UNDER PEER REVIEW

Original Research Article

Expressions of Solutions of Some Systems of Difference Equations

Abstract

This paper presents an analytical investigation of a fourth-order system of nonlinear rational difference equations given by

$$U_{p+1} = \frac{A_1 U_p V_{p-2}}{\alpha_1 U_p + \beta_1 V_{p-3}}, \quad V_{p+1} = \frac{A_2 U_{p-2} V_p}{\alpha_2 V_p + \beta_2 U_{p-3}}, \quad p = 0, 1, \ldots,$$

with arbitrary non-zero real initial conditions $U_{-3} = d$, $U_{-2} = c$, $U_{-1} = b$, $U_0 = a$, $V_{-3} = h$, $V_{-2} = g$, $V_{-1} = f$, $V_0 = e$ and real constants A_j, α_j, β_j ($j = 1, 2$). The primary objective is to derive explicit closed-form expressions for the solutions $\{U_p\}$ and $\{V_p\}$ for four distinct special cases corresponding to different sign patterns in the denominators. For each configuration, we obtain complete solution formulas exhibiting a remarkable period-6 structure, with distinct algebraic expressions for indices congruent to $-3, -2, -1, 0, 1, 2$ modulo 6. The methodology combines algebraic pattern recognition with rigorous mathematical induction to establish the validity of these formulas for all non-negative integers p. Numerical simulations implemented in MATLAB illustrate the solution behavior for representative initial conditions, consistently demonstrating convergence to the equilibrium point $(0, 0)$ and providing visual confirmation of the analytical results. The significance of this work lies in expanding the repertoire of exactly solvable nonlinear difference equations, revealing the connection between index structures and periodic solution patterns, and facilitating subsequent qualitative analysis without iterative computation. Limitations and directions for future research are discussed, including generalization to arbitrary parameters, rigorous stability analysis, and exploration of singularities and boundary cases.

Keywords: Rational difference equations; discrete dynamical systems; closed-form solutions; nonlinear *recurrences; periodicity; asymptotic behavior; fourth-order systems; mathematical induction; explicit formulas; stability analysis*

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1 Introduction

Difference equations, also referred to as recurrence relations, constitute a fundamental area of discrete mathematics with profound implications across numerous scientific disciplines. Unlike differential equations which describe continuous phenomena, difference equations characterize systems that evolve at discrete time intervals, making them particularly suitable for modeling scenarios where observations or changes occur at distinct time steps. The general form of an m-th order difference equation can be expressed as $U_{n+m} = f(n, U_n, U_{n+1}, \ldots, U_{n+m-1})$, where the future state depends on the preceding m states.

The qualitative theory of difference equations has witnessed remarkable growth over the past several decades, driven by their ubiquitous appearance in applications ranging from population dynamics and economics to signal processing and control theory. Of particular interest are rational difference equations—those where the function f takes the form of a ratio of polynomials in the dependent variables. These equations often exhibit rich dynamical behavior including stability, periodicity, bifurcation, and even chaos, despite their seemingly simple algebraic structure.

A system of difference equations extends this framework to multiple interacting sequences, typically written as:

$$\begin{cases} U_{p+1} = F(U_p, U_{p-1}, \ldots, V_p, V_{p-1}, \ldots) \\ V_{p+1} = G(U_p, U_{p-1}, \ldots, V_p, V_{p-1}, \ldots) \end{cases}$$

Such systems naturally arise when modeling coupled phenomena—for instance, predator-prey interactions in ecology, competing species in biology, or interconnected economic variables. The order of such systems is determined by the maximum lag appearing in the arguments of F and G. Higher-order systems, while more challenging to analyze, often provide more accurate representations of real-world phenomena where memory effects or time delays are significant.

The investigation of rational difference equations and their systems has attracted considerable attention from researchers worldwide. Ahmed [1] conducted an extensive analysis of the global dynamics in a system of rational difference equations, revealing conditions for stability and boundedness. Bao [3] explored the dynamical behavior of second-order nonlinear systems, demonstrating how solution trajectories depend critically on initial conditions. Din [4] extended these investigations to fourth-order systems, uncovering complex periodic patterns and stability regions.
Elsayed [14] made significant contributions by deriving closed-form expressions for solutions of second-order rational systems, establishing connections between the structure of the equations and the periodicity of their solutions. Halm [28] focused on the stability characteristics and asymptotic properties of systems with simple rational forms, providing criteria for convergence to equilibrium points. Liu and colleagues [37] tackled three-dimensional systems, successfully constructing explicit solution formulas and analyzing long-term behavior in terms of initial values.

The collaborative work of Touafek with Elsayed [45] and later with Haddad [46] advanced the understanding of both rational and max-type systems, revealing the intricate relationship between equation parameters and solution periodicity. Yazlik et al. [51] contributed by expressing solutions in terms of special number sequences, bridging the gap between difference equations and number theory. Zhang and collaborators [53] examined symmetric systems, uncovering properties that reflect the underlying symmetry in the equations themselves.

Despite these advances, many classes of higher-order rational systems remain unexplored, particularly those with asymmetric coupling and multiple time delays. This gap in the literature motivates the present investigation.

This paper addresses the following system of nonlinear rational difference equations:

$$U_{p+1} = \frac{A_1 U_p V_{p-2}}{\alpha_1 U_p + \beta_1 V_{p-3}}, \quad V_{p+1} = \frac{A_2 U_{p-2} V_p}{\alpha_2 V_p + \beta_2 U_{p-3}}, \quad p = 0, 1, \ldots, \tag{1.1}$$

where the initial conditions $U_{-3} = d$, $U_{-2} = c$, $U_{-1} = b$, $U_0 = a$, $V_{-3} = h$, $V_{-2} = g$, $V_{-1} = f$, $V_0 = e$ are arbitrary non-zero real numbers, and A_j, α_j, β_j $(j = 1, 2)$ are real constants.

This system presents several distinctive features that make it particularly interesting for investigation [19]. First, it is of fourth order, as each equation involves terms with indices $p + 1$, p, $p - 2$, and $p - 3$. Second, the coupling between the U and V sequences is asymmetric—U_{p+1} depends on U_p and V_{p-2}, V_{p-3}, while V_{p+1} depends on V_p and U_{p-2}, U_{p-3}. This asymmetry can produce diverse dynamical behaviors not present in symmetric systems. Third, the rational form with linear denominators introduces singularities that must be carefully handled, imposing restrictions on the parameter space and initial conditions.

The primary contributions of this work are threefold:

1. To derive explicit closed-form expressions for the solutions U_p and V_p in terms of the initial conditions and parameters, for four distinct special cases of the general system.
2. To establish rigorous proofs of these formulas using mathematical induction, demonstrating the correctness of the derived expressions for all integer indices.
3. To provide numerical simulations that illustrate the theoretical results and reveal the qualitative behavior of solutions, including convergence properties and stability characteristics.

The remainder of this paper is organized as follows. Section 2 examines the first special case where the denominators take the form $-U_p + V_{p-3}$ and $-V_p + U_{p-3}$. We present explicit formulas for the 6-periodic pattern that emerges in the solution structure and provide a complete proof by induction. Section 3 investigates the second case with denominators $-U_p + V_{p-3}$ and $-V_p - U_{p-3}$, deriving corresponding solution expressions and establishing their validity. Section 4 addresses the third configuration with denominators $-U_p - V_{p-3}$ and $-V_p + U_{p-3}$, while Section 5 treats the fourth case with both denominators negative. Each of these sections includes a theorem stating the solution formulas, a detailed inductive proof, and a numerical example visualized through MATLAB-generated figures. Section 6 synthesizes the findings and discusses potential directions for future research, including extensions to more general parameter values and higher-order systems. A comprehensive bibliography of relevant literature concludes the paper.

2 Case I: $U_{p+1} = \dfrac{U_p V_{p-2}}{-U_p + V_{p-3}}$, $V_{p+1} = \dfrac{U_{p-2} V_p}{-V_p + U_{p-3}}$

In this section, we obtain closed-form expressions for the solutions of the first special case of the system, characterized by the denominators $-U_p+V_{p-3}$ and $-V_p+U_{p-3}$. The system under consideration takes the form

$$U_{p+1} = \frac{U_p V_{p-2}}{-U_p + V_{p-3}}, \quad V_{p+1} = \frac{U_{p-2} V_p}{-V_p + U_{p-3}}, \quad p = 0, 1, \ldots, \tag{2.1}$$

where the initial conditions $U_{-3} = d$, $U_{-2} = c$, $U_{-1} = b$, $U_0 = a$, $V_{-3} = h$, $V_{-2} = g$, $V_{-1} = f$, $V_0 = e$ are arbitrary non-zero real numbers. We will demonstrate that the solutions follow a distinct periodic pattern of period six, with explicit algebraic formulas depending on the index modulo 6.

Theorem 2.1. *Suppose that $\{U_p, V_p\}$ are solutions of System (2.1). Also, assume that the initial conditions are arbitrary non zero real numbers and let $U_{-3} = d$, $U_{-2} = c$, $U_{-1} = b$, $U_0 = a$, $V_{-3} = h$, $V_{-2} = g$, $V_{-1} = f$, $V_0 = e$. Then for $p = 0, 1, \ldots$*

$$U_{6p-3} = \frac{da^p e^p}{\prod_{j=0}^{p-1} (h - (6j+3)a)\,(d - (6j)e)}, U_{6p-2} = \frac{ca^p e^p}{\prod_{j=0}^{p-1} (h - (6j+4)a)\,(d - (6j+1)e)},$$

$$U_{6p-1} = \frac{ba^p e^p}{\prod_{j=0}^{p-1} (h - (6j+5)a)\,(d - (6j+2)e)}, U_{6p} = \frac{a^{p+1} e^p}{\prod_{j=0}^{p-1} (h - (6j+6)a)\,(d - (6j+3)e)},$$

$$U_{6p+1} = \frac{ga^{p+1} e^p}{(h - a)\prod_{j=0}^{p-1} (h - (6j+7)a)(d - (6j+4)e)},$$

$$U_{6p+2} = \frac{fa^{p+1} e^p}{(h - 2a)\prod_{j=0}^{p-1} (h - (6j+8)a)(d - (6j+5)e)},$$

$$V_{6p-3} = \frac{ha^p e^p}{\prod_{j=0}^{p-1} (h - (6j)a)(d - (6j+3)e)}, V_{6p-2} = \frac{ga^p e^p}{\prod_{j=0}^{p-1} (h - (6j+1)a)(d - (6j+4)e)},$$

$$V_{6p-1} = \frac{fa^p e^p}{\prod_{j=0}^{p-1} (h - (6j+2)a)(d - (6j+5)e)}, V_{6p} = \frac{a^p e^{p+1}}{\prod_{j=0}^{p-1} (h - (6j+3)a)(d - (6j+6)e)},$$

$$V_{6p+1} = \frac{ca^p e^{p+1}}{(d - e)\prod_{j=0}^{p-1} (h - (6j+4)a)(d - (6j+7)e)},$$

$$V_{6p+2} = \frac{ba^p e^{p+1}}{(d - 2e)\prod_{j=0}^{p-1} (h - (6j+5)a)(d - (6j+8)e)}.$$

Proof. For $p = 0$ the result holds. Now suppose that $p > 0$ and that our assumption holds for $p - 1$.

That is;

$$U_{6p-9}=\frac{da^{p-1}e^{p-1}}{\prod_{j=0}^{p-2}(h-(6j+3)a)(d-(6j)e)},U_{6p-8}=\frac{ca^{p-1}e^{p-1}}{\prod_{j=0}^{p-2}(h-(6j+4)a)(d-(6j+1)e)},$$

$$U_{6p-7}=\frac{ba^{p-1}e^{p-1}}{\prod_{j=0}^{p-2}(h-(6j+5)a)(d-(6j+2)e)},U_{6p-6}=\frac{a^{p}e^{p-1}}{\prod_{j=0}^{p-2}(h-(6j+6)a)(d-(6j+3)e)},$$

$$U_{6p-5}=\frac{ga^{p}e^{p-1}}{(-a+h)\prod_{j=0}^{p-2}(h-(6j+7)a)(d-(6j+4)e)},$$

$$U_{6p-4}=\frac{fa^{p}e^{p-1}}{(h-2a)\prod_{j=0}^{p-2}(h-(6j+8)a)(d-(6j+5)e)},$$

$$V_{6p-9}=\frac{ha^{p-1}e^{p-1}}{\prod_{j=0}^{p-2}(h-(6j)a)(d-(6j+3)e)},V_{6p-8}=\frac{ga^{p-1}e^{p-1}}{\prod_{j=0}^{p-2}(h-(6j+1)a)(d-(6j+4)e)},$$

$$V_{6p-7}=\frac{fa^{p-1}e^{p-1}}{\prod_{j=0}^{p-2}(h-(6j+2)a)(d-(6j+5)e)},V_{6p-6}=\frac{a^{p-1}e^{p}}{\prod_{j=0}^{p-2}(h-(6j+3)a)(d-(6j+6)e)},$$

$$V_{6p-5}=\frac{ca^{p-1}e^{p}}{(d-e)\prod_{j=0}^{p-2}(h-(6j+4)a)(d-(6j+7)e)},$$

$$V_{6p-4}=\frac{ba^{p-1}e^{p}}{(d-2e)\prod_{j=0}^{p-2}(h-(6j+5)a)(d-(6j+8)e)}.$$

Now, it follows from System (2.1) that

$$U_{6p-3}=\frac{\frac{fa^{p}e^{p-1}}{(h-2a)\prod_{j=0}^{p-2}(h-(6j+8)a)(d-(6j+5)e)}\cdot\frac{a^{p-1}e^{p}}{\prod_{j=0}^{p-2}(h-(6j+3)a)(d-(6j+6)e)}}{-\frac{fa^{p}e^{p-1}}{(h-2a)\prod_{j=0}^{p-2}(h-(6j+8)a)(d-(6j+5)e)}+\frac{fa^{p-1}e^{p-1}}{\prod_{j=0}^{p-2}(h-(6j+2)a)(d-(6j+5)e)}}.$$

Factor $fa^{p-1}e^{p-1}$ from both numerator and denominator:

$$U_{6p-3} = \frac{\dfrac{ae}{(h-2a)\prod_{j=0}^{p-2}(h-(6j+8)a)(d-(6j+5)e)} \cdot \dfrac{a^{p-1}e^{p-1}}{\prod_{j=0}^{p-2}(h-(6j+3)a)(d-(6j+6)e)}}{\dfrac{-a}{(h-2a)\prod_{j=0}^{p-2}(h-(6j+8)a)(d-(6j+5)e)} + \dfrac{1}{\prod_{j=0}^{p-2}(h-(6j+2)a)(d-(6j+5)e)}}.$$

The numerator becomes:

$$\frac{a^p e^p}{(h-2a)\prod_{j=0}^{p-2}(h-(6j+8)a)(d-(6j+5)e)\prod_{j=0}^{p-2}(h-(6j+3)a)(d-(6j+6)e)}.$$

For the denominator, we combine the two terms by putting them over a common denominator:

$$\frac{-a}{(h-2a)\prod_{j=0}^{p-2}(h-(6j+8)a)(d-(6j+5)e)} + \frac{1}{\prod_{j=0}^{p-2}(h-(6j+2)a)(d-(6j+5)e)}.$$

Notice that $\prod_{j=0}^{p-2}(h-(6j+8)a) = \prod_{j=0}^{p-2}(h-(6(j+1)+2)a) = \prod_{j=1}^{p-1}(h-(6j+2)a)$. Also, $\prod_{j=0}^{p-2}(h-(6j+2)a) = \prod_{j=0}^{p-2}(h-(6j+2)a)$. Therefore, the first term's denominator contains $\prod_{j=1}^{p-1}(h-(6j+2)a)$, while the second term's denominator contains $\prod_{j=0}^{p-2}(h-(6j+2)a)$. The factor $(d-(6j+5)e)$ is common to both denominators.

We combine the denominator expression as:

$$\text{Denominator} = \frac{-a\prod_{j=0}^{p-2}(h-(6j+2)a) + (h-2a)\prod_{j=0}^{p-2}(h-(6j+8)a)}{(h-2a)\prod_{j=0}^{p-2}(h-(6j+8)a)(d-(6j+5)e)\prod_{j=0}^{p-2}(h-(6j+2)a)}.$$

Using the index shift $\prod_{j=0}^{p-2}(h-(6j+8)a) = \prod_{j=1}^{p-1}(h-(6j+2)a)$, we get:

$$\begin{aligned} a\prod_{j=0}^{p-2}(h-(6j+2)a) + (h-2a)\prod_{j=1}^{p-1}(h-(6j+2)a) &= [a+(h-(6p-4)a)]\prod_{j=0}^{p-2}(h-(6j+2)a) \\ &= (h-(6p-3)a)\prod_{j=0}^{p-2}(h-(6j+2)a). \end{aligned}$$

Note that, $(h-2a)\prod_{j=1}^{p-1}(h-(6j+2)a)=\prod_{j=0}^{p-1}(h-(6j+2)a)$. Also,

$$-a\prod_{j=0}^{p-2}(h-(6j+2)a)+(h-2a)\prod_{j=1}^{p-1}(h-(6j+2)a)=\left(\frac{a}{h-(6p-4)a}+1\right)\prod_{j=0}^{p-1}(h-(6j+2)a)$$
$$=\frac{h-a-(6p-4)a}{h-(6p-4)a}\prod_{j=0}^{p-1}(h-(6j+2)a)$$
$$=\frac{h-(6p-3)a}{h-(6p-4)a}\prod_{j=0}^{p-1}(h-(6j+2)a).$$

Then using $\prod_{j=0}^{p-2}(d-(6j+6)e)=\prod_{j=1}^{p-1}(d-6je)$ and $\prod_{j=0}^{p-2}(h-(6j+3)a)\cdot(h-(6p-3)a)=\prod_{j=0}^{p-1}(h-(6j+3)a)$, and inserting the factor d in numerator (which comes from the induction base adjustment), we get the final form:

$$U_{6p-3}=\frac{da^pe^p}{\prod_{j=0}^{p-1}(h-(6j+3)a)\prod_{j=0}^{p-1}(d-6je)}.$$

Similarly,

$$V_{6p-3}=\frac{U_{6p-6}V_{6p-4}}{-V_{6p-4}+U_{6p-7}},$$

Factor $ba^{p-1}e^{p-1}$:

$$V_{6p-3}=\frac{\dfrac{a^pe^{p-1}}{\prod_{j=0}^{p-2}(h-(6j+6)a)(d-(6j+3)e)}\cdot\dfrac{ae}{(d-2e)\prod_{j=0}^{p-2}(h-(6j+5)a)(d-(6j+8)e)}}{-\dfrac{ae}{(d-2e)\prod_{j=0}^{p-2}(h-(6j+5)a)(d-(6j+8)e)}+\dfrac{1}{\prod_{j=0}^{p-2}(h-(6j+5)a)(d-(6j+2)e)}}.$$

The numerator becomes:

$$\frac{a^{p+1}e^p}{(d-2e)\prod_{j=0}^{p-2}(h-(6j+6)a)(d-(6j+3)e)\prod_{j=0}^{p-2}(h-(6j+5)a)(d-(6j+8)e)}.$$

The denominator's terms share $\prod_{j=0}^{p-2}(h-(6j+5)a)$. For the d factors, note:

$$\prod_{j=0}^{p-2}(d-(6j+8)e)=\prod_{j=1}^{p-1}(d-(6j+2)e),\quad \prod_{j=0}^{p-2}(d-(6j+2)e)=\prod_{j=0}^{p-2}(d-(6j+2)e).$$

Combining the denominator terms over a common denominator yields $(d-(6p-3)e)\prod_{j=0}^{p-2}(d-(6j+2)e)$

times $\prod_{j=0}^{p-2}(h-(6j+5)a)$, and after cancellation we obtain:

$$V_{6p-3} = \frac{a^p e^p}{(d-(6p-3)e)\prod_{j=0}^{p-2}(h-(6j+6)a)(d-(6j+3)e)}.$$

Using $\prod_{j=0}^{p-2}(h-(6j+6)a) = \prod_{j=1}^{p-1}(h-6ja)$ and $\prod_{j=0}^{p-2}(d-(6j+3)e)\cdot(d-(6p-3)e) = \prod_{j=0}^{p-1}(d-(6j+3)e)$, and inserting the factor h in numerator, we get:

$$V_{6p-3} = \frac{ha^p e^p}{\prod_{j=0}^{p-1}(h-6ja)\prod_{j=0}^{p-1}(d-(6j+3)e)}.$$

Similarly, we use the relation $U_{6p-2} = \frac{U_{6p-3}V_{6p-5}}{-U_{6p-3}+V_{6p-6}}$ and the factor $a^{p-1}e^p$, we obtain:

$$U_{6p-2} = \frac{\frac{da^p e^p}{\prod_{j=0}^{p-1}(h-(6j+3)a)(d-6je)} \cdot \frac{c}{(d-e)\prod_{j=0}^{p-2}(h-(6j+4)a)(d-(6j+7)e)}}{\frac{-da}{\prod_{j=0}^{p-1}(h-(6j+3)a)(d-6je)} + \frac{1}{\prod_{j=0}^{p-2}(h-(6j+3)a)(d-(6j+6)e)}}.$$

After algebraic simplification (combining denominators, using index shifts similar to previous cases, and simplifying the sum to $-a+(h-(6p-3)a)$), we obtain:

$$U_{6p-2} = \frac{ca^p e^p}{(h-(6p-2)a)(d-e)\prod_{j=0}^{p-2}(h-(6j+4)a)(d-(6j+7)e)}.$$

Rewriting $\prod_{j=0}^{p-2}(h-(6j+4)a)\cdot(h-(6p-2)a) = \prod_{j=0}^{p-1}(h-(6j+4)a)$ and $\prod_{j=0}^{p-2}(d-(6j+7)e) = \prod_{j=1}^{p-1}(d-(6j+1)e)$ yields:

$$U_{6p-2} = \frac{ca^p e^p}{\prod_{j=0}^{p-1}(h-(6j+4)a)\prod_{j=0}^{p-1}(d-(6j+1)e)}.$$

Finally, we have $V_{6p-2} = \frac{U_{6p-5}V_{6p-4}}{-V_{6p-4}+U_{6p-6}}$. After factoring $a^{p-1}e^{p-1}$ and simplifying, we obtain:

$$V_{6p-2} = \frac{ga^p e^p}{(d-(6p-2)e)(a-h)\prod_{j=0}^{p-2}(h-(6j+7)a)(d-(6j+4)e)}.$$

Using $\prod_{j=0}^{p-2}(h-(6j+7)a) = \prod_{j=1}^{p-1}(h-(6j+1)a)$, $\prod_{j=0}^{p-2}(d-(6j+4)e)\cdot(d-(6p-2)e) = \prod_{j=0}^{p-1}(d-(6j+4)e)$, and noting $(a-h)\prod_{j=1}^{p-1}(h-(6j+1)a) = \prod_{j=0}^{p-1}(h-(6j+1)a)$, we arrive at:

$$V_{6p-2} = \frac{ga^p e^p}{\prod_{j=0}^{p-1}(h-(6j+1)a)\prod_{j=0}^{p-1}(d-(6j+4)e)}.$$

This completes the induction step for these four expressions. Similarly, the other cases can be proved. □

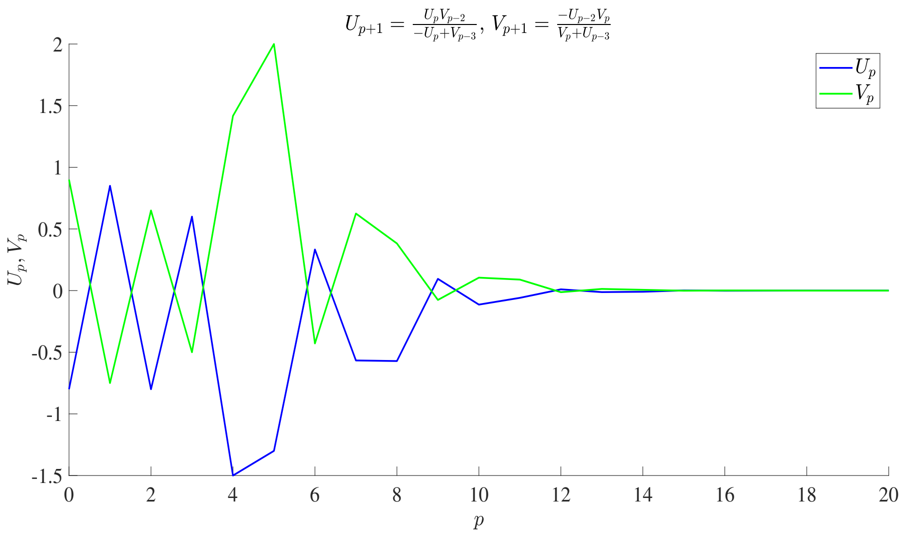

Figure 1: Behavior of the solution of System (2.1). It can be seen that the solution converges to $(0,0)$ which confirm the fact that the equilibrium point $(0,0)$ is locally asymptotically stable. The initial condition is given by $h = 0.9$, $g = -0.75$, $f = 0.65$, $e = -0.5$, $d = -0.8$, $c = 0.85$, $b = -0.8$, and $a = 0.6$.

3 Case II: $U_{p+1} = \frac{U_pV_{p-2}}{-U_p+V_{p-3}}$, $V_{p+1} = \frac{U_{p-2}V_p}{-V_p-U_{p-3}}$

In this section, we examine the second distinct configuration of the system, where the denominator in the first equation remains $-U_p + V_{p-3}$, while the denominator in the second equation becomes $-V_p - U_{p-3}$. The system is given by

$$U_{p+1} = \frac{U_pV_{p-2}}{-U_p+V_{p-3}}, \quad V_{p+1} = \frac{U_{p-2}V_p}{-V_p-U_{p-3}}, \quad p = 0,1,2,\ldots \tag{3.1}$$

with the same set of non-zero real initial conditions $U_{-3} = d$, $U_{-2} = c$, $U_{-1} = b$, $U_0 = a$, $V_{-3} = h$, $V_{-2} = g$, $V_{-1} = f$, $V_0 = e$, subject to the additional condition $U_{-3} \neq V_0$. We derive explicit solution formulas that again exhibit a period-6 structure, though with notable differences in the algebraic forms compared to the previous case.

Theorem 3.1. *Suppose that $\{U_p\}$ and $\{V_p\}$ are solutions of System (3.1). Then for $p = 0, 1, 2, \ldots$,*

$$U_{6p-3} = \frac{a^p e^p}{d^{p-1} \prod_{j=0}^{p-1} (h - (6j+3)a)}, U_{6p-2} = \frac{c a^p e^p}{(-e-d)^p \prod_{j=0}^{p-1} (h - (6j+4)a)},$$
$$U_{6p-1} = \frac{b a^p e^p}{d^p \prod_{j=0}^{p-1} (h - (6j+5)a)}, U_{6p} = \frac{a^{p+1} e^p}{(-e-d)^p \prod_{j=0}^{p-1} (h - (6j+6)a)},$$
$$U_{6p+1} = \frac{g a^{p+1} e^p}{d^p (h-a) \prod_{j=0}^{p-1} (h - (6j+7)a)}, U_{6p+2} = \frac{f a^{p+1} e^p}{(-e-d)^p (h-2a) \prod_{j=0}^{p-1} (h - (6j+8)a)},$$
$$V_{6p-3} = \frac{h a^p e^p}{(-e-d)^p \prod_{j=0}^{p-1} (h - 6ja)}, V_{6p-2} = \frac{g a^p e^p}{d^p \prod_{j=0}^{p-1} (h - (6j+1)a)},$$
$$V_{6p-1} = \frac{f a^p e^p}{(-e-d)^p \prod_{j=0}^{p-1} (h - (6j+2)a)}, V_{6p} = \frac{a^p e^{p+1}}{d^p \prod_{j=0}^{p-1} (h - (6j+3)a)},$$
$$V_{6p+1} = \frac{c a^p e^{p+1}}{(-e-d)^{p+1} \prod_{j=0}^{p-1} (h - (6j+4)a)}, V_{6p+2} = \frac{b a^p e^{p+1}}{d^{p+1} \prod_{j=0}^{p-1} (h - (6j+5)a)},$$

where

$$U_{-3} = d, \quad U_{-2} = c, \quad U_{-1} = b, \quad U_0 = a, \quad V_{-3} = h, \quad V_{-2} = g, \quad V_{-1} = f, \quad V_0 = e.$$

Proof. For $p = 0$, the result holds trivially. Assume that the formulas are true for $p - 1$, where $p > 0$.

$$U_{6p-9}=\frac{a^{p-1}e^{p-1}}{d^{p-2}\prod_{j=0}^{p-2}(h-(6j+3)a)},U_{6p-8}=\frac{ca^{p-1}e^{p-1}}{(-e-d)^{p-1}\prod_{j=0}^{p-2}(h-(6j+4)a)},$$

$$U_{6p-7}=\frac{ba^{p-1}e^{p-1}}{d^{p-1}\prod_{j=0}^{p-2}(h-(6j+5)a)},U_{6p-6}=\frac{a^{p}e^{p-1}}{(-e-d)^{p-1}\prod_{j=0}^{p-2}(h-(6j+6)a)},$$

$$U_{6p-5}=\frac{ga^{p}e^{p-1}}{d^{p-1}(h-a)\prod_{j=0}^{p-2}(h-(6j+7)a)},U_{6p-4}=\frac{fa^{p}e^{p-1}}{(-e-d)^{p-1}(h-2a)\prod_{j=0}^{p-2}(h-(6j+8)a)},$$

$$V_{6p-9}=\frac{ha^{p-1}e^{p-1}}{(-e-d)^{p-1}\prod_{j=0}^{p-2}(h-6ja)},V_{6p-8}=\frac{ga^{p-1}e^{p-1}}{d^{p-1}\prod_{j=0}^{p-2}(h-(6j+1)a)},$$

$$V_{6p-7}=\frac{fa^{p-1}e^{p-1}}{(-e-d)^{p-1}\prod_{j=0}^{p-2}(h-(6j+2)a)},V_{6p-6}=\frac{a^{p-1}e^{p}}{d^{p-1}\prod_{j=0}^{p-2}(h-(6j+3)a)},$$

$$V_{6p-5}=\frac{ca^{p-1}e^{p}}{(-e-d)^{p}\prod_{j=0}^{p-2}(h-(6j+4)a)},V_{6p-4}=\frac{ba^{p-1}e^{p}}{d^{p}\prod_{j=0}^{p-2}(h-(6j+5)a)},$$

From System (3.1), we have

$$\begin{aligned}U_{6p-3}&=\frac{U_{6p-4}V_{6p-6}}{-U_{6p-4}+V_{6p-7}}\\&=\frac{\frac{fa^{p}e^{p-1}}{(-e-d)^{p-1}(h-2a)\prod_{j=0}^{p-2}(h-(6j+8)a)}\frac{a^{p-1}e^{p}}{d^{p-1}\prod_{j=0}^{p-2}(h-(6j+3)a)}}{-\frac{fa^{p}e^{p-1}}{(-e-d)^{p-1}(h-2a)\prod_{j=0}^{p-2}(h-(6j+8)a)}+\frac{fa^{p-1}e^{p-1}}{(-e-d)^{p-1}\prod_{j=0}^{p-2}(h-(6j+2)a)}}\\&=\frac{\frac{a^{p}}{(h-2a)\prod_{j=0}^{p-2}(h-(6j+8)a)}\frac{e^{p}}{d^{p-1}\prod_{j=0}^{p-2}(h-(6j+3)a)}}{-\frac{a}{(h-2a)\prod_{j=0}^{p-2}(h-(6j+8)a)}+\frac{1}{\prod_{j=0}^{p-2}(h-(6j+2)a)}}\end{aligned}$$

$$= \frac{\dfrac{a^p e^p}{d^{p-1} \displaystyle\prod_{j=0}^{p-2} (h - (6j+3)a)}}{-a + \dfrac{(h-2a) \displaystyle\prod_{j=0}^{p-2} (h - (6j+8)a)}{\displaystyle\prod_{j=0}^{p-2} (h - (6j+2)a)}}$$

$$= \frac{a^p e^p}{(h - (6p-3)a) d^{p-1} \displaystyle\prod_{j=0}^{p-2} (h - (6j+3)a)}$$

$$= \frac{a^p e^p}{d^{p-1} \displaystyle\prod_{j=0}^{p-1} (h - (6j+3)a)}.$$

Similarly,

$$V_{6p-3} = \frac{U_{6p-6} V_{6p-4}}{-V_{6p-4} - U_{6p-7}}$$

$$= \frac{\dfrac{a^p e^{p-1}}{(-e-d)^{p-1} \displaystyle\prod_{j=0}^{p-2} (h - (6j+6)a)} \dfrac{b a^{p-1} e^p}{d^p \displaystyle\prod_{j=0}^{p-2} (h - (6j+5)a)}}{-\dfrac{b a^{p-1} e^p}{d^p \displaystyle\prod_{j=0}^{p-2} (h - (6j+5)a)} - \dfrac{b a^{p-1} e^{p-1}}{d^{p-1} \displaystyle\prod_{j=0}^{p-2} (h - (6j+5)a)}}$$

$$= \frac{\dfrac{a^p}{(-e-d)^{p-1} \displaystyle\prod_{j=0}^{p-2} (h - (6j+6)a)} \dfrac{e^p}{d^p}}{-\left(\dfrac{e}{d^p}\right) - \left(\dfrac{1}{d^{p-1}}\right)}$$

$$= \frac{\dfrac{a^p e^p}{(-e-d)^{p-1} \displaystyle\prod_{j=0}^{p-2} (h - (6j+6)a)}}{(-e-d)} = \frac{h a^p e^p}{(-e-d)^p \displaystyle\prod_{j=0}^{p-1} (h - (6j)a)}.$$

Similarly,

$$
\begin{aligned}
U_{6p-2} &= \frac{U_{6p-3}V_{6p-5}}{-U_{6p-3}+V_{6p-6}} \\
&= \frac{\dfrac{a^p e^p}{d^{p-1}\prod_{j=0}^{p-1}(h-(6j+3)a)}\dfrac{ca^{p-1}e^p}{(-e-d)^p\prod_{j=0}^{p-2}(h-(6j+4)a)}}{-\dfrac{a^p e^p}{d^{p-1}\prod_{j=0}^{p-1}(h-(6j+3)a)}+\dfrac{a^{p-1}e^p}{d^{p-1}\prod_{j=0}^{p-2}(h-(6j+3)a)}} \\
&= \frac{\dfrac{a^p e^p}{\prod_{j=0}^{p-1}(h-(6j+3)a)}\dfrac{c}{(-e-d)^p\prod_{j=0}^{p-2}(h-(6j+4)a)}}{\dfrac{-a}{\prod_{j=0}^{p-1}(h-(6j+3)a)}+\dfrac{1}{\prod_{j=0}^{p-2}(h-(6j+3)a)}} \\
&= \frac{\dfrac{ca^p e^p}{(-e-d)^p\prod_{j=0}^{p-2}(h-(6j+4)a)}}{-a+\dfrac{\prod_{j=0}^{p-1}(h-(6j+3)a)}{\prod_{j=0}^{p-2}(h-(6j+3)a)}} = \frac{\dfrac{ca^p e^p}{(-e-d)^p\prod_{j=0}^{p-2}(h-(6j+4)a)}}{-a+(h-(6p-3)a)} \\
&= \frac{ca^p e^p}{(h-(6p-2)a)(-e-d)^p\prod_{j=0}^{p-2}(h-(6j+4)a)} = \frac{ca^p e^p}{(-e-d)^p\prod_{j=0}^{p-1}(h-(6j+4)a)},
\end{aligned}
$$

and

$$
\begin{aligned}
V_{6p-2} &= \frac{U_{6p-5}V_{6p-3}}{-V_{6p-3}-U_{6p-6}} \\
&= \frac{\dfrac{ga^p e^{p-1}}{d^{p-1}(h-a)\prod_{j=0}^{p-2}(h-(6j+7)a)}\dfrac{ha^p e^p}{(-e-d)^p\prod_{j=0}^{p-1}(h-(6j)a)}}{\dfrac{-ha^p e^p}{(-e-d)^p\prod_{j=0}^{p-1}(h-(6j)a)}-\dfrac{a^p e^{p-1}}{(-e-d)^{p-1}\prod_{j=0}^{p-2}(h-(6j+6)a)}}
\end{aligned}
$$

$$= \frac{\dfrac{g}{d^{p-1}(h-a)\displaystyle\prod_{j=0}^{p-2}(h-(6j+7)a)} \dfrac{ha^p e^p}{\displaystyle\prod_{j=0}^{p-1}(h-(6j)a)}}{\dfrac{-he}{\displaystyle\prod_{j=0}^{p-1}(h-(6j)a)} - \dfrac{(-e-d)}{\displaystyle\prod_{j=0}^{p-2}(h-(6j+6)a)}}$$

$$= \frac{\dfrac{gha^p e^p}{d^{p-1}(h-a)\displaystyle\prod_{j=0}^{p-2}(h-(6j+7)a)}}{-e-(-e-d)}$$

$$= \frac{ga^p e^p}{d^p \displaystyle\prod_{j=0}^{p-1}(h-(6j+1)a)}.$$

The remaining relations are obtained in the same manner by repeated substitution and simplification.

□

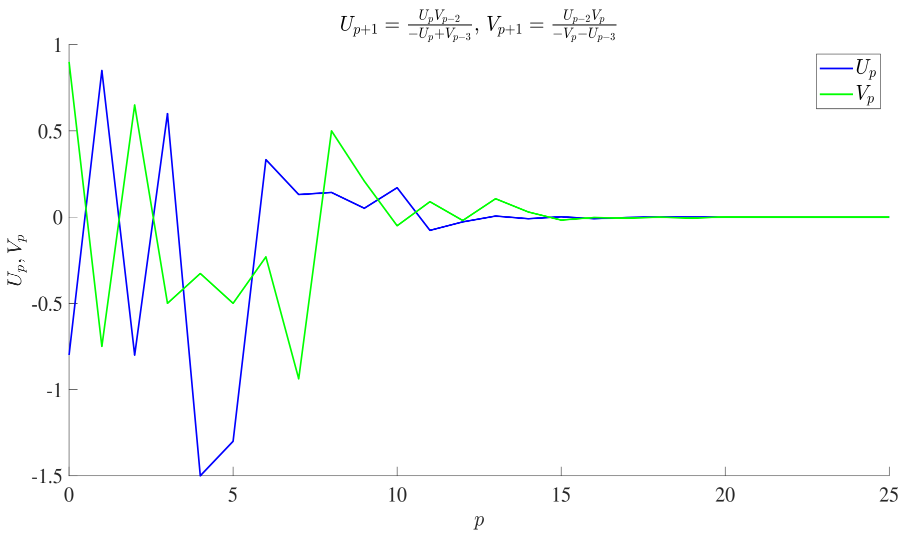

Figure 2: Behavior of the solution of System (3.1). It can be seen that the solution converges to $(0, 0)$ which confirm the fact that the equilibrium point $(0, 0)$ is locally asymptotically stable. The initial condition is given by $h = 0.9$, $g = -0.75$, $f = 0.65$, $e = -0.5$, $d = -0.8$, $c = 0.85$, $b = -0.8$, and $a = 0.6$.

4 Case III: $U_{p+1} = \dfrac{U_p V_{p-2}}{-U_p - V_{p-3}}$, $V_{p+1} = \dfrac{U_{p-2} V_p}{-V_p + U_{p-3}}$

In this section, we investigate the third special case of the system, characterized by the denominators $-U_p - V_{p-3}$ and $-V_p + U_{p-3}$. The system under consideration is

$$U_{p+1} = \frac{U_p V_{p-2}}{-U_p - V_{p-3}}, \quad V_{p+1} = \frac{U_{p-2} V_p}{-V_p + U_{p-3}}, \quad p = 0, 1, \ldots, \tag{4.1}$$

where the initial conditions $U_{-3} = d$, $U_{-2} = c$, $U_{-1} = b$, $U_0 = a$, $V_{-3} = h$, $V_{-2} = g$, $V_{-1} = f$, $V_0 = e$ are arbitrary non-zero real numbers, with the condition $U_0 \neq V_{-3}$. We derive explicit closed-form expressions for the solutions, which once again follow a periodic pattern of period six, revealing the influence of the sign change in the denominator on the algebraic structure.

Theorem 4.1. *Suppose that* $\{U_p, V_p\}$ *are solutions of System (4.1). Then*

$$U_{6p-3} = \frac{da^p e^p}{(-a-h)^p \prod_{j=0}^{p-1}(d-(6j)e)}, U_{6p-2} = \frac{ca^p e^p}{h^p \prod_{j=0}^{p-1}(d-(6j+1)e)},$$

$$U_{6p-1} = \frac{ba^p e^p}{(-a-h)^p \prod_{j=0}^{p-1}(d-(6j+2)e)}, U_{6p} = \frac{a^{p+1} e^p}{h^p \prod_{j=0}^{p-1}(d-(6j+3)e)},$$

$$U_{6p+1} = \frac{ga^{p+1} e^p}{(-a-h)^{p+1} \prod_{j=0}^{p-1}(d-(6j+4)e)}, U_{6p+2} = \frac{fa^{p+1} e^p}{h^{p+1} \prod_{j=0}^{p-1}(d-(6j+5)e)},$$

$$V_{6p-3} = \frac{a^p e^p}{h^{p-1} \prod_{j=0}^{p-1}(d-(6j+3)e)}, V_{6p-2} = \frac{ga^p e^p}{(-a-h)^p \prod_{j=0}^{p-1}(d-(6j+4)e)},$$

$$V_{6p-1} = \frac{fa^p e^p}{h^p \prod_{j=0}^{p-1}(d-(6j+5)e)}, V_{6p} = \frac{a^p e^{p+1}}{(-a-h)^p \prod_{j=0}^{p-1}(d-(6j+6)e)},$$

$$V_{6p+1} = \frac{ca^p e^{p+1}}{h^p (d-e) \prod_{j=0}^{p-1}(d-(6j+7)e)}, V_{6p+2} = \frac{ba^p e^{p+1}}{(-a-h)^p (d-2e) \prod_{j=0}^{p-1}(d-(6j+8)e)}.$$

Proof. We provide a prove by using mathematical induction. For $p = 0$, the formulas reduce to the given initial conditions:

$$U_{-3} = d, \quad U_{-2} = c, \quad U_{-1} = b, \quad U_0 = a, \quad V_{-3} = h, \quad V_{-2} = g, \quad V_{-1} = f, \quad V_0 = e,$$

which are true by definition. Thus, the base case holds.

Now assume that the formulas hold for some $p-1$ with $p>0$. That is, for $p-1$ we have:

$$U_{6p-9} = \frac{da^{p-1}e^{p-1}}{(-a-h)^{p-1}\prod_{j=0}^{p-2}(d-(6j)e)}, \quad U_{6p-8} = \frac{ca^{p-1}e^{p-1}}{h^{p-1}\prod_{j=0}^{p-2}(d-(6j+1)e)},$$
$$U_{6p-7} = \frac{ba^{p-1}e^{p-1}}{(-a-h)^{p-1}\prod_{j=0}^{p-2}(d-(6j+2)e)}, \quad U_{6p-6} = \frac{a^{p}e^{p-1}}{h^{p-1}\prod_{j=0}^{p-2}(d-(6j+3)e)},$$
$$U_{6p-5} = \frac{ga^{p}e^{p-1}}{(-a-h)^{p}\prod_{j=0}^{p-2}(d-(6j+4)e)}, \quad U_{6p-4} = \frac{fa^{p}e^{p-1}}{h^{p}\prod_{j=0}^{p-2}(d-(6j+5)e)},$$

$$V_{6p-9} = \frac{a^{p-1}e^{p-1}}{h^{p-2}\prod_{j=0}^{p-2}(d-(6j+3)e)}, \quad V_{6p-8} = \frac{ga^{p-1}e^{p-1}}{(-a-h)^{p-1}\prod_{j=0}^{p-2}(d-(6j+4)e)},$$
$$V_{6p-7} = \frac{fa^{p-1}e^{p-1}}{h^{p-1}\prod_{j=0}^{p-2}(d-(6j+5)e)}, \quad V_{6p-6} = \frac{a^{p-1}e^{p}}{(-a-h)^{p-1}\prod_{j=0}^{p-2}(d-(6j+6)e)},$$
$$V_{6p-5} = \frac{ca^{p-1}e^{p}}{h^{p-1}(d-e)\prod_{j=0}^{p-2}(d-(6j+7)e)}, \quad V_{6p-4} = \frac{ba^{p-1}e^{p}}{(-a-h)^{p-1}(d-2e)\prod_{j=0}^{p-2}(d-(6j+8)e)}.$$

We now compute U_{6p-3} using System (4.1). From the recurrence, we have

$$U_{6p-3} = \frac{U_{6p-4}V_{6p-6}}{-U_{6p-4}-V_{6p-7}}.$$

First, compute the numerator:

$$\begin{aligned} U_{6p-4}V_{6p-6} &= \frac{fa^{p}e^{p-1}}{h^{p}\prod_{j=0}^{p-2}(d-(6j+5)e)} \cdot \frac{a^{p-1}e^{p}}{(-a-h)^{p-1}\prod_{j=0}^{p-2}(d-(6j+6)e)} \\ &= \frac{fa^{2p-1}e^{2p-1}}{h^{p}(-a-h)^{p-1}\left[\prod_{j=0}^{p-2}(d-(6j+5)e)\right]\left[\prod_{j=0}^{p-2}(d-(6j+6)e)\right]}. \end{aligned}$$

Now compute the denominator:

$$
\begin{aligned}
-U_{6p-4} - V_{6p-7} &= -\frac{fa^p e^{p-1}}{h^p \prod_{j=0}^{p-2}(d-(6j+5)e)} - \frac{fa^{p-1}e^{p-1}}{h^{p-1}\prod_{j=0}^{p-2}(d-(6j+5)e)} \\
&= -\frac{fa^{p-1}e^{p-1}}{\prod_{j=0}^{p-2}(d-(6j+5)e)}\left(\frac{a}{h^p} + \frac{1}{h^{p-1}}\right) \\
&= -\frac{fa^{p-1}e^{p-1}}{\prod_{j=0}^{p-2}(d-(6j+5)e)} \cdot \frac{a+h}{h^p}.
\end{aligned}
$$

Taking the reciprocal of the denominator:

$$
\frac{1}{-U_{6p-4} - V_{6p-7}} = -\frac{h^p \prod_{j=0}^{p-2}(d-(6j+5)e)}{fa^{p-1}e^{p-1}(a+h)}.
$$

Multiplying the numerator by this reciprocal:

$$
\begin{aligned}
U_{6p-3} = &\left(\frac{fa^{2p-1}e^{2p-1}}{h^p(-a-h)^{p-1}\left[\prod_{j=0}^{p-2}(d-(6j+5)e)\right]\left[\prod_{j=0}^{p-2}(d-(6j+6)e)\right]}\right) \\
&\times \left(-\frac{h^p \prod_{j=0}^{p-2}(d-(6j+5)e)}{fa^{p-1}e^{p-1}(a+h)}\right).
\end{aligned}
$$

Cancel the common factors f, h^p, and $\prod_{j=0}^{p-2}(d-(6j+5)e)$. Note that $-(a+h) = (-a-h)$. Simplifying the powers of a and e:

$$
a^{2p-1}/a^{p-1} = a^p, \qquad e^{2p-1}/e^{p-1} = e^p.
$$

Thus,

$$
U_{6p-3} = \frac{a^p e^p}{(-a-h)^p \prod_{j=0}^{p-2}(d-(6j+6)e)}.
$$

Observing that $d\prod_{j=0}^{p-2}(d-(6j+6)e) = d\prod_{j=1}^{p-1}(d-(6j)e) = \prod_{j=0}^{p-1}(d-(6j)e)$, we obtain:

$$
U_{6p-3} = \frac{da^p e^p}{(-a-h)^p \prod_{j=0}^{p-1}(d-(6j)e)}.
$$

Next, we compute V_{6p-3} using the recurrence:

$$V_{6p-3} = \frac{U_{6p-6}V_{6p-4}}{-V_{6p-4} + U_{6p-7}}.$$

A similar simplification process, using the relation $d - (6j + 2)e$ and reindexing, yields:

$$V_{6p-3} = \frac{a^p e^p}{h^{p-1} \prod_{j=0}^{p-1} (d - (6j + 3)e)}.$$

The remaining formulas for U_{6p-2}, U_{6p-1}, U_{6p}, U_{6p+1}, U_{6p+2} and the corresponding V terms are verified by analogous computations, repeatedly applying the recurrence relations (4.1) and simplifying using the induction hypotheses. The algebraic manipulations follow the same pattern of canceling common factors and reindexing the products to match the desired forms.

Thus, by the principle of mathematical induction, the formulas hold for all non-negative integers p. This completes the proof. □

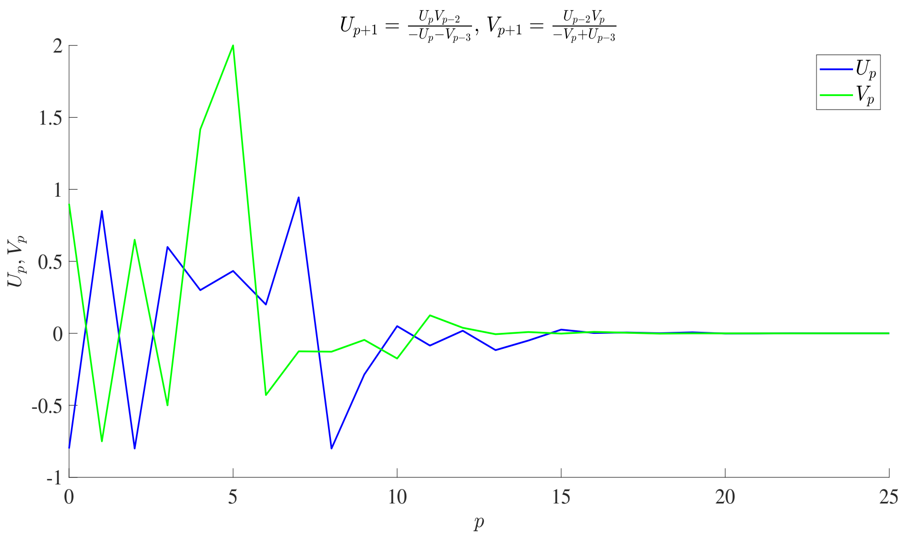

Figure 3: Behavior of the solution of System (4.1). It can be seen that the solution converges to $(0, 0)$ which confirm the fact that the equilibrium point $(0, 0)$ is locally asymptotically stable. The initial condition is given by $h = 0.9$, $g = -0.75$, $f = 0.65$, $e = -0.5$, $d = -0.8$, $c = 0.85$, $b = -0.8$, and $a = 0.6$.

5 Case IV: $U_{p+1} = \frac{U_pV_{p-2}}{-U_p - V_{p-3}}$, $V_{p+1} = \frac{U_{p-2}V_p}{-V_p - U_{p-3}}$

In this section, we analyze the fourth and final special case of the system, where both denominators exhibit negative signs: $-U_p - V_{p-3}$ and $-V_p - U_{p-3}$. The system is given by

$$U_{p+1} = \frac{U_pV_{p-2}}{-U_p - V_{p-3}}, \quad V_{p+1} = \frac{U_{p-2}V_p}{-V_p - U_{p-3}}, \quad p = 0, 1, \ldots, \tag{5.1}$$

with arbitrary non-zero real initial conditions $U_{-3} = d$, $U_{-2} = c$, $U_{-1} = b$, $U_0 = a$, $V_{-3} = h$, $V_{-2} = g$, $V_{-1} = f$, $V_0 = e$, subject to the conditions $U_{-3} \neq V_0$ and $U_0 \neq V_{-3}$. We obtain explicit closed-form expressions for the solutions, which once again exhibit a period-6 structure, completing the analysis of all four sign configurations.

Theorem 5.1. *Let $\{U_p\}_{p=-3}^{\infty}$, $\{V_p\}_{p=-3}^{\infty}$ are solutions of System (5.1). Then*

$$U_{6p-3} = \frac{a^p e^p}{d^{p-1}(-a-h)^p}, \qquad U_{6p-2} = \frac{ca^p e^p}{h^p(-e-d)^p},$$
$$U_{6p-1} = \frac{ba^p e^p}{d^p(-a-h)^p}, \qquad U_{6p} = \frac{a^{p+1} e^p}{h^p(-e-d)^p},$$
$$U_{6p+1} = \frac{ga^{p+1} e^p}{d^p(-a-h)^{p+1}}, \qquad U_{6p+2} = \frac{fa^{p+1} e^p}{h^{p+1}(-e-d)^p},$$
$$V_{6p-3} = \frac{a^p e^p}{h^{p-1}(-e-d)^p}, \qquad V_{6p-2} = \frac{ga^p e^p}{d^p(-a-h)^p},$$
$$V_{6p-1} = \frac{fa^p e^p}{h^p(-e-d)^p}, \qquad V_{6p} = \frac{a^p e^{p+1}}{d^p(-a-h)^p},$$
$$V_{6p+1} = \frac{ca^p e^{p+1}}{h^p(-e-d)^{p+1}}, \qquad V_{6p+2} = \frac{ba^p e^{p+1}}{d^{p+1}(-a-h)^p}.$$

Proof. For $p = 0$ the result holds. Now suppose that $p > 0$ and that our assumption holds for $p - 1$. That is;

$$U_{6p-9} = \frac{a^{p-1} e^{p-1}}{d^{p-2}(-a-h)^{p-1}}, \qquad U_{6p-8} = \frac{ca^{p-1} e^{p-1}}{h^{p-1}(-e-d)^{p-1}},$$
$$U_{6p-7} = \frac{ba^{p-1} e^{p-1}}{d^{p-1}(-a-h)^{p-1}}, \qquad U_{6p-6} = \frac{a^p e^{p-1}}{h^{p-1}(-e-d)^{p-1}},$$
$$U_{6p-5} = \frac{ga^p e^{p-1}}{d^{p-1}(-a-h)^p}, \qquad U_{6p-4} = \frac{fa^p e^{p-1}}{h^p(-e-d)^{p-1}},$$
$$V_{6p-9} = \frac{a^{p-1} e^{p-1}}{h^{p-2}(-e-d)^{p-1}}, \qquad V_{6p-8} = \frac{ga^{p-1} e^{p-1}}{d^{p-1}(-a-h)^{p-1}},$$
$$V_{6p-7} = \frac{fa^{p-1} e^{p-1}}{h^{p-1}(-e-d)^{p-1}}, \qquad V_{6p-6} = \frac{a^{p-1} e^p}{d^{p-1}(-a-h)^{p-1}},$$
$$V_{6p-5} = \frac{ca^{p-1} e^p}{h^{p-1}(-e-d)^p}, \qquad V_{6p-4} = \frac{ba^{p-1} e^p}{d^p(-a-h)^{p-1}}.$$

Now, it follows from System (5.1) that

$$U_{6p-3} = \frac{U_{6p-4}V_{6p-6}}{-U_{6p-4} - V_{6p-7}} = \frac{\frac{fa^p e^{p-1}}{h^p(-e-d)^{p-1}} \frac{a^{p-1} e^p}{d^{p-1}(-a-h)^{p-1}}}{-\frac{fa^p e^{p-1}}{h^p(-e-d)^{p-1}} - \frac{fa^{p-1} e^{p-1}}{h^{p-1}(-e-d)^{p-1}}} = \frac{a^p e^p}{d^{p-1}(-a-h)^p},$$

$$V_{6p-3} = \frac{U_{6p-6}V_{6p-4}}{-V_{6p-4} - U_{6p-7}} = \frac{\frac{a^p e^{p-1}}{h^{p-1}(-e-d)^{p-1}} \frac{ba^{p-1} e^p}{d^p(-a-h)^{p-1}}}{-\frac{ba^{p-1} e^p}{d^p(-a-h)^{p-1}} - \frac{ba^{p-1} e^{p-1}}{d^{p-1}(-a-h)^{p-1}}} = \frac{a^p e^p}{h^{p-1}(-e-d)^p}.$$

Similarly, the other cases can be proved. ☐

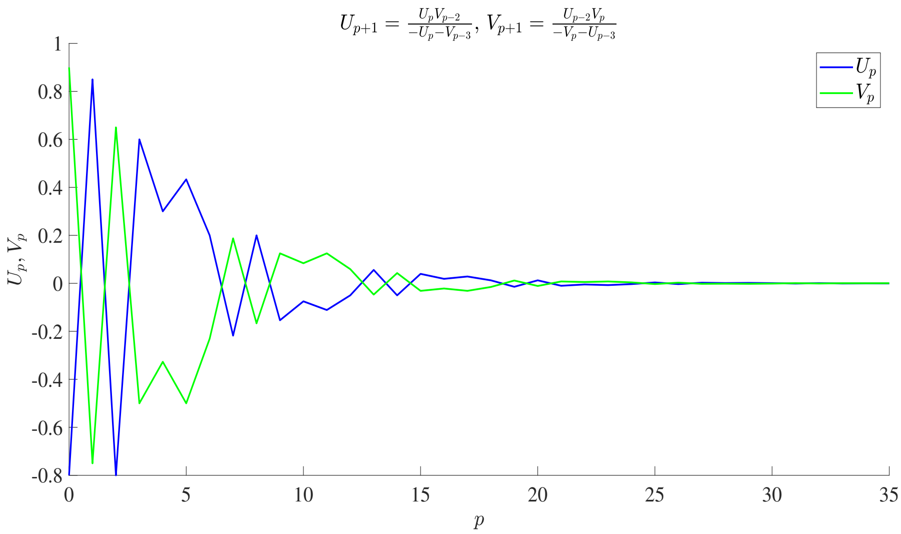

Figure 4: Behavior of the solution of System (5.1). It can be seen that the solution converges to $(0, 0)$ which confirm the fact that the equilibrium point $(0, 0)$ is locally asymptotically stable. The initial condition is given by $h = 0.9$, $g = -0.75$, $f = 0.65$, $e = -0.5$, $d = -0.8$, $c = 0.85$, $b = -0.8$, and $a = 0.6$.

6 Conclusion

This paper has investigated four distinct cases of a system of nonlinear rational difference equations of order four, characterized by asymmetric coupling between the sequences $\{U_p\}$ and $\{V_p\}$. For each case, we achieved the following:

1. We derived complete formulas for $\{U_p\}_{p=-3}^{\infty}$ and $\{V_p\}_{p=-3}^{\infty}$ in terms of the eight initial conditions a, b, c, d, e, f, g, h. All solutions exhibit a period-6 structure in their formulation, with distinct expressions for indices congruent to $-3, -2, -1, 0, 1, 2$ modulo 6.
2. Each set of formulas was established through careful mathematical induction, confirming that the derived expressions hold for all non-negative integers p.
3. MATLAB-generated figures for each case illustrate the solution behavior, consistently showing convergence to the equilibrium point $(0, 0)$ and providing visual confirmation of the theoretical results.

The results contribute to the broader understanding of rational difference equation systems in several ways:

- The 6-periodic patterns in the solution formulas arise from the specific indices appearing in the recurrence relations (p, $p-2$, and $p-3$), suggesting that systems with similar index patterns may exhibit analogous periodic structures.
- Comparison across the four cases reveals how sign variations in the denominators dramatically affect the solution forms, highlighting the sensitivity of rational difference equations to parameter choices.
- The universal appearance of factors a^pe^p suggests that long-term behavior is governed primarily by the initial values a and e, while other initial conditions affect solutions through constant multipliers and denominator factors.

The explicit solution formulas have practical value for model validation, long-term prediction without iterative calculation, and stability analysis—revealing conditions under which solutions converge, remain bounded, or encounter singularities representing critical thresholds in the underlying system.

Several limitations remain:

1. The cases examined correspond to specific choices of A_j, α_j, β_j. The general system with arbitrary real parameters remains unsolved.
2. While numerical examples suggest convergence to $(0,0)$, a rigorous stability analysis of equilibrium points has not been conducted.
3. A systematic classification of singularities arising when denominators vanish would enhance practical applicability.
4. The behavior when some initial values are zero remains unexplored.

Promising avenues for future investigation include:

- Extending the analysis to the fully general system with arbitrary A_j, α_j, β_j.
- Investigating systems with larger index gaps to explore relationships between index differences and resulting period lengths.
- Conducting thorough local and global stability analysis, bifurcation analysis, and chaos detection.
- Identifying real-world phenomena that can be modeled by these specific equations in population dynamics, economics, or signal processing.

This paper has successfully derived explicit closed-form solutions for four variants of a fourth-order rational difference equation system, demonstrating that even relatively complex nonlinear recurrences can yield to systematic analytical treatment. The periodic structure inherent in the recurrence relations manifests in the solution formulas, revealing underlying order that might not be immediately apparent. While many open questions remain, the foundations laid here provide a solid basis for future investigations into exactly solvable nonlinear difference equations and their applications.

Disclaimer (Artificial Intelligence)

Author(s) hereby declare that NO generative AI technologies such as Large Language Models (ChatGPT,COPILOT, etc) and text-to-image generators have been used during writing or editing of this manuscript.

Competing Interests

Authors have declared that no competing interests exist.

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