# Original Research Article
# Wang-Type Fixed Point Theorems for Expansive Maps in Hemi-Metric Spaces

**Abstract**

This paper develops new fixed point results for expansive mappings within the framework of hemi-metric spaces. We show that every surjective expansive mapping defined on an $h$-complete hemi-metric space possesses a unique fixed point. A corresponding local version is also established on invariant $h$-complete subsets. The proposed results extend classical expansion-type fixed point theory from pairwise distance settings to multi-point $(m+1)$-dimensional structures. The approach is based on an inverse contraction technique, which converts an expansive mapping into a contractive inverse mapping. Several examples are included to demonstrate the effectiveness of the theory. In addition, an application to a Fredholm integral equation is presented, illustrating the usefulness of the obtained results in functional analysis.These findings provide a natural generalization of existing results in generalized metric spaces.

**Keywords:** Fixed point theory; expansive mappings; hemi-metric space; inverse contraction; integral equations.

**MSC (2020):** 47H10, 54E35, 54H25, 45B05.

## 1 Introduction

Fixed point theory is a core area of nonlinear analysis with wide-ranging applications in differential equations, optimization, economics, and dynamical systems [7, 33]. A fixed point of a mapping is an element that remains unchanged under the mapping, and the existence of such points often plays a crucial role in ensuring the solvability and stability of mathematical models.

A fundamental result in this area is the Banach contraction principle, which establishes the existence and uniqueness of fixed points in complete metric spaces.

**Theorem 1.1** (Banach Contraction Principle [7]). *Let $(\mathcal{X}, \delta)$ be a complete metric space and let $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ be a mapping satisfying*

$$\delta(\mathcal{T}\xi, \mathcal{T}\eta) \leq k\, \delta(\xi, \eta), \quad \textit{for all } \xi, \eta \in \mathcal{X},$$

*where $0 < k < 1$. Then $\mathcal{T}$ admits a unique fixed point $\xi^* \in \mathcal{X}$.*

Over the years, this principle has been generalized in various directions, including partially ordered metric spaces, nonlinear contractive conditions, and hybrid mappings [11, 15, 16, 24]. More recently, approaches involving simulation functions, control functions, and $\alpha$-admissible mappings have attracted considerable attention [18, 23, 28, 32, 36].

In contrast, expansive mappings act by enlarging distances rather than contracting them. Although such mappings do not directly fit into the classical framework, fixed point results can still be derived under suitable conditions. The systematic study of expansive mappings was initiated by Wang et al. [6], and further developments were carried out by Daffer and Kaneko [5], Rhoades [8], and Ćirić [12].

A representative result in this setting is the following.

**Theorem 1.2** (Wang-type Expansive Mapping Principle [6]). *Let $(\mathcal{X}, \delta)$ be a complete metric space and let $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ be a surjective mapping such that*

$$\delta(\mathcal{T}\xi, \mathcal{T}\eta) \geq a\,\delta(\xi, \eta),$$

*for some constant $a > 1$. Then $\mathcal{T}$ has a unique fixed point.*

Alongside these developments, various generalizations of metric spaces have been introduced to extend the applicability of fixed point theory. Early examples include generalized metric structures due to Dhage [3, 4] and Gähler [1, 2]. Subsequently, Mustafa and Sims [10] proposed $G$-metric spaces, while other frameworks such as dislocated and parametric metric spaces were investigated in [14, 17, 25].

More recently, attention has shifted toward generalized metric-type spaces including partial metric spaces and $b$-metric spaces, where both expansive and nonlinear fixed point results have been established [13, 21, 30, 29]. In addition, unified approaches based on control functions and generalized contractions have been developed in [20, 22, 27, 26, 31, 35].

Despite these advances, most existing theories are formulated in terms of pairwise distances. However, many practical problems involve interactions among several elements simultaneously, which cannot be adequately captured by two-point distance functions. This observation motivates the introduction of hemi-metric spaces, where distance is defined on $(m + 1)$-tuples.

Hemi-metric spaces have recently been introduced as a natural extension of classical metric structures. In particular, Ozturk and Radenovic [34] established foundational results in this setting, demonstrating their suitability for fixed point analysis involving multi-point interactions.

Nevertheless, the theory of expansive mappings in hemi-metric spaces remains largely undeveloped. In particular, extending Wang-type expansive principles to this multi-point framework presents a nontrivial challenge.

The purpose of this paper is to address this issue. By employing an inverse contraction approach—where the inverse of an expansive mapping behaves as a contractive mapping—we establish both global and local fixed point results in hemi-metric spaces. In addition, we develop a nonlinear extension based on control functions, thereby significantly expanding the scope of the theory.

## 2 Preliminaries

In this section, we recall the essential notions required for our study. Hemi-metric spaces extend the classical concept of metric spaces by introducing distance functions depending on multiple points rather than pairs. The following definitions are based on recent developments in the literature (see, e.g., [34, 31, 35]).

**Definition 2.1** ([34]). *Let $\mathcal{H}$ be a nonempty set and let $m \in \mathbb{N}$. A function*

$$\delta_h : \mathcal{H}^{m+1} \to [0, \infty)$$

*is called an* $m$-hemi-metric *on* $\mathcal{H}$ *if, for any* $\xi_1, \xi_2, \dots, \xi_{m+1} \in \mathcal{H}$*, the following conditions hold:*

*(i)* $\delta_h(\xi_1, \dots, \xi_{m+1}) \geq 0$*;*

*(ii)* $\delta_h(\xi, \dots, \xi) = 0$ *for every* $\xi \in \mathcal{H}$*, and* $\delta_h(\xi_1, \dots, \xi_{m+1}) = 0$ *implies* $\xi_1 = \xi_2 = \cdots = \xi_{m+1}$*;*

*(iii)* $\delta_h$ *remains unchanged under any permutation of its arguments;*

*(iv) for all* $\xi_1, \dots, \xi_{m+1}, \eta \in \mathcal{H}$*,*

$$\delta_h(\xi_1, \dots, \xi_{m+1}) \leq \sum_{i=1}^{m+1} \delta_h(\xi_1, \dots, \xi_{i-1}, \eta, \xi_{i+1}, \dots, \xi_{m+1}).$$

*The pair* $(\mathcal{H}, \delta_h)$ *is referred to as a hemi-metric space.*

**Remark 2.2.** *Unlike standard metric spaces, where distance is defined between two points, a hemi-metric captures the joint dispersion of* $(m+1)$ *elements. This multi-point perspective is useful in modeling situations involving collective interactions, such as multi-agent systems or higher-dimensional dependencies.*

**Definition 2.3** ([34])**.** *A sequence* $(\xi_n) \subset \mathcal{H}$ *is said to converge to a point* $\xi \in \mathcal{H}$ *if*

$$\delta_h(\xi_n, \xi, \xi, \dots, \xi) \to 0 \quad \text{as } n \to \infty.$$

**Definition 2.4** ([34])**.** *A sequence* $(\xi_n) \subset \mathcal{H}$ *is called* hemi-Cauchy *if for every* $\varepsilon > 0$*, there exists* $N \in \mathbb{N}$ *such that*

$$\delta_h(\xi_{n_1}, \xi_{n_2}, \dots, \xi_{n_{m+1}}) < \varepsilon$$

*whenever* $n_1, n_2, \dots, n_{m+1} \geq N$*.*

**Definition 2.5** ([34])**.** *A hemi-metric space* $(\mathcal{H}, \delta_h)$ *is said to be* $h$-complete *if every hemi-Cauchy sequence converges to an element of* $\mathcal{H}$*.*

**Definition 2.6.** *Let* $(\mathcal{H}, \delta_h)$ *be a hemi-metric space. A point* $\xi \in \mathcal{H}$ *is called a* fixed point *of a mapping* $\mathcal{F} : \mathcal{H} \to \mathcal{H}$ *if*

$$\mathcal{F}(\xi) = \xi.$$

**Definition 2.7.** *A mapping* $\mathcal{F} : \mathcal{H} \to \mathcal{H}$ *is said to be* sequentially continuous *if* $\xi_n \to \xi$ *implies* $\mathcal{F}(\xi_n) \to \mathcal{F}(\xi)$*.*

**Definition 2.8.** *A mapping* $\mathcal{F} : \mathcal{H} \to \mathcal{H}$ *is called* surjective *if for every* $\eta \in \mathcal{H}$ *there exists* $\xi \in \mathcal{H}$ *such that* $\mathcal{F}(\xi) = \eta$*.*

## 3 Examples of Hemi-Metric Spaces

In this section, we provide several illustrative examples of hemi-metric spaces in order to demonstrate the flexibility and general applicability of the concept.

**Example 3.1.** *Let* $\mathcal{H} = \mathbb{R}$ *and define*

$$\delta_h(\xi_1, \dots, \xi_{m+1}) = \sum_{1 \leq i < j \leq m+1} |\xi_i - \xi_j|.$$

*Then* $(\mathcal{H}, \delta_h)$ *is a hemi-metric space.*

*Proof.* The function $\delta_h$ is nonnegative since it is a finite sum of absolute values. If $\xi_1 = \cdots = \xi_{m+1} = \xi$, then each term vanishes and hence $\delta_h(\xi, \dots, \xi) = 0$. Conversely, if $\delta_h(\xi_1, \dots, \xi_{m+1}) = 0$, then all pairwise differences are zero, which implies $\xi_1 = \cdots = \xi_{m+1}$.

The symmetry of $\delta_h$ follows immediately from the fact that it depends only on pairwise differences, which are invariant under permutations.

Let $\eta \in \mathcal{H}$. By the triangle inequality,

$$|\xi_i - \xi_j| \le |\xi_i - \eta| + |\eta - \xi_j| \quad \text{for all } i < j.$$

Summing over all pairs yields

$$\delta_h(\xi_1, \dots, \xi_{m+1}) \le \sum_{1 \le i < j \le m+1} \big( |\xi_i - \eta| + |\eta - \xi_j| \big).$$

Each term $|\xi_k - \eta|$ appears exactly $m$ times in the sum, and therefore

$$\delta_h(\xi_1, \dots, \xi_{m+1}) \le m \sum_{k=1}^{m+1} |\xi_k - \eta|.$$

Now, for each index $i$, consider replacing $\xi_i$ by $\eta$. The corresponding value of $\delta_h$ contains all terms $|\eta - \xi_k|$ for $k \neq i$, hence

$$\delta_h(\xi_1, \dots, \xi_{i-1}, \eta, \xi_{i+1}, \dots, \xi_{m+1}) \ge \sum_{k \neq i} |\xi_k - \eta|.$$

Summing over $i = 1, \dots, m+1$, each $|\xi_k - \eta|$ again appears exactly $m$ times, so that

$$\sum_{i=1}^{m+1} \delta_h(\xi_1, \dots, \xi_{i-1}, \eta, \xi_{i+1}, \dots, \xi_{m+1}) \ge m \sum_{k=1}^{m+1} |\xi_k - \eta|.$$

Combining the above inequalities gives

$$\delta_h(\xi_1, \dots, \xi_{m+1}) \le \sum_{i=1}^{m+1} \delta_h(\xi_1, \dots, \xi_{i-1}, \eta, \xi_{i+1}, \dots, \xi_{m+1}),$$

which verifies the required condition. Hence $(\mathcal{H}, \delta_h)$ is a hemi-metric space. □

**Example 3.2.** *Let $\mathcal{H} = \mathbb{R}^n$ and define*

$$\delta_h(\xi_1, \dots, \xi_{m+1}) = \max_{1 \le i,j \le m+1} \|\xi_i - \xi_j\|,$$

*where $\|\cdot\|$ denotes the Euclidean norm. Then $(\mathcal{H}, \delta_h)$ is a hemi-metric space.*

*Proof.* The mapping $\delta_h$ is clearly nonnegative. If all points coincide, then $\delta_h = 0$, and conversely, if $\delta_h = 0$, then $\|\xi_i - \xi_j\| = 0$ for all $i, j$, which implies that all points are equal. Permutation invariance follows directly from the definition.

Let $\eta \in \mathcal{H}$. Using the triangle inequality,

$$\|\xi_i - \xi_j\| \le \|\xi_i - \eta\| + \|\eta - \xi_j\|.$$

Taking the maximum over all indices $i, j$, we obtain

$$\delta_h(\xi_1, \dots, \xi_{m+1}) \le 2 \max_k \|\xi_k - \eta\|.$$

For each $i$, the quantity $\delta_h(\xi_1, \ldots, \eta, \ldots, \xi_{m+1})$ is at least $\max_k \|\xi_k - \eta\|$. Summing over $i = 1, \ldots, m+1$, we get

$$\sum_{i=1}^{m+1} \delta_h(\xi_1, \ldots, \eta, \ldots, \xi_{m+1}) \geq (m+1) \max_k \|\xi_k - \eta\|.$$

Since $m + 1 \geq 2$, it follows that

$$\delta_h(\xi_1, \ldots, \xi_{m+1}) \leq \sum_{i=1}^{m+1} \delta_h(\xi_1, \ldots, \eta, \ldots, \xi_{m+1}),$$

which establishes the required inequality. Therefore, $(\mathcal{H}, \delta_h)$ is a hemi-metric space. □

**Example 3.3.** *Let $\mathcal{H} = \mathbb{R}$ and define*

$$\delta_h(\xi_1, \ldots, \xi_{m+1}) = \sum_{i=1}^{m+1} \left|\xi_i - \bar{\xi}\right|,$$

*where*

$$\bar{\xi} = \frac{1}{m+1} \sum_{k=1}^{m+1} \xi_k$$

*denotes the arithmetic mean of the $(m+1)$-tuple. Then $(\mathcal{H}, \delta_h)$ is a hemi-metric space.*

*Proof.* The mapping $\delta_h$ is clearly nonnegative since it is a finite sum of absolute values. If all components coincide, say $\xi_1 = \cdots = \xi_{m+1} = \xi$, then the mean satisfies $\bar{\xi} = \xi$, and hence each term $|\xi_i - \bar{\xi}| = 0$, which implies $\delta_h(\xi, \ldots, \xi) = 0$. Conversely, if $\delta_h(\xi_1, \ldots, \xi_{m+1}) = 0$, then each term $|\xi_i - \bar{\xi}| = 0$, so $\xi_i = \bar{\xi}$ for all $i$, and therefore $\xi_1 = \cdots = \xi_{m+1}$.

The function $\delta_h$ is invariant under permutations, since the mean $\bar{\xi}$ is symmetric in its arguments and the sum of absolute deviations does not depend on the order of the elements.

Let $\eta \in \mathcal{H}$ and consider replacing one component $\xi_i$ by $\eta$. Denote by $\bar{\xi}^{(i)}$ the mean of the modified tuple. By properties of averages, we have

$$\bar{\xi}^{(i)} = \bar{\xi} + \frac{\eta - \xi_i}{m+1}.$$

Using the triangle inequality, each term satisfies

$$|\xi_j - \bar{\xi}| \leq |\xi_j - \bar{\xi}^{(i)}| + |\bar{\xi}^{(i)} - \bar{\xi}|.$$

Summing over all indices and using the fact that $|\bar{\xi}^{(i)} - \bar{\xi}| = \frac{|\eta - \xi_i|}{m+1}$, we obtain

$$\delta_h(\xi_1, \ldots, \xi_{m+1}) \leq \delta_h(\xi_1, \ldots, \xi_{i-1}, \eta, \xi_{i+1}, \ldots, \xi_{m+1}) + |\eta - \xi_i|.$$

Summing this inequality over all $i = 1, \ldots, m+1$ yields

$$\delta_h(\xi_1, \ldots, \xi_{m+1}) \leq \sum_{i=1}^{m+1} \delta_h(\xi_1, \ldots, \xi_{i-1}, \eta, \xi_{i+1}, \ldots, \xi_{m+1}),$$

which verifies the generalized triangle inequality. Hence $(\mathcal{H}, \delta_h)$ is a hemi-metric space. □

## 4 Main Results

In this section, we introduce the notion of expansive mappings in hemi-metric spaces and establish global and local fixed point results.

**Definition 4.1.** *Let $(\mathcal{H}, \delta_h)$ be a hemi-metric space. A mapping $\mathcal{F} : \mathcal{H} \to \mathcal{H}$ is said to be* expansive *if there exists a constant $a > 1$ such that*

$$\delta_h\big(\mathcal{F}(\xi_1), \dots, \mathcal{F}(\xi_{m+1})\big) \geq a\, \delta_h(\xi_1, \dots, \xi_{m+1})$$

*for all $(\xi_1, \dots, \xi_{m+1}) \in \mathcal{H}^{m+1}$.*

**Theorem 4.2.** *Let $(\mathcal{H}, \delta_h)$ be an $h$-complete hemi-metric space of order $m$, and let $\mathcal{F} : \mathcal{H} \to \mathcal{H}$ be a surjective expansive mapping. Then $\mathcal{F}$ has a unique fixed point in $\mathcal{H}$.*

*Proof.* We begin by proving that $\mathcal{F}$ is injective. Suppose that $\mathcal{F}(\xi) = \mathcal{F}(\eta)$ for some $\xi, \eta \in \mathcal{H}$. Consider the $(m+1)$-tuple $(\xi, \dots, \xi, \eta)$. Then

$$\delta_h(\mathcal{F}(\xi), \dots, \mathcal{F}(\xi), \mathcal{F}(\eta)) = 0.$$

By the expansivity condition, we obtain

$$0 \geq a\, \delta_h(\xi, \dots, \xi, \eta).$$

Since $a > 1$ and $\delta_h \geq 0$, it follows that $\delta_h(\xi, \dots, \xi, \eta) = 0$, and hence $\xi = \eta$. Therefore, $\mathcal{F}$ is injective. Combined with surjectivity, this shows that $\mathcal{F}$ is bijective.

Let $\mathcal{G} = \mathcal{F}^{-1}$. For any $(\zeta_1, \dots, \zeta_{m+1}) \in \mathcal{H}^{m+1}$, define $\xi_i = \mathcal{G}(\zeta_i)$. Then

$$\delta_h(\zeta_1, \dots, \zeta_{m+1}) = \delta_h(\mathcal{F}(\xi_1), \dots, \mathcal{F}(\xi_{m+1})) \geq a\, \delta_h(\xi_1, \dots, \xi_{m+1}),$$

which implies

$$\delta_h(\mathcal{G}(\zeta_1), \dots, \mathcal{G}(\zeta_{m+1})) \leq \frac{1}{a}\, \delta_h(\zeta_1, \dots, \zeta_{m+1}).$$

Thus $\mathcal{G}$ is a contraction with constant $k = \frac{1}{a} \in (0, 1)$.

Choose an arbitrary $\xi_0 \in \mathcal{H}$ and define an iterative sequence by $\xi_{n+1} = \mathcal{G}(\xi_n)$. Repeated application of the above inequality yields

$$\delta_h(\xi_n, \xi_{n+1}, \dots, \xi_{n+p}) \leq k^n\, \delta_h(\xi_0, \xi_1, \dots, \xi_p),$$

for all $p \geq 1$. Hence $(\xi_n)$ is a hemi-Cauchy sequence.

Since $(\mathcal{H}, \delta_h)$ is $h$-complete, there exists $\xi^* \in \mathcal{H}$ such that $\xi_n \to \xi^*$. Passing to the limit in $\xi_{n+1} = \mathcal{G}(\xi_n)$, we obtain $\mathcal{G}(\xi^*) = \xi^*$, which implies $\mathcal{F}(\xi^*) = \xi^*$. Thus $\xi^*$ is a fixed point of $\mathcal{F}$.

To prove uniqueness, let $\eta^* \in \mathcal{H}$ be another fixed point. Then

$$\delta_h(\xi^*, \eta^*, \dots) = \delta_h(\mathcal{F}(\xi^*), \mathcal{F}(\eta^*), \dots) \geq a\, \delta_h(\xi^*, \eta^*, \dots).$$

Since $a > 1$, this is possible only if $\delta_h(\xi^*, \eta^*, \dots) = 0$, which implies $\xi^* = \eta^*$. □

**Remark 4.3.** *The preceding theorem generalizes the classical Wang-type expansive mapping principle to the setting of hemi-metric spaces, where distance is defined on $(m+1)$-tuples. The key idea is the use of an inverse contraction approach, which transforms an expansive mapping into a contractive one, allowing the application of iterative techniques in a multi-point framework.*

**Theorem 4.4.** *Let $(\mathcal{H}, \delta_h)$ be a hemi-metric space and let $\mathcal{K} \subset \mathcal{H}$ be a nonempty subset satisfying $\mathcal{F}(\mathcal{K}) \subset \mathcal{K}$. Assume that:*

*(i) there exists $a > 1$ such that*

$$\delta_h\big(\mathcal{F}(\xi_1), \dots, \mathcal{F}(\xi_{m+1})\big) \geq a\, \delta_h(\xi_1, \dots, \xi_{m+1})$$

*for all $(\xi_1, \dots, \xi_{m+1}) \in \mathcal{K}^{m+1}$;*

*(ii) $\mathcal{F} : \mathcal{K} \to \mathcal{K}$ is surjective;*

*(iii) $(\mathcal{K}, \delta_h)$ is h-complete.*

*Then $\mathcal{F}$ admits a unique fixed point in $\mathcal{K}$.*

*Proof.* We first show that $\mathcal{F}$ is injective on $\mathcal{K}$. Suppose $\mathcal{F}(\xi) = \mathcal{F}(\eta)$ for some $\xi, \eta \in \mathcal{K}$. Consider the $(m+1)$-tuple $(\xi, \dots, \xi, \eta)$. Then

$$\delta_h(\mathcal{F}(\xi), \dots, \mathcal{F}(\xi), \mathcal{F}(\eta)) = 0.$$

Using condition (i), we obtain

$$0 \geq a\, \delta_h(\xi, \dots, \xi, \eta).$$

Since $a > 1$ and $\delta_h \geq 0$, it follows that $\delta_h(\xi, \dots, \xi, \eta) = 0$, and hence $\xi = \eta$. Thus $\mathcal{F}$ is injective. Together with surjectivity, this shows that $\mathcal{F}$ is bijective on $\mathcal{K}$.

Let $\mathcal{G} = \mathcal{F}^{-1} : \mathcal{K} \to \mathcal{K}$. For any $(\zeta_1, \dots, \zeta_{m+1}) \in \mathcal{K}^{m+1}$, set $\xi_i = \mathcal{G}(\zeta_i)$. Then

$$\delta_h(\zeta_1, \dots, \zeta_{m+1}) = \delta_h(\mathcal{F}(\xi_1), \dots, \mathcal{F}(\xi_{m+1})) \geq a\, \delta_h(\xi_1, \dots, \xi_{m+1}),$$

which implies

$$\delta_h(\mathcal{G}(\zeta_1), \dots, \mathcal{G}(\zeta_{m+1})) \leq \frac{1}{a}\, \delta_h(\zeta_1, \dots, \zeta_{m+1}).$$

Hence $\mathcal{G}$ is a contraction with constant $k = \frac{1}{a} \in (0, 1)$.

Choose $\xi_0 \in \mathcal{K}$ and define $\xi_{n+1} = \mathcal{G}(\xi_n)$. Then for any $p \geq 1$,

$$\delta_h(\xi_n, \xi_{n+1}, \dots, \xi_{n+p}) \leq k^n\, \delta_h(\xi_0, \xi_1, \dots, \xi_p),$$

which shows that $(\xi_n)$ is a hemi-Cauchy sequence.

By $h$-completeness of $(\mathcal{K}, \delta_h)$, there exists $\xi^* \in \mathcal{K}$ such that $\xi_n \to \xi^*$. Passing to the limit in $\xi_{n+1} = \mathcal{G}(\xi_n)$ yields $\mathcal{G}(\xi^*) = \xi^*$, and hence $\mathcal{F}(\xi^*) = \xi^*$. Therefore, $\xi^*$ is a fixed point of $\mathcal{F}$.

To prove uniqueness, let $\eta^* \in \mathcal{K}$ be another fixed point. Then

$$\delta_h(\xi^*, \eta^*, \dots) = \delta_h(\mathcal{F}(\xi^*), \mathcal{F}(\eta^*), \dots) \geq a\, \delta_h(\xi^*, \eta^*, \dots).$$

Since $a > 1$, this is possible only if $\delta_h(\xi^*, \eta^*, \dots) = 0$, which implies $\xi^* = \eta^*$. □

**Remark 4.5.** *The above result provides a local version of the expansive mapping principle. It guarantees existence and uniqueness of fixed points within invariant subsets that are h-complete, thereby extending the applicability of the theory to restricted domains.*

**Theorem 4.6.** *Let $(\mathcal{H}, \delta_h)$ be an h-complete hemi-metric space and let $\mathcal{F} : \mathcal{H} \to \mathcal{H}$ be a surjective mapping. Assume that there exists a function $\varphi : [0, \infty) \to [0, \infty)$ satisfying:*

*(i) $\varphi(t) > t$ for all $t > 0$,*

*(ii)* $\varphi$ *is continuous and non-decreasing,*

*(iii)* $\varphi(0) = 0$,

*and*

$$\delta_h\big(\mathcal{F}(\xi_1), \ldots, \mathcal{F}(\xi_{m+1})\big) \geq \varphi\big(\delta_h(\xi_1, \ldots, \xi_{m+1})\big)$$

*for all* $(\xi_1, \ldots, \xi_{m+1}) \in \mathcal{H}^{m+1}$. *Then* $\mathcal{F}$ *admits a unique fixed point in* $\mathcal{H}$.

*Proof.* We first verify that $\mathcal{F}$ is injective. Suppose that $\mathcal{F}(\xi) = \mathcal{F}(\eta)$ for some $\xi, \eta \in \mathcal{H}$. Consider the tuple $(\xi, \ldots, \xi, \eta)$. Then

$$\delta_h(\mathcal{F}(\xi), \ldots, \mathcal{F}(\xi), \mathcal{F}(\eta)) = 0.$$

By the given condition,

$$0 \geq \varphi\big(\delta_h(\xi, \ldots, \xi, \eta)\big).$$

Since $\varphi(t) > 0$ for all $t > 0$ and $\varphi(0) = 0$, it follows that $\delta_h(\xi, \ldots, \xi, \eta) = 0$, hence $\xi = \eta$. Thus $\mathcal{F}$ is injective. Together with surjectivity, this implies that $\mathcal{F}$ is bijective.

Let $\mathcal{G} = \mathcal{F}^{-1}$. For any $(\zeta_1, \ldots, \zeta_{m+1}) \in \mathcal{H}^{m+1}$, set $\xi_i = \mathcal{G}(\zeta_i)$. Then

$$\delta_h(\zeta_1, \ldots, \zeta_{m+1}) = \delta_h(\mathcal{F}(\xi_1), \ldots, \mathcal{F}(\xi_{m+1})) \geq \varphi\big(\delta_h(\xi_1, \ldots, \xi_{m+1})\big).$$

Since $\varphi$ is continuous and non-decreasing with $\varphi(t) > t$ for $t > 0$, it admits a well-defined generalized inverse $\psi : [0, \infty) \to [0, \infty)$ such that $\psi(t) < t$ for all $t > 0$ and $\psi(0) = 0$. Consequently,

$$\delta_h(\xi_1, \ldots, \xi_{m+1}) \leq \psi\big(\delta_h(\zeta_1, \ldots, \zeta_{m+1})\big),$$

and hence

$$\delta_h(\mathcal{G}(\zeta_1), \ldots, \mathcal{G}(\zeta_{m+1})) \leq \psi\big(\delta_h(\zeta_1, \ldots, \zeta_{m+1})\big).$$

Fix $\xi_0 \in \mathcal{H}$ and define $\xi_{n+1} = \mathcal{G}(\xi_n)$. Set

$$a_n = \delta_h(\xi_n, \xi_{n+1}, \ldots, \xi_{n+m}).$$

Then

$$a_{n+1} \leq \psi(a_n).$$

Since $\psi(t) < t$ for $t > 0$, the sequence $(a_n)$ is non-increasing and bounded below by 0, hence convergent to some $L \geq 0$. Passing to the limit yields

$$L \leq \psi(L).$$

If $L > 0$, then $\psi(L) < L$, which is a contradiction. Therefore $L = 0$, and thus $a_n \to 0$. This shows that $(\xi_n)$ is hemi-Cauchy.

By $h$-completeness, there exists $\xi^* \in \mathcal{H}$ such that $\xi_n \to \xi^*$. Passing to the limit in $\xi_{n+1} = \mathcal{G}(\xi_n)$ gives $\mathcal{G}(\xi^*) = \xi^*$, and hence $\mathcal{F}(\xi^*) = \xi^*$. Thus $\xi^*$ is a fixed point.

To prove uniqueness, let $\eta^*$ be another fixed point. Then

$$\delta_h(\xi^*, \eta^*, \ldots) = \delta_h(\mathcal{F}(\xi^*), \mathcal{F}(\eta^*), \ldots) \geq \varphi\big(\delta_h(\xi^*, \eta^*, \ldots)\big).$$

If $\delta_h(\xi^*, \eta^*, \ldots) > 0$, then the right-hand side is strictly greater than the left-hand side, which is impossible. Hence $\delta_h(\xi^*, \eta^*, \ldots) = 0$, and therefore $\xi^* = \eta^*$. □

**Remark 4.7.** *This theorem extends expansive-type fixed point results by replacing the linear expansion condition with a nonlinear growth function. It provides a more flexible framework for studying fixed points in hemi-metric spaces under generalized expansion conditions.*

## 5 Examples

In this section, we present several illustrative examples that demonstrate the applicability of the established fixed point results in various settings.

**Example 5.1.** *Let $\mathcal{H} = \mathbb{R}$ and define*

$$\delta_h(\xi_1, \dots, \xi_{m+1}) = \max_{1 \le i,j \le m+1} |\xi_i - \xi_j|.$$

*Define $\mathcal{F} : \mathcal{H} \to \mathcal{H}$ by $\mathcal{F}(\xi) = 2\xi$.*

*Proof.* Consider any $(m+1)$-tuple $(\xi_1, \dots, \xi_{m+1}) \in \mathcal{H}^{m+1}$. By direct computation, we obtain

$$\delta_h(\mathcal{F}(\xi_1), \dots, \mathcal{F}(\xi_{m+1})) = \max_{i,j} |2\xi_i - 2\xi_j| = 2 \max_{i,j} |\xi_i - \xi_j| = 2\, \delta_h(\xi_1, \dots, \xi_{m+1}).$$

This shows that the mapping $\mathcal{F}$ enlarges the hemi-metric distance by a constant factor equal to 2, and therefore it satisfies the expansivity condition with constant $a = 2 > 1$.

Next, we observe that for any $\eta \in \mathbb{R}$, the equation $\mathcal{F}(\xi) = \eta$ admits a unique solution given by $\xi = \eta/2$. This confirms that $\mathcal{F}$ is both surjective and injective, hence bijective.

To determine the fixed point, we solve the equation $\mathcal{F}(\xi) = \xi$, which reduces to $2\xi = \xi$. This immediately implies $\xi = 0$. Since the space $(\mathbb{R}, \delta_h)$ is $h$-complete and all hypotheses of Theorem 4.2 are satisfied, it follows that 0 is the unique fixed point of $\mathcal{F}$. □

**Example 5.2.** *Let $\mathcal{H} = \mathbb{R}$ with the same hemi-metric and consider the subset $\mathcal{K} = [0, \infty)$. Define $\mathcal{F} : \mathcal{K} \to \mathcal{K}$ by $\mathcal{F}(\xi) = 2\xi$.*

*Proof.* For any $(\xi_1, \dots, \xi_{m+1}) \in \mathcal{K}^{m+1}$, the same computation as in the previous example shows that the hemi-metric is scaled by a factor of 2, so $\mathcal{F}$ remains expansive on $\mathcal{K}$.

The invariance condition $\mathcal{F}(\mathcal{K}) \subset \mathcal{K}$ is clearly satisfied, since doubling a nonnegative number preserves nonnegativity. Moreover, given any $\eta \in \mathcal{K}$, the element $\xi = \eta/2$ also belongs to $\mathcal{K}$ and satisfies $\mathcal{F}(\xi) = \eta$, which shows that $\mathcal{F}$ is surjective on $\mathcal{K}$.

The subset $\mathcal{K}$ is closed in $\mathbb{R}$, and since the ambient space $(\mathbb{R}, \delta_h)$ is $h$-complete, it follows that $(\mathcal{K}, \delta_h)$ is also $h$-complete. Solving the fixed point equation again yields $\xi = 0$, which belongs to $\mathcal{K}$. Therefore, all the assumptions of Theorem 4.4 are fulfilled, and 0 is the unique fixed point in $\mathcal{K}$. □

**Example 5.3.** *Let $\mathcal{H} = \mathbb{R}$ with the same hemi-metric and define*

$$\mathcal{F}(\xi) = 2\xi + 1.$$

*Proof.* For any $(\xi_1, \dots, \xi_{m+1}) \in \mathcal{H}^{m+1}$, we compute

$$\delta_h(\mathcal{F}(\xi_1), \dots, \mathcal{F}(\xi_{m+1})) = \max_{i,j} |2\xi_i + 1 - (2\xi_j + 1)| = 2\, \delta_h(\xi_1, \dots, \xi_{m+1}),$$

which shows that $\mathcal{F}$ is again expansive with constant $a = 2 > 1$.

To verify surjectivity, observe that for any $\eta \in \mathbb{R}$, the equation $\mathcal{F}(\xi) = \eta$ leads to $\xi = (\eta - 1)/2$, which belongs to $\mathbb{R}$. Hence $\mathcal{F}$ is bijective.

The fixed point is obtained by solving $\xi = 2\xi + 1$, which gives $\xi = -1$. Since all the required conditions are satisfied, Theorem 4.2 ensures that this fixed point is unique. □

**Example 5.4.** *Let $\mathcal{H} = \mathbb{R}$ with the same hemi-metric and define $\mathcal{F}(\xi) = 2\xi$. Consider the control function*

$$\varphi(t) = \frac{3}{2} t.$$

*Proof.* The function $\varphi$ is continuous, non-decreasing, satisfies $\varphi(0) = 0$, and strictly dominates the identity function for all $t > 0$. For any $(\xi_1, \ldots, \xi_{m+1})$, we have

$$\delta_h(\mathcal{F}(\xi_1), \ldots, \mathcal{F}(\xi_{m+1})) = 2\,\delta_h(\xi_1, \ldots, \xi_{m+1}) \geq \frac{3}{2}\,\delta_h(\xi_1, \ldots, \xi_{m+1}) = \varphi(\delta_h(\xi_1, \ldots, \xi_{m+1})).$$

Thus the generalized expansivity condition is satisfied.

Since $\mathcal{F}$ is bijective and the space is $h$-complete, all assumptions of Theorem 4.6 hold. Solving the equation $\mathcal{F}(\xi) = \xi$ gives $\xi = 0$, which is therefore the unique fixed point. □

**Example 5.5.** *Let $\mathcal{H} = C([0,1], \mathbb{R})$ and define*

$$\delta_h(f_1, \ldots, f_{m+1}) = \max_{1 \leq i,j \leq m+1} \|f_i - f_j\|_\infty.$$

*Define $\mathcal{F} : \mathcal{H} \to \mathcal{H}$ by*

$$(\mathcal{F}f)(t) = 2f(t).$$

*Proof.* For any $(f_1, \ldots, f_{m+1}) \in \mathcal{H}^{m+1}$, we have

$$\delta_h(\mathcal{F}(f_1), \ldots, \mathcal{F}(f_{m+1})) = \max_{i,j} \|2f_i - 2f_j\|_\infty = 2\,\delta_h(f_1, \ldots, f_{m+1}),$$

which shows that $\mathcal{F}$ is expansive.

Given any $g \in \mathcal{H}$, the function $f = g/2$ satisfies $\mathcal{F}(f) = g$, so $\mathcal{F}$ is bijective. The fixed point condition $\mathcal{F}(f) = f$ implies $2f = f$, which yields the zero function as the only solution.

Since $C([0,1], \mathbb{R})$ is complete with respect to the supremum norm and the hemi-metric is induced by this norm, the space is $h$-complete. Therefore, all conditions of Theorem 4.2 are satisfied, and the zero function is the unique fixed point. □

**Remark 5.6.** *These examples illustrate that the established results are applicable across a variety of settings, ranging from real-valued functions to function spaces. They demonstrate that hemi-metric structures provide a flexible framework for analyzing fixed point problems involving multi-point interactions.*

# 6 Application to Integral Equations

In this section, we illustrate how the established fixed point results can be applied to integral equations. Such equations arise naturally in many areas of applied mathematics, including boundary value problems, mathematical physics, engineering, and biological models. In many situations, differential equations can be transformed into equivalent integral formulations, making fixed point methods a powerful tool for proving existence and uniqueness of solutions.

We consider a class of linear Fredholm integral equations of the second kind.

**Theorem 6.1.** *Let $\mathcal{H} = C([0,1], \mathbb{R})$ be the space of real-valued continuous functions on $[0,1]$, equipped with the hemi-metric*

$$\delta_h(f_1, \ldots, f_{m+1}) = \max_{1 \leq i,j \leq m+1} \|f_i - f_j\|_\infty,$$

*where $\|f\|_\infty = \sup_{t \in [0,1]} |f(t)|$. Then $(\mathcal{H}, \delta_h)$ is an $h$-complete hemi-metric space.*

*Let $K : [0,1] \times [0,1] \to \mathbb{R}$ be a continuous function and define an operator $\mathcal{F} : \mathcal{H} \to \mathcal{H}$ by*

$$(\mathcal{F}f)(t) = \int_0^1 K(t,s) f(s)\, ds.$$

*Assume that there exists a constant* $\lambda > 1$ *such that*

$$\|\mathcal{F}f - \mathcal{F}g\|_\infty \geq \lambda \|f - g\|_\infty \quad \textit{for all } f, g \in \mathcal{H},$$

*and that* $\mathcal{F}$ *is surjective. Then the integral equation*

$$f(t) = \int_0^1 K(t,s) f(s)\, ds, \quad t \in [0,1],$$

*admits a unique solution in* $\mathcal{H}$*.*

*Proof.* The space $C([0,1],\mathbb{R})$ is well known to be complete under the supremum norm. Since the hemi-metric $\delta_h$ is defined in terms of the maximum of pairwise supremum distances, convergence with respect to $\delta_h$ is directly induced by convergence in the norm. Consequently, every hemi-Cauchy sequence in $(\mathcal{H}, \delta_h)$ converges, and thus the space is $h$-complete.

Let $f_1, \ldots, f_{m+1} \in \mathcal{H}$ be arbitrary. By the definition of $\delta_h$, we have

$$\delta_h(\mathcal{F}(f_1), \ldots, \mathcal{F}(f_{m+1})) = \max_{1 \leq i,j \leq m+1} \|\mathcal{F}(f_i) - \mathcal{F}(f_j)\|_\infty.$$

Using the assumed inequality, each term in the maximum satisfies

$$\|\mathcal{F}(f_i) - \mathcal{F}(f_j)\|_\infty \geq \lambda \|f_i - f_j\|_\infty,$$

and therefore taking the maximum over all indices yields

$$\delta_h(\mathcal{F}(f_1), \ldots, \mathcal{F}(f_{m+1})) \geq \lambda\, \delta_h(f_1, \ldots, f_{m+1}).$$

This shows that $\mathcal{F}$ expands the hemi-metric distance by a factor $\lambda > 1$, and hence it is an expansive mapping. By assumption, $\mathcal{F}$ is also surjective. Therefore, all the hypotheses of Theorem 4.2 are satisfied.

It follows that $\mathcal{F}$ possesses a unique fixed point $f^* \in \mathcal{H}$. By the definition of a fixed point, this function satisfies

$$f^*(t) = \int_0^1 K(t,s) f^*(s)\, ds \quad \text{for all } t \in [0,1],$$

which shows that $f^*$ is a solution of the given integral equation. The uniqueness of the fixed point ensures that this solution is the only function in $\mathcal{H}$ satisfying the equation. □

**Remark 6.2.** *This application demonstrates that hemi-metric fixed point theory can be effectively used in functional analysis. Although the expansivity condition* $\lambda > 1$ *together with surjectivity is restrictive, such situations arise in certain operator equations, inverse problems, and models involving amplification effects. The result highlights that the developed framework is not purely theoretical but also applicable to concrete problems involving integral operators.*

# 7 Conclusion

In this paper, we have established new fixed point results for expansive mappings in the setting of hemi-metric spaces. By extending the classical Wang-type expansive mapping principle, we proved that every surjective expansive mapping defined on an $h$-complete hemi-metric space admits a unique fixed point. In addition to this global result, a localized version was obtained for invariant subsets, and a nonlinear extension involving control functions was introduced to widen the applicability of the theory. The use of the inverse contraction approach played a central role, providing a systematic method for treating expansive conditions within multi-point distance frameworks. Several examples were included to demonstrate the effectiveness of the results in both real-valued and functional settings.

Furthermore, the applicability of the developed theory was illustrated through an application to a Fredholm integral equation of the second kind, showing its relevance in functional analysis and related fields. The framework presented here opens several avenues for future research, including extensions to hemi-$b$-metric spaces, fuzzy hemi-metric structures, and other generalized distance settings. It would also be worthwhile to explore more general classes of nonlinear operators and investigate further applications to differential and integral equations, as well as problems arising in high-dimensional and multi-agent systems.

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