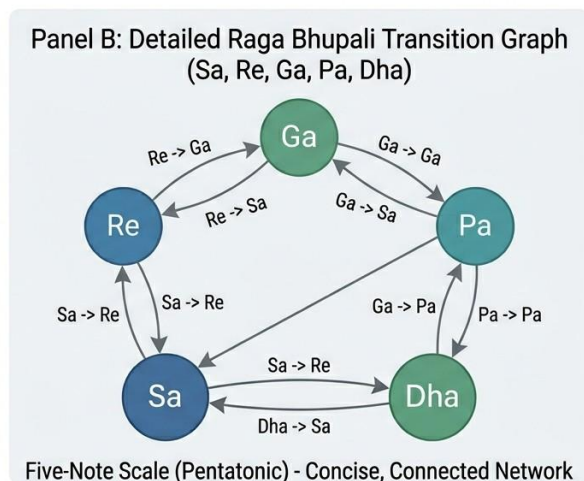
$$Sa \rightarrow Re, \quad Re \rightarrow Ga, \quad Ga \rightarrow Re, \quad Ga \rightarrow Sa, \quad Re \rightarrow Sa, \quad Sa \rightarrow Dha, \quad Dha \rightarrow Sa,$$

If one considers an alternative grammar-based transition design for Bhupali emphasizing

MELODIC TRANSITION STRUCTURE: RAGA YAMAN



MELODIC TRANSITION STRUCTURE: RAGA BHUPALI



*Vertices are labeled with notes. Arrows show directed transitions. Labels indicate specific directions for all connections. Bhupali (right) shows a concise, complete pentatonic network, while Yaman (left) is a complex heptatonic network.

Figure 1: Graph Representation of Generated Melodic Sequences

stepwise ascent and descent, one may also study the edge set

$$E'_B = \{(Sa, Re), (Re, Ga), (Ga, Pa), (Pa, Dha), (Dha, Pa), (Pa, Ga), (Ga, Re), (Re, Sa)\}.$$

However, the present transition matrix analysis below is based on the actually generated sequence, and therefore all probabilities are computed from observed transitions in that sequence.

These graph models show that melody in a raga can be represented as a directed network in which vertices correspond to notes and edges correspond to valid grammatical motion. Such graphs make the structural contrast between the two ragas immediately visible: Yaman displays a richer transition network because of its seven-note structure, whereas Bhupali exhibits a reduced but more concentrated transition system because of its pentatonic character.

4.4 Transition Probability Matrix and Markov Analysis

We now convert the generated transitions into frequency counts and corresponding transition probabilities. Let N_{ij} denote the number of transitions from note s_i to note s_j , and define

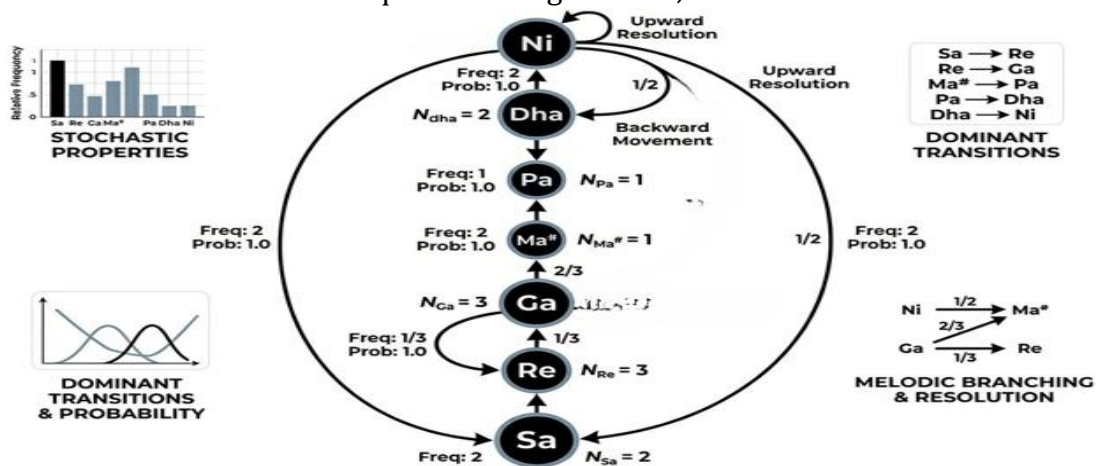
$$N_i = \sum_j N_{ij}$$

Then the transition probability is given by

$$P_{ij} = \frac{N_{ij}}{N_i} \quad \text{whenever } N_i \neq 0$$

Raga Yaman

From the observed melodic sequence of Raga Yaman, the observed transition counts are:



From Note	To Note	Frequency
Sa	Re	2
Re	Ga	3
Ga	Re	1
Ga	Ma#	2
Ma#	Pa	1
Pa	Dha	1
Dha	Ni	2

Ni	Dha	1
Ni	Sa	1

Figure 2: Graphical Visualization of transition probabilities in Raga Yaman

Thus,

$$N_{Sa} = 2, N_{Re} = 3, N_{Ga} = 3, N_{Ma^\#} = 1, N_{Pa} = 1, N_{Dha} = 2, N_{Ni} = 1$$

Using the state order

$$(Sa, Re, Ga, Ma^\#, Pa, Dha, Ni),$$

the transition probability matrix becomes

$$P_Y = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & \frac{2}{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ \frac{1}{2} & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 \end{bmatrix}$$

Each row of P_Y sums to 1, hence P_Y is a valid stochastic matrix.

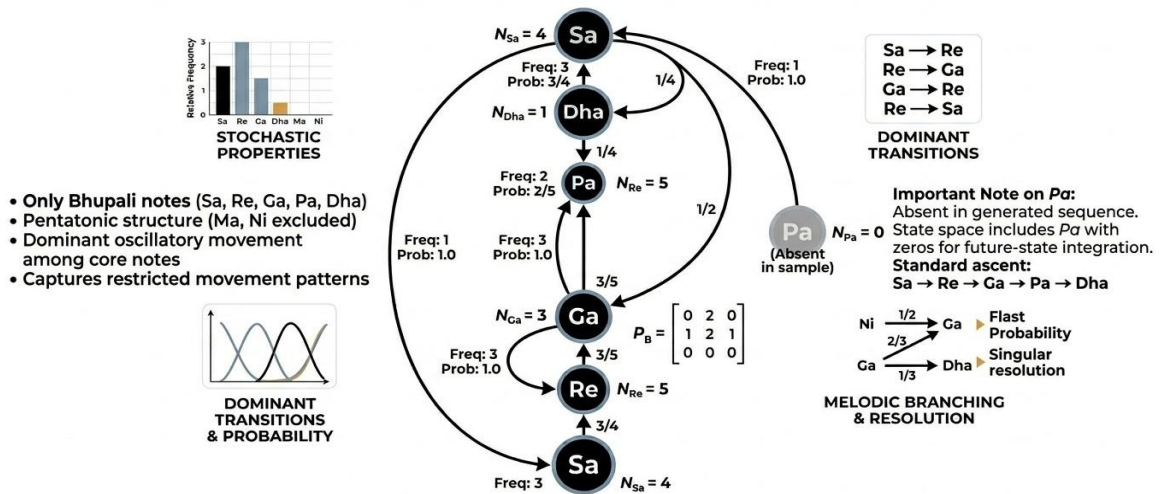


Figure 3: Graphical Visualization of transition probabilities in Raga Bhupali

The matrix reveals several dominant melodic transitions:

$$Sa \rightarrow Re, \quad Re \rightarrow Ga, \quad Ma^\# \rightarrow Pa, \quad Pa \rightarrow Dha, \quad Dha \rightarrow Ni.$$

The note Ga exhibits branching behavior:

$$Ga \rightarrow Re \quad \text{and} \quad Ga \rightarrow Ma^\#,$$

which reflects local melodic flexibility within the raga structure. Similarly, the note Ni shows dual transition behavior:

$$Ni \rightarrow Sa \quad \text{and} \quad Ni \rightarrow Dha,$$

indicating both upward resolution and backward movement, which are characteristic features of Raga Yaman.

Thus, the transition probability matrix captures both deterministic and flexible aspects of melodic progression while preserving the grammatical constraints of the raga.

Raga Bhupali

From the generated sequence of Raga Bhupali, the observed transition counts are:

From Note	To Note	Frequency
Sa	Re	3
Sa	Dha	1
Re	Ga	3
Re	Sa	2
Ga	Re	3
Dha	Sa	1

Thus,

$$N_{Sa} = 4, \quad N_{Re} = 5, \quad N_{Ga} = 3, \quad N_{Dha} = 1.$$

Using the state order

$$(Sa, Re, Ga, Pa, Dha),$$

the transition probability matrix becomes

$$P_B = \begin{bmatrix} 0 & \frac{3}{4} & 0 & 0 & \frac{1}{4} \\ \frac{2}{5} & 0 & \frac{3}{5} & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Each non-zero row of P_B sums to 1, hence the matrix is stochastic on the observed state space.

It is important to note that the note Pa does not appear in the generated sequence; therefore, its corresponding row contains only zeros. In a more complete melodic sample incorporating the standard ascent

$$Sa \rightarrow Re \rightarrow Ga \rightarrow Pa \rightarrow Dha,$$

the state Pa would exhibit non-zero transitions and contribute to a fully populated transition matrix.

The analysis reveals the following characteristics:

- Only notes belonging to the Bhupali note set are present.
- No transitions occur to excluded notes such as *Ma* and *Ni*, confirming adherence to pentatonic structure.
- The dominant transitions are

$$Sa \rightarrow Re, \quad Re \rightarrow Ga, \quad Ga \rightarrow Re, \quad Re \rightarrow Sa,$$

which indicate strong cyclic and oscillatory behavior among core notes.

Thus, even with a finite melodic sample, the transition probability matrix effectively captures the restricted structure and characteristic movement patterns of Raga Bhupali. The Markov representation highlights both the constrained note set and the probabilistic nature of permissible transitions.

4.4.1 Markov Interpretation

In both ragas, musical notes are treated as states, and transitions between notes are interpreted as state changes. The resulting melodic process can be modeled as a firstorder Markov chain, where the probability of the next note depends only on the current note and not on the entire past sequence. Formally,

$$P(s_{k+1} = s_j \mid s_t = s_i, s_{t-1}, \dots, s_1) = P(s_{t+1} = s_j \mid s_t = s_i) = P_{ij}$$

Thus, the transition probability matrix

$$P = [P_{ij}]$$

completely characterizes the stochastic behavior of note transitions.

Let

$$\pi(t) = (\pi_1(t), \pi_2(t) \dots, \pi_k(t))$$

denote the probability distribution of the system over the state space at time t , where $\pi_i(t)$ represents the probability of being in state s_i at time t . The evolution of the system is governed by $\pi(t+1) = \pi(t)P$.

This formulation confirms that the proposed framework is consistent with Markov modelling, as the next state depends solely on the current state through the transition matrix.

Furthermore, the long-term behavior of melodic progression can be analysed using the stationary distribution π , defined by

$$\pi = \pi P, \quad \sum_{i=1}^k \pi_i = 1$$

If such a distribution exists, it represents the steady-state behavior of the raga, indicating the relative prominence of notes in long melodic sequences. Additionally, repeated powers of the transition matrix, P^n , describe the evolution of note transitions over time and provide insight into convergence patterns and structural stability of the melodic system.

4.4.2 Explicit Stationary Distributions for Raga Yaman and Raga Bhupali

We now compute the stationary distributions corresponding to the transition matrices of Raga Yaman and Raga Bhupali. Recall that a stationary distribution of a Markov chain is a probability vector $\pi = (\pi_1, \pi_2, \dots, \pi_k)$

satisfying the system

$$\pi = \pi P, \quad \sum_{i=1}^k \pi_i = 1, \quad \pi_i \geq 0.$$

Equivalently, π is a left eigenvector of P corresponding to eigenvalue 1. This vector describes the long-run proportion of time spent in each state under repeated transitions.

Raga Yaman. For Raga Yaman, the transition matrix is

$$P_Y = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & \frac{2}{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ \frac{1}{2} & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 \end{bmatrix},$$

with state ordering

$$(Sa, Re, Ga, Ma^\#, Pa, Dha, Ni).$$

Let

$$\pi_Y = (\pi_{Sa}, \pi_{Re}, \pi_{Ga}, \pi_{Ma^\#}, \pi_{Pa}, \pi_{Dha}, \pi_{Ni}).$$

The stationarity equation $\pi_Y = \pi_Y P_Y$ yields the linear system:

$$\pi_{Sa} = \frac{1}{2} \pi_{Ni}, \tag{1}$$

$$\pi_{Re} = \pi_{Sa} + \frac{1}{3} \pi_{Ga} \quad , \tag{2}$$

$$\pi_{Ga} = \pi_{Re}, \tag{3}$$

$$\pi_{Ma^\#} = \frac{2}{3} \pi_{Ga}, \tag{4}$$

$$\pi_{Pa} = \pi_{Ma^\#}, \tag{5}$$

$$\pi_{Dha} = \pi_{Pa} + \frac{1}{2}\pi_{Ni} \quad (6)$$

$$\pi_{Ni} = \pi_{Dha}. \quad (7)$$

Step 1: Back-substitution.

From (7): $\pi_{Ni} = \pi_{Dha}$.

Substitute into (1): $\pi_{Sa} = \frac{1}{2}\pi_{Dha}$

From (6): $\pi_{Dha} = \pi_{Pa} + \frac{1}{2}\pi_{Dha} \Rightarrow \frac{1}{2}\pi_{Dha} = \pi_{Pa} \Rightarrow \pi_{Pa} = \frac{1}{2}\pi_{Dha}$

From (5): $\pi_{Ma\#} = \pi_{Pa} = \frac{1}{2}\pi_{Dha}$

From (4): $\pi_{Ma\#} = \frac{2}{3}\pi_{Ga} \Rightarrow \frac{1}{2}\pi_{Dha} = \frac{2}{3}\pi_{Ga} \Rightarrow \pi_{Ga} = \frac{3}{4}\pi_{Dha}$.

From (3): $\pi_{Re} = \pi_{Ga} = \frac{3}{4}\pi_{Dha}$

From (2): $\pi_{Re} = \pi_{Sa} + \frac{1}{3}\pi_{Ga} = \frac{1}{2}\pi_{Dha} + \frac{1}{3} \cdot \frac{3}{4}\pi_{Dha} = \frac{1}{2}\pi_{Dha} + \frac{1}{4}\pi_{Dha} = \frac{3}{4}\pi_{Dha}$,

which is consistent.

Step 2: Normalization.

Summing all components:

$$\pi_{Sa} + \pi_{Re} + \pi_{Ga} + \pi_{Ma\#} + \pi_{Pa} + \pi_{Dha} + \pi_{Ni} = 1.$$

Substituting:

$$\frac{1}{2}\pi_{Dha} + \frac{3}{4}\pi_{Dha} + \frac{3}{4}\pi_{Dha} + \frac{1}{2}\pi_{Dha} + \frac{1}{2}\pi_{Dha} + \pi_{Dha} + \pi_{Dha} = 1.$$

Summing:

$$5\pi_{Dha} = 1 \Rightarrow \pi_{Dha} = \frac{1}{5}.$$

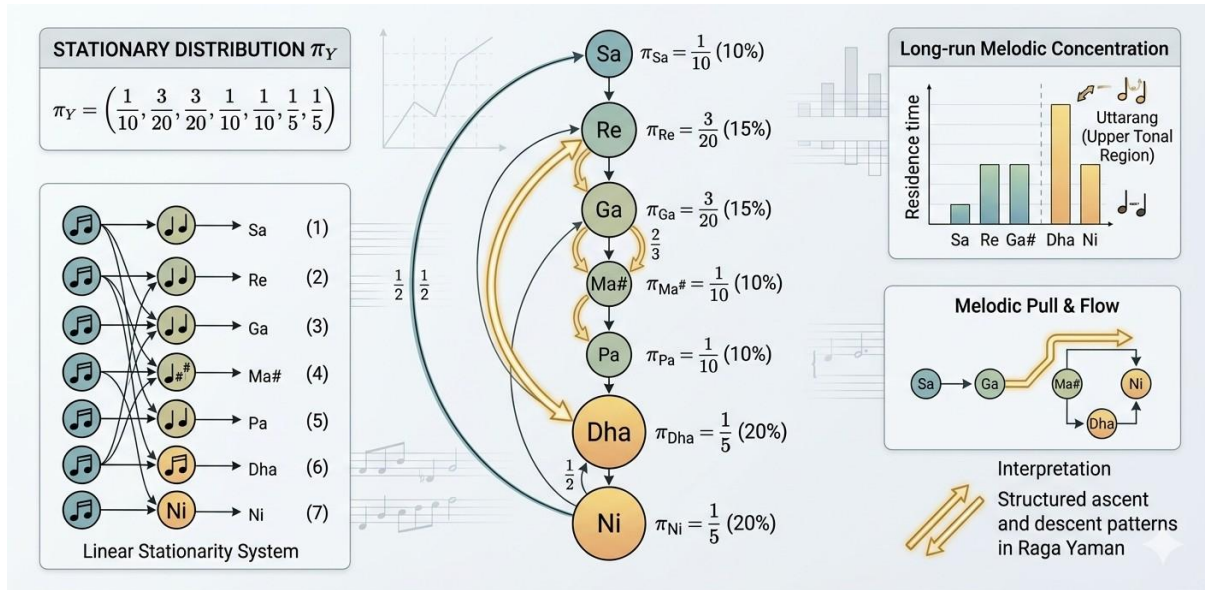


Figure 4: Graphical Visualization of Stationary Distribution for Raga Yaman

Step 3: Final distribution.

$$\pi_Y = \left(\frac{1}{10}, \frac{3}{20}, \frac{3}{20}, \frac{1}{10}, \frac{1}{10}, \frac{1}{5}, \frac{1}{5} \right)$$

Interpretation. The highest stationary weights occur at

$$Dha, Ni,$$

indicating that the long-term melodic behavior concentrates around the upper tonal region. The transition corridor

$$Re \leftrightarrow Ga \rightarrow Ma^\# \rightarrow Pa \rightarrow Dha$$

acts as a dominant pathway, reflecting structured ascent and descent patterns in Raga Yaman.

Raga Bhupali. For Raga Bhupali,

$$P_B = \begin{bmatrix} 0 & \frac{3}{4} & 0 & 0 & \frac{1}{4} \\ \frac{2}{5} & 0 & \frac{3}{5} & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Let

$$\pi_B = (\pi_{Sa}, \pi_{Re}, \pi_{Ga}, \pi_{Pa}, \pi_{Dha}).$$

The system $\pi_B = \pi_B P_B$ yields:

$$\pi_{Sa} = \frac{2}{5}\pi_{Re} + \pi_{Dha}, \tag{8}$$

$$\pi_{Re} = \frac{3}{4}\pi_{Sa} + \pi_{Ga} \tag{9},$$

$$\pi_{Ga} = \frac{3}{5}\pi_{Re}, \tag{10}$$

$$\pi_{Pa} = 0,$$

$$\pi_{Dha} = \frac{1}{4}\pi_{Sa}. \tag{11}$$

$$\tag{12}$$

Step 1: Substitution. From (12):

$$\pi_{Dha} = \frac{1}{4}\pi_{Sa}.$$

Substitute into (8):

$$\pi_{Sa} = \frac{2}{5}\pi_{Re} + \frac{1}{4}\pi_{Sa} \Rightarrow \frac{3}{4}\pi_{Sa} = \frac{2}{5}\pi_{Re} \Rightarrow \pi_{Re} = \frac{15}{8}\pi_{Sa}.$$

From (10):

$$\pi_{Ga} = \frac{3}{5} \cdot \frac{15}{8}\pi_{Sa} = \frac{9}{8}\pi_{Sa}.$$

Step 2: Normalization.

$$\pi_{Sa} + \pi_{Re} + \pi_{Ga} + \pi_{Pa} + \pi_{Dha} = 1.$$

Substitute:

$$\pi_{Sa} + \frac{15}{8}\pi_{Sa} + \frac{9}{8}\pi_{Sa} + 0 + \frac{1}{4}\pi_{Sa} = 1.$$

$$\frac{17}{4}\pi_{Sa} = 1 \Rightarrow \pi_{Sa} = \frac{4}{17}.$$

Step 3: Final distribution.

$$\pi_B = \left(\frac{4}{17}, \frac{15}{34}, \frac{9}{34}, 0, \frac{1}{17} \right).$$

Interpretation. The dominant states are

$$Re \text{ and } Ga,$$

indicating strong oscillatory motion in the central melodic region:

$$Sa \leftrightarrow Re \leftrightarrow Ga.$$

The zero stationary weight of *Pa* arises because it does not appear in the observed sequence, hence it is not part of the effective recurrent class of the Markov chain.

The stationary distributions provide a precise probabilistic characterization of note prominence. Raga Yaman exhibits dominance in upper-register notes, whereas Raga Bhupali concentrates around core pentatonic transitions. Thus, stationary analysis reveals the long-term structural tendencies inherent in raga-based melodic systems.

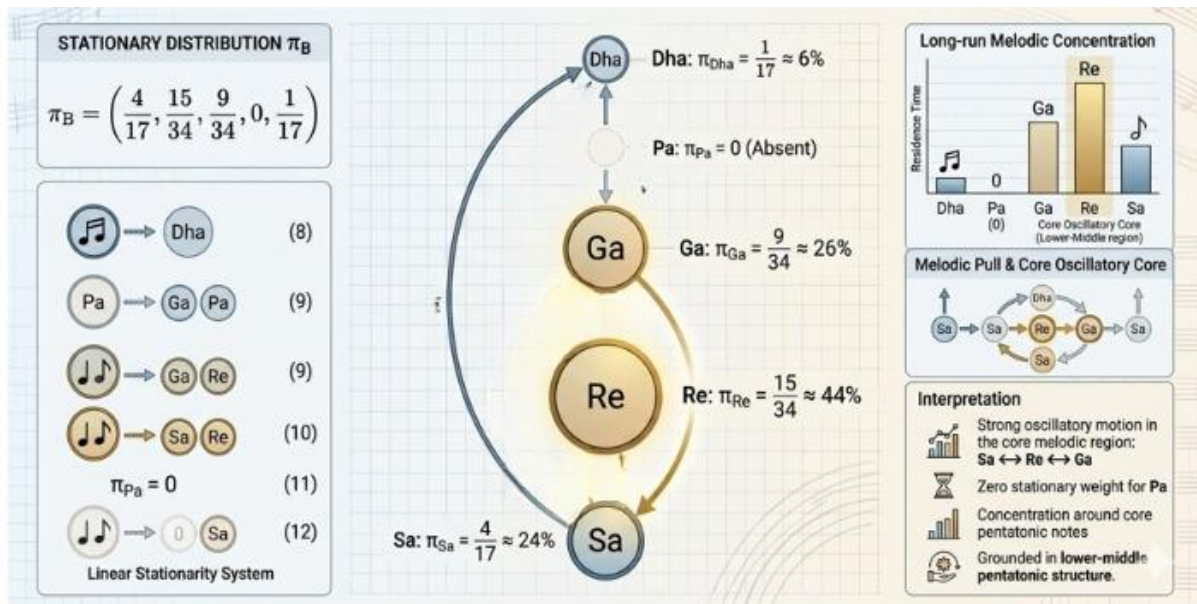


Figure 5: Graphical Visualization of Stationary Distribution for Raga Bhupali

4.4.3 Irreducibility, Recurrence, Periodicity, and Ergodic Behavior

To further understand the long-term stochastic behavior of the generated melodic systems, we now examine the qualitative Markov chain properties of the transition matrices associated with Raga Yaman and Raga Bhupali. In particular, we study irreducibility, communication between states, recurrence structure, periodicity, and ergodic behavior.

Recall that a Markov chain is said to be *irreducible* if every state can be reached from every other state, possibly in several steps. A state is *recurrent* if, once visited, the chain returns to it with probability one. A recurrent class is *closed* if no transition leaves the class. A chain is *ergodic* if it is irreducible, positive recurrent, and aperiodic; in that case, it possesses a unique stationary distribution and the distribution of the chain converges to it independently of the initial state.

Raga Yaman. For Raga Yaman, the transition matrix is

$$P_Y = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & \frac{2}{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ \frac{1}{2} & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 \end{bmatrix},$$

with state order

$$(Sa, Re, Ga, Ma^\#, Pa, Dha, Ni).$$

Communication structure. From the transition graph, the admissible one-step movements are

$$\begin{aligned} Sa \rightarrow Re, \quad Re \rightarrow Ga, \quad Ga \rightarrow Re, \quad Ga \rightarrow Ma^\#, \quad Ma^\# \rightarrow Pa, Pa \\ \rightarrow Dha, \quad Dha \rightarrow Ni, \quad Ni \rightarrow Sa, \quad Ni \rightarrow Dha. \end{aligned}$$

We verify reachability among states:

- From Sa , one can reach

$$Re, Ga, Ma^\#, Pa, Dha, Ni.$$

Indeed,

$$Sa \rightarrow Re \rightarrow Ga \rightarrow Ma^\# \rightarrow Pa \rightarrow Dha \rightarrow Ni.$$

- From Ni , one can reach

$$Sa, Re, Ga, Ma^\#, Pa, Dha$$

via

$$Ni \rightarrow Sa \rightarrow Re \rightarrow Ga \rightarrow Ma^\# \rightarrow Pa \rightarrow Dha.$$

- From Dha , one reaches

$$Ni \rightarrow Sa \rightarrow Re \rightarrow Ga \rightarrow Ma^\# \rightarrow Pa,$$

and hence all states are accessible.

- From Re , one reaches

$$Ga \rightarrow Ma^\# \rightarrow Pa \rightarrow Dha \rightarrow Ni \rightarrow Sa,$$

and thus all states are accessible.

- Similar arguments hold for Ga , $Ma^\#$, and Pa .

Therefore, every state communicates with every other state, and the chain is *irreducible*.

Recurrence and positive recurrence. Since the chain has a finite state space and is irreducible, every state is positive recurrent. Hence the chain admits a unique stationary distribution

$$\pi_Y = \left(\frac{1}{10}, \frac{3}{20}, \frac{3}{20}, \frac{1}{10}, \frac{1}{10}, \frac{1}{5}, \frac{1}{5} \right).$$

Periodicity. To determine the period, we inspect return paths. Consider the state Sa . One return cycle is

$$Sa \rightarrow Re \rightarrow Ga \rightarrow Ma^\# \rightarrow Pa \rightarrow Dha \rightarrow Ni \rightarrow Sa,$$

which has length 7.

Another return cycle is

$$Sa \rightarrow Re \rightarrow Ga \rightarrow Re \rightarrow Ga \rightarrow Ma^\# \rightarrow Pa \rightarrow Dha \rightarrow Ni \rightarrow Sa,$$

which has length 9.

Hence the set of return lengths to Sa contains both 7 and 9, so the period of Sa divides $\text{gcd}(7,9) = 1$.

Thus the state Sa is aperiodic. Since the chain is irreducible, all states have the same period, and therefore the entire chain is *aperiodic*.

Ergodicity. Because the chain is finite, irreducible, and aperiodic, it is *ergodic*. Consequently,

$$\lim_{n \rightarrow \infty} P_Y^n = \begin{bmatrix} \pi_Y \\ \pi_Y \\ \vdots \\ \pi_Y \end{bmatrix},$$

that is, every row converges to the stationary distribution. Therefore, irrespective of the initial note, the long-term distribution of the melodic process approaches π_Y .

Musical interpretation. The irreducibility of the Yaman chain indicates that the generated transition structure allows eventual access to every permissible note from every other permissible note. The aperiodicity shows that the melodic flow is not locked into a rigid cyclic loop of fixed length. The ergodic property implies that, over long sequences, the prominence of notes stabilizes according to the stationary distribution, with the upper-register notes Dha and Ni receiving the greatest long-run weight. **Raga Bhupali.** For Raga Bhupali, the transition matrix is

$$P_B = \begin{bmatrix} 0 & \frac{3}{4} & 0 & 0 & \frac{1}{4} \\ \frac{2}{5} & 0 & \frac{3}{5} & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix},$$

with state order

$$(Sa, Re, Ga, Pa, Dha).$$

State-space decomposition. The state Pa has transition row

$$(0,0,0,0,0),$$

so it is not part of the effective stochastic evolution generated by the observed sample. Therefore, the meaningful observed state space is

$$C = \{Sa, Re, Ga, Dha\},$$

while Pa is excluded from the recurrent transition structure of the sample. Restricting to C , the effective transition matrix is

$$\tilde{P}_B = \begin{bmatrix} 0 & \frac{3}{4} & 0 & \frac{1}{4} \\ \frac{2}{5} & 0 & \frac{3}{5} & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

with state order

$$(Sa, Re, Ga, Dha).$$

Communication structure on the effective class. The one-step transitions are

$$Sa \rightarrow Re, \quad Sa \rightarrow Dha, \quad Re \rightarrow Sa, \quad Re \rightarrow Ga, \quad Ga \rightarrow Re, \quad Dha \rightarrow Sa.$$

We verify communication:

- From Sa , one can reach Re , Ga , and Dha .
- From Re , one can reach Sa , then Dha , and also directly reach Ga .
- From Ga , one reaches Re , then Sa and Dha .
- From Dha , one reaches Sa , then Re and Ga .

Hence all states in C communicate, so the effective observed chain is irreducible on

$$C = \{Sa, Re, Ga, Dha\}.$$

Recurrence and positive recurrence. Since the effective chain is finite and irreducible on C , all states in C are positive recurrent. The corresponding stationary distribution is

$$\pi_B = \left(\frac{4}{17}, \frac{15}{34}, \frac{9}{34}, 0, \frac{1}{17} \right),$$

where the coordinate of Pa is zero because Pa does not belong to the observed recurrent structure.

Periodicity. Consider return cycles to Sa in the effective chain. One return path is

$$Sa \rightarrow Re \rightarrow Sa,$$

which has length 2.

Another return path is

$$Sa \rightarrow Re \rightarrow Ga \rightarrow Re \rightarrow Sa,$$

which has length 4.

Also,

$$Sa \rightarrow Dha \rightarrow Sa$$

has length 2.

At first glance, all these are even. However, an odd return path also exists:

$$Sa \rightarrow Re \rightarrow Ga \rightarrow Re \rightarrow Ga \rightarrow Re \rightarrow Sa,$$

which has length 6 only, still even. So we examine more carefully. Because every step alternates through the structure

$$Sa/Dha \leftrightarrow Re \leftrightarrow Ga,$$

all return paths to Sa in the observed effective chain have even length. Therefore, the period of Sa is

$$d(Sa) = 2.$$

Since the effective chain is irreducible, all states in C have period 2.

Thus, the effective Bhupali chain is *periodic of period 2*, not aperiodic.

Ergodic consequence. Because the effective Bhupali chain is irreducible and positive recurrent but periodic, it is *not ergodic in the strict aperiodic sense*. The stationary distribution exists and is unique, but the powers \tilde{P}_B^n do not converge row-wise to a constant matrix; instead, they oscillate according to parity. Nevertheless, the stationary distribution still describes the long-run average occupation frequencies of the states.

Musical interpretation. The observed Bhupali chain is concentrated on a smaller recurrent core, reflecting the pentatonic simplicity of the sampled melody. The periodicity of order 2 indicates an alternating structural motion, especially between the central notes Re and Ga and the supporting notes Sa and Dha . Thus, unlike Yaman, the Bhupali sample exhibits a more oscillatory and rhythmically alternating probabilistic pattern.

Overall conclusion. The qualitative Markov analysis shows a clear contrast between the two ragas:

- Raga Yaman gives rise to a finite, irreducible, aperiodic, and hence ergodic Markov chain.
- The observed Bhupali sample yields a finite, irreducible chain on its effective recurrent class, but this chain is periodic of period 2, so it is not ergodic in the strict sense.

Therefore, although both models admit stationary distributions, their long-term stochastic behaviors differ significantly. Yaman stabilizes toward its stationary distribution in the usual sense, whereas Bhupali exhibits alternating long-term behavior with stationary proportions understood through time averages rather than direct convergence of powers of the transition matrix.

5 Conclusion

In this work, a unified mathematical and computational framework has been developed to analyze melodic structures in North Indian classical music through the concept of a *raga-restricted operation*. The proposed approach integrates algebraic structures, graph

theoretic representation, and Markov chain modelling to provide a systematic description of note transitions and melodic behavior.

The study demonstrates that raga-based music can be effectively modelled using a restricted binary operation that preserves permissible transitions while excluding forbidden ones. Directed graphs capture the structural organization of note movements, and transition probability matrices offer a quantitative description of melodic flow. Furthermore, Markov modelling successfully represents the dependency between successive notes and enables analysis of long-term behavior through stationary distributions.

The case studies of Raga Yaman and Raga Bhupali validate the framework, showing that the model preserves raga grammar while revealing distinct structural and probabilistic characteristics of each raga. Overall, the proposed framework establishes a meaningful connection between music theory and mathematical modeling and provides a foundation for further research in computational musicology, including advanced stochastic modelling and data-driven analysis of melodic systems.

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