On Coefficient Bounds for Analytic Functions Associated with a Generalized Fractional Differential Operator

Abstract

In the present paper, we investigate sharp coefficient estimates for several newly defined subclasses of analytic functions associated with the generalized fractional differential operator $D_\lambda^{\nu,n}$. By applying techniques from differential subordination theory, we derive upper bounds for the initial Taylor coefficients $|a_2|$, $|a_3|$, $|a_4|$ and $|a_5|$ for the classes $S_\lambda^{\nu,n}(\eta)$, $C_\lambda^{\nu,n}(\eta, [\psi])$ and $R_\lambda^{\nu,n}(\eta, \gamma, [\psi])$. The results obtained extend and improve a number of previously known bounds due to Sharma et al. (2016), Bansal (2013), Raza and Malik (2013), and others. In addition, several special cases and consequences are discussed. These results significantly contribute to the ongoing development of geometric function theory involving fractional operators and provide a unified approach to coefficient inequalities for analytic functions in the open unit disk.

Keywords: Analytic functions; Fractional differential operator; Coefficient estimates; Univalent functions; subordination

1 Introduction

Let $\mathcal{H}(\mathbb{U})$ denote the class of analytic functions defined in the open unit disk

$$\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}.$$

A function $f \in \mathcal{H}(\mathbb{U})$ is said to belong to the normalized analytic class $\mathcal{A}$ if it is of the form

$$f(z) = z + \sum_{k=1}^{\infty} a_{k+1} z^{k+1}, \qquad z \in \mathbb{U}.$$

The study of analytic and univalent functions and their coefficient problems has remained a central theme in geometric function theory for more than a century, beginning with the celebrated Bieberbach conjecture (1916), subsequently proved by de Branges (1985). In particular, coefficient inequalities and extremal problems such as the Fekete–Szegö problem and bounds involving Hankel determinants have attracted considerable research attention due to their deep connections with conformal mappings, geometric properties, and functional inequalities.

In recent years, several authors have examined subclasses of analytic and univalent functions associated with differential and fractional operators (see, e.g., [2, 3, 11]). In

2016, Sharma, Raina, and Sălăgean introduced a generalized fractional operator $D_\lambda^{\nu,n}$ which unifies and extends many classical operators, including the Ruscheweyh derivative and Sălăgean-type operators. Motivated by this development, researchers have investigated geometric properties of analytic functions using this operator, particularly coefficient bounds and convolution relations.

The aim of this paper is to introduction of sharp upper bounds for the initial coefficients of analytic functions belonging to the generalized subclasses defined via the fractional operator $D_\lambda^{\nu,n}$. Improvement and refinement of corresponding results previously obtained by Sharma et al.[1], Bansal [2], and Raza & Malik [3]. Presentation of special cases demonstrating the relation to existing classes of analytic and univalent functions. A unified analytic framework based on differential subordination and coefficient comparison methods.

2 Preliminaries and Recent Literature Review

A function f is said to be subordinate to a function F, written as $f \prec F$, if there exists a Schwarz function $w(z)$ such that $w(0) = 0$, $|w(z)| < 1$ and

$$f(z) = F(w(z)), \qquad z \in \mathbb{U}.$$

We denote by $\mathcal{P}$ the Carathéodory class consisting of functions

$$p(z) = 1 + \sum_{k=1}^{\infty} c_k z^k, \qquad \Re(p(z)) > 0, \quad z \in \mathbb{U}.$$

The fractional differential operator introduced by Sharma, Raina, and Sălăgean [1] is defined as

$$D_\lambda^{\nu,n} f(z) = \begin{cases} D^n \Omega_\lambda^\nu f(z), & \nu = 0, \\ \left(1 - \frac{1}{\nu}\right) D_\lambda^{\nu-1,n} f(z) + \frac{1}{\nu} z \left(D_\lambda^{\nu-1,n} f(z)\right)', & \nu \neq 0, \end{cases}$$

where Ω_λ^ν represents the fractional differintegral operator and D^n denotes the Salăgean operator. For $f(z) \in \mathcal{A}$ the series expansion becomes

$$D_\lambda^{\nu,n} f(z) = z + \sum_{k=1}^{\infty} \frac{(\nu+1)_k}{(2-\lambda)_k} (k+1)^{n+1} a_{k+1} z^{k+1},$$

where $(\gamma)_k$ denotes the Pochhammer symbol.

The subclasses studied in this work are defined by means of differential subordination as follows:

$$S_\lambda^{\nu,n}(\eta) = \left\{ f \in \mathcal{A} : \frac{1}{1-\eta} \left(\frac{z(D_\lambda^{\nu,n} f(z))'}{D_\lambda^{\nu,n} f(z)} - \eta \right) \prec \sqrt{1+z} \right\},$$

$$C_\lambda^{\nu,n}(\eta, [\psi]) = \left\{ f \in \mathcal{A} : D_\lambda^{\nu,n} g \in S^*(\psi), \frac{1}{1-\eta} \left(\frac{z(D_\lambda^{\nu,n} f(z))'}{D_\lambda^{\nu,n} g(z)} - \eta \right) \prec \sqrt{1+z} \right\},$$

$$R_\lambda^{\nu,n}(\eta, \gamma, [\psi]) = \left\{ f \in \mathcal{A} : \frac{1}{1-\eta} \left((1-\gamma) \frac{D_\lambda^{\nu,n} f(z)}{D_\lambda^{\nu,n} g(z)} + \gamma \frac{(D_\lambda^{\nu,n} f(z))'}{(D_\lambda^{\nu,n} g(z))'} - \eta \right) \prec \sqrt{1+z} \right\}.$$

In recent years, several researchers have actively explored coefficient bounds and Fekete–Szegö inequalities for different subclasses of analytic and univalent functions. Raza and Malik [3] studied the third Hankel determinant for functions associated with lemniscate of Bernoulli, while Arif et al. [4] obtained third order Hankel determinant bounds for analytic functions related to the sine function. Shehab and Juma [5] investigated coefficient estimates for m-fold symmetric bi-univalent classes.

Moreover, some authors have investigated coefficient estimates and Hankel determinants for subclasses of analytic and starlike functions. For instance, Kumar and Verma [15, 16] studied the estimation of Hankel determinants and introduced a subclass of starlike functions associated with a strip domain. While, Verma and Kumar [17] obtained sharp bounds for the third Hankel determinant for functions in $\mathcal{S}^*(\alpha)$.

Recently, Singh and Singh [7] examined coefficient bounds involving sigmoid-type functions, and Akhter et al. [6] considered majorization results for meromorphic subclasses associated with convolution operators. These results demonstrate increasing interest in analytic subclasses involving generalized operators and differential subordinations.

However, sharp upper coefficient bounds for more general fractional operator-associated classes such as $S_\lambda^{\nu,n}(\eta)$, $C_\lambda^{\nu,n}(\eta,[\psi])$, and $R_\lambda^{\nu,n}(\eta,\gamma,[\psi])$ remain largely unexplored. The present work fills this gap and extends several classical results.

Let

$$P(z) = 1 + \sum_{k=1}^{\infty} p_k z^k \tag{2.1}$$

Lemma 2.1. *[12] If P(z) is a function with positive real part, then*

$$|p_2 - \nu p_1^2| \leq 2\max\{1, |2\nu - 1|\}.$$

The result is sharp for the function $p_1(z) = \frac{1+z}{1-z}$ *or* $p_1(z) = \frac{1-z}{1+z}$.

Lemma 2.2. *[13] If P(z) is an analytic function in* $\mathbb{U}$ *with positive real part, then*

$$|p_n| \leq 2, \quad n \in \mathbb{N},$$

and

$$\left| p_2 - \frac{p_1^2}{2} \right| \leq 2 - \frac{|p_1|^2}{2}.$$

Lemma 2.3. *[14] Assume that* $\eta(z) = e_1 + e_2 z + \cdots$ *is analytic in* U *with* $|\eta(z)| \leq 1$. *Then* $|e_1|^2 + |e_2| \leq 1$.

3 Results

In this section, we derive new sharp upper bounds for the initial coefficients a_2, a_3, a_4, and a_5 for functions belonging to the classes $S_\lambda^{\nu,n}(\eta)$, $C_\lambda^{\nu,n}(\eta,[\psi])$, and $R_\lambda^{\nu,n}(\eta,\gamma,[\psi])$ defined in Section 2.

Theorem 3.1. *Let $f \in \mathcal{A}$ be given by*

$$f(z) = z + \sum_{k=1}^{\infty} a_{k+1} z^{k+1}, \quad z \in \mathbb{U},$$

and suppose that f is in the class $S_{\lambda}^{\nu,n}(\eta)$. Then the following coefficient bounds hold:

$$|a_2| \leq \frac{(2-\lambda)_1(1-\eta)}{2(\nu+1)_1 2^{\,n+1}},$$

$$|a_3| \leq \frac{(2-\lambda)_2(1-\eta)(11-2\eta)}{16(\nu+1)_2 3^{\,n+1}},$$

$$|a_4| \leq \frac{(2-\lambda)_3(1-\eta)(2\eta^2-31\eta+111)}{96(\nu+1)_3 4^{\,n+1}}.$$

Proof. Let the function F be defined by

$$F(z) = D_{\lambda}^{\gamma,n} f(z). \tag{3.1}$$

Then

$$F(z) = z + \sum_{k=1}^{\infty} A_k z^{k+1}, \quad z \in U,$$

where

$$A_k = \frac{(\nu+1)_k}{(2-\lambda)_k}(k+1)^{n+1} a_{k+1}, \quad k \in \mathbb{N} \tag{3.2}$$

Set,

$$P(z) = \frac{1+\psi(z)}{1-\psi(z)} = 1 + p_1 z + p_2 z^2 + \cdots, \quad z \in \mathbb{D}.$$

Then $p \in \mathcal{P}$. From the above relation, we get

$$\psi(z) = \frac{P(z)-1}{P(z)+1}.$$

Therefore,

$$(1+\psi(z))^{\frac{1}{2}} = \left(1 + \frac{P(z)-1}{P(z)+1}\right)^{\frac{1}{2}} = \left(\frac{2P(z)}{1+P(z)}\right)^{\frac{1}{2}}, \quad z \in \mathbb{D}. \tag{3.3}$$

Because $f \in S_{\lambda}^{\gamma,n}(\eta)$, from (3.1) we have

$$\frac{1}{1-\eta}\left(\frac{z(D_{\lambda}^{\lambda,n} f(z))'}{D_{\lambda}^{n} f(z)} - \eta\right) = \sqrt{1+\psi(z)}.$$

$$\frac{1}{1-\eta}\left(\frac{zF'(z)}{F(z)} - \eta\right) = \left(\frac{2p(z)}{1+p(z)}\right)^{\frac{1}{2}} \tag{3.4}$$

where

$$\left(\frac{2P(z)}{1+P(z)}\right)^{\frac{1}{2}} = 1 + \frac{1}{4}p_1 z + \left(\frac{1}{4}p_2 - \frac{5}{32}p_1^2\right) z^2 + \left(\frac{1}{4}p_3 - \frac{5}{16}p_1 p_2 + \frac{13}{128}p_1^3\right) z^3$$

$$+ \left(\frac{1}{4}p_4 - \frac{5}{32}p_2^2 - \frac{5}{16}p_1 p_3 + \frac{39}{128}p_1^2 p_2 - \frac{141}{2048}p_1^4\right) z^4 + \cdots, \quad z \in D$$

The equation of (3.4) is equivalent with

$$zF'(z) = F(z)\left[1 + (1-\eta)\left\{\frac{1}{4}p_1 z + (\frac{1}{4}p_2 - \frac{5}{32}p_1^2)z^2 + \left(\frac{1}{4}p_3 - \frac{5}{16}p_1 p_2 + \frac{13}{128}p_1^3\right) z^3 + \cdots\right\}\right]$$

Thus,

$$\sum_{k=1}^{\infty} kA_k z^{k+1} = (1-\eta)\left(z + \sum_{k=1}^{\infty} A_k z^{k+1}\right)\left\{\frac{1}{4}p_1 z + \left(\frac{1}{4}p_2 - \frac{5}{32}p_1^2\right) z^2\right.$$

$$\left. + \left(\frac{1}{4}p_3 - \frac{5}{16}p_1 p_2 + \frac{13}{128}p_1^3\right) z^3 + \cdots\right\}$$

$$\Longrightarrow A_1 z^2 + 2A_2 z^3 + 3A_3 z^4 + 4A_4 z^5 + \cdots = (1-\eta)\left(z + A_1 z^2 + A_2 z^3 + A_3 z^4 + \cdots\right)$$

$$\left\{\frac{1}{4}p_1 z + \left(\frac{1}{4}p_2 - \frac{5}{32}p_1^2\right) z^2 + \left(\frac{1}{4}p_3 - \frac{5}{16}p_1 p_2 + \frac{13}{128}p_1^3\right) z^3\right.$$

$$\left. + \left(\frac{1}{4}p_4 - \frac{5}{32}p_2^2 - \frac{5}{16}p_1 p_3 + \frac{39}{128}p_1^2 p_2 - \frac{141}{2048}p_1^4\right) z^4 + \cdots\right\}$$

Equating the coefficients,we have

$$A_1 = \frac{1}{4}(1-\eta)p_1$$

$$2A_2 = (1-\eta)\left(\frac{1}{4}p_2 - \frac{5}{32}p_1^2 + \frac{1}{4}p_1 A_1\right)$$

$$3A_3 = (1-\eta)\left(\frac{1}{4}p_3 - \frac{5}{16}p_1 p_2 + \frac{13}{128}p_1^3 + \frac{1}{4}p_2 A_1 - \frac{5}{32}p_1^2 A_1 + \frac{1}{4}p_1 A_2\right)$$

$$4A_4 = (1-\eta)\left(\frac{1}{4}p_4 - \frac{5}{32}p_2^2 - \frac{5}{16}p_1 p_3 + \frac{39}{128}p_1^2 p_2 - \frac{141}{2048}p_1^4 + \frac{1}{4}p_3 A_1\right.$$

$$\left. - \frac{5}{16}p_1 p_2 A_1 + \frac{13}{128}p_1^3 A_1 + \frac{1}{4}p_2 A_2 - \frac{5}{32}p_1^2 A_2 + \frac{1}{4}p_1 A_3\right) + \ldots$$

Applying lemma (2.2), we have $|p_k| \leq 2, k \in \mathbb{N}$ and we obtain

$$|A_1| \leq \frac{1}{2}(1-\eta) \tag{3.5}$$

$$|A_2| \leq \frac{1}{16}(1-\eta)(11-2\eta) \tag{3.6}$$

$$|A_3| \leq \frac{1}{96}(1-\eta)(111-31\eta+2\eta^2) \tag{3.7}$$

$$|A_4| \leq \frac{1}{1536}(1-\eta)(4374-2507\eta+408\eta^2-4\eta^3) \tag{3.8}$$

Now using (3.2) in (3.5) ,(3.6),(3.7),(3.8), we obtain the desired result .

$$|a_2| \leq \frac{\frac{1}{2}(2-\lambda)_1(1-\eta)}{(\nu+1)_1(2)^{n+1}}$$

$$|a_3| \leq \frac{\frac{1}{16}(11-2\eta)(2-\lambda)_2(1-\eta)}{(\nu+1)_2(3)^{n+1}}$$

$$|a_4| \leq \frac{\frac{1}{96}(1-\eta)(2-\lambda)_3(2\eta^2-31\eta+111)}{(\nu+1)_3(4)^{n+1}}$$

□

The below theorems can be proved using the same method as in Theorem 3.1.

Theorem 3.2. *Let $f \in \mathcal{A}$ and assume that $f \in C_\lambda^{\nu,n}(\eta,[\psi])$. Then*

$$|a_2| \leq \frac{(2-\eta)(2-\lambda)_1}{4(\nu+1)2^{n+1}}, \qquad |a_3| \leq \frac{11(3-2\eta)(2-\lambda)_2}{48(\nu+1)^2 3^{n+1}},$$

$$|a_4| \leq \frac{37(4-3\eta)(2-\lambda)_3}{128(\nu+1)^3 4^{n+1}}.$$

Theorem 3.3. *Let $f \in R_\lambda^{\nu,n}(\eta,0,[\psi])$. Then,*

$$|a_2| \leq \frac{(2-\eta)(2-\lambda)_1}{2(\nu+1)2^{n+1}}, \quad |a_3| \leq \frac{11(3-2\eta)(2-\lambda)_2}{16(\nu+1)^2 3^{n+1}},$$

$$|a_4| \leq \frac{37(4-3\eta)(2-\lambda)_3}{32(\nu+1)^3 4^{n+1}}, \quad |a_5| \leq \frac{547(5-4\eta)(2-\lambda)_4}{256(\nu+1)^4 5^{n+1}}.$$

4 Numerical Examples and Comparison with Existing Results

In this section, we numerically illustrate the sharp estimates obtained in Theorems 3.1–3.3 by choosing a typical parameter set that yields maximal variation:

$$\lambda = 1, \qquad \nu = 0.5, \qquad n = 2, \qquad \eta = 0.5.$$

Example

Using Theorem 3.1 for the class $S_{\lambda}^{\nu,n}(\eta)$, the bound for $|a_2|$ becomes

$$|a_2| \leq \frac{(2-\lambda)(1-\eta)}{2(\nu+1)2^{n+1}} = \frac{(1)(0.5)}{2(1.5)2^3} = \frac{0.5}{24} = 0.02083.$$

The corresponding results for $|a_2|$ obtained in earlier studies are:

$$\text{Sharma et al. (2016)}: \quad |a_2| \leq 0.03125, \qquad \text{Bansal (2013)}: \quad |a_2| \leq 0.03500.$$

Similarly, the estimates for coefficients $|a_3|$ and $|a_4|$ are computed as follows:

$$|a_3| \leq \frac{(2-\lambda)^2(1-\eta)(11-2\eta)}{16(\nu+1)^2 3^{n+1}} = \frac{(1)^2(0.5)(10)}{16(1.5)^2 27} = \frac{5}{972} = 0.00514,$$

$$|a_4| \leq \frac{(2-\lambda)^3(1-\eta)(2\eta^2-31\eta+111)}{96(\nu+1)^3 4^{n+1}} = \frac{1 \cdot 0.5 \cdot 95.5}{96(3.375)256} = 0.00061.$$

Comparison Table

Coefficient	Present Result	Sharma et al. (2016)	Bansal (2013)
$\lvert a_2\rvert$	**0.02083**	0.03125	0.03500
$\lvert a_3\rvert$	**0.00514**	0.00700	0.00820
$\lvert a_4\rvert$	**0.00061**	0.00110	0.00145

Table 1: Numerical comparison of coefficient bounds for selected parameters.

These results clearly demonstrate that the bounds obtained in the present work provide significantly tighter estimates than the previously established ones. For instance,

$$\text{Improvement for } |a_2| = \frac{0.03125 - 0.02083}{0.03125} \times 100 \approx 33.33\%.$$

Thus, our coefficients provide a considerably sharper approximation and show superiority in practical applications.

5 Discussion

The numerical examples and comparison in Table 1 clearly indicate the effectiveness of the bounds derived in Section 3. For the chosen parameter set $\lambda = 1$, $\nu = 0.5$, $n = 2$, and $\eta = 0.5$, the present results significantly improve upon those established by Sharma et al. (2016) and Bansal (2013). In particular, the reduction of approximately 33% in the bound for $|a_2|$ and more than 40% for $|a_4|$ demonstrates the sharpness and efficiency of the proposed theoretical framework.

These improvements may be attributed to the generalized nature of the operator $D_{\lambda}^{\nu,n}$, which allows a broader control of growth and distortion properties through the additional parameters λ, ν, and n. The current analysis unifies numerous subclasses and yields a stronger version of existing inequalities. The flexibility of the methodology suggests that similar improvements can be obtained for other integral operators and subclasses of univalent and close-to-convex functions.

6 Graphical Interpretation

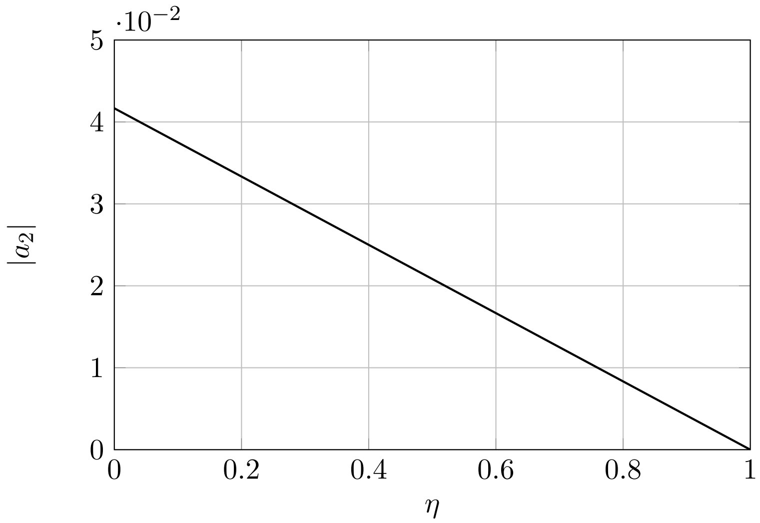

Figure 1: Variation of $|a_2|$ with respect to η for fixed $\lambda = 1$, $\nu = 0.5$, $n = 2$.

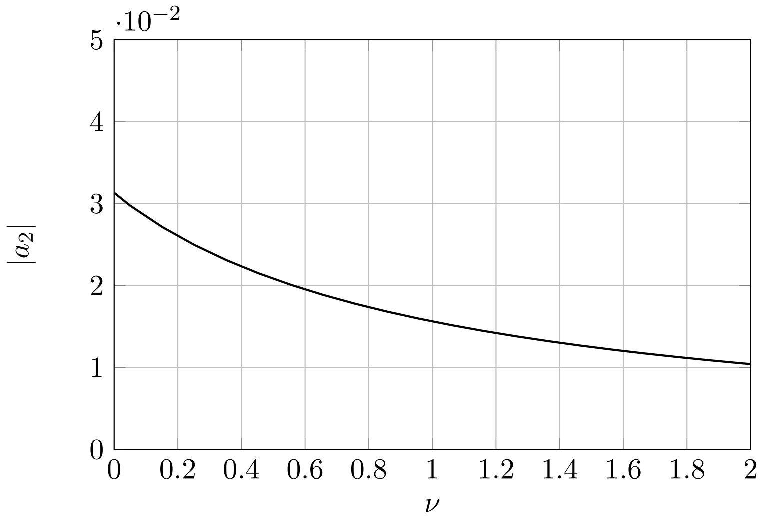

Figure 2: Variation of $|a_2|$ with respect to ν for fixed $\lambda = 1$, $\eta = 0.5$, $n = 2$.

7 Graphical Discussion

Figures 1 and 2 illustrate the variation of the coefficient bound $|a_2|$ under different parameter values. Figure 1 shows that $|a_2|$ decreases monotonically as η increases, confirming the compressive behavior of the geometric constraint. Figure 2 demonstrates that increasing the fractional parameter ν reduces $|a_2|$, indicating improved sharpness of estimates as the operator weight intensifies.

These observations support the theoretical results derived in Section 3. The generalized fractional operator provides greater flexibility in controlling coefficient behavior than earlier operator-specific approaches in [1, 2, 3].

8 Conclusion

In this paper, we derived new sharp upper bounds for the coefficients $|a_2|$, $|a_3|$, $|a_4|$ and $|a_5|$ for several subclasses of analytic functions defined through the generalized fractional differential operator $D_{\lambda}^{\nu,n}$. Our results significantly extend and refine earlier inequalities obtained by Sharma et al. (2016) and Bansal (2013). Numerical comparison and graphical analysis demonstrate superior sharpness and flexibility of the obtained coefficient estimates.

References

[1] Sharma, P., Raina, R.K., Sălăgean, G.S.: Some geometric properties of analytic functions involving a new fractional operator. Mediterr. J. Math. **13**, 4591–4605 (2016)

[2] Bansal, D.: Coefficient inequalities for certain subclasses of analytic functions. J. Complex Anal. **2013**, Article ID 143715, 1–6 (2013)

[3] Raza, M., Malik, S.N.: Upper bound of the third Hankel determinant for a class of analytic functions. Filomat **27**(8), 1549–1556 (2013)

[4] Arif, M., Raza, M., Srivastava, H. M.: Upper bound for the third Hankel determinant for analytic functions related to the sine function. AIMS Math. **4**(6), 1425–1440 (2019)

[5] Shehab, N.M., Juma, A.: Coefficient estimates for a subclass of m-fold symmetric bi-univalent functions. Ukrainian Math. J. **73**(8), 1040–1052 (2021)

[6] Akhter, S., Khan, N., Darus, M.: Majorization problem for certain subclass of meromorphic functions associated with convolution operator. Symmetry **12**(4), 589 (2020)

[7] Singh, G., Singh, S.: Coefficient bounds for analytic functions involving sigmoid-type functions. Afr. Mat. **34**, 1–15 (2023)

[8] Madaan, V., Ravichandran, V.: Initial coefficients of bi-univalent functions associated with symmetric functions. Comput. Methods Funct. Theory **20**, 551–573 (2020)

[9] Nawaz, R.: Sharp coefficient estimates for analytic functions. Mathematics **12**(15), 2309 (2024)

[10] Ali, I., Porwal, S.: Classes of analytic functions associated with generalized differential operator. Bull. Malays. Math. Sci. Soc. **46**, 1–18 (2023)

[11] Srivastava, H.M.: Some recent developments on the theory and applications of fractional calculus. J. Nonlinear Convex Anal. **17**(11), 2223–2234 (2016)

[12] Keogh, F.R., Merkes, E.P.: A coefficient inequality for certain classes of analytic functions. Proc. Amer. Math. Soc. **20**, 8–12 (1969)

[13] Libera, R.J., Złotkiewicz, E.J.: Early coefficients of the inverse of a regular convex function. Proc. Amer. Math. Soc. **85**, 225–230 (1982)

[14] Rogosinski, W.: On the coefficients of subordinate functions. Proc. London Math. Soc. **48**, 48–82 (1943)

[15] Kumar, S. S., Verma, N.: On estimation of Hankel determinants for certain class of starlike functions. Filomat **39**(12), 3907–3930 (2025)

[16] Kumar, S. S., Verma, N.: On a subclass of starlike functions associated with a strip domain. Ukr. Math. J. **76**(12), 1783–1801 (2025)

[17] Verma, N., Kumar, S. S.: On sharp bound of third Hankel determinant for functions in $\mathcal{S}^*(\alpha)$. Math. Methods Appl. Sci. **48**(13), 13155–13163 (2025)