Journal of Advances in Mathematics and Computer Science

xx(x): ....., 20yy; Article no.JAMCS.xxxxx

ISSN: 2456-9968

(Past name: British Journal of Mathematics & Computer Science, Past ISSN: 2231-0851)

SOME RESULTS ON HYPERBOLIC SOMBOR INDEX

Original Research Article

Received: XX December 20XX
Accepted: XX December 20XX
Online Ready: XX December 20XX

Abstract

Let $G(V,E)$ be a simple connected graph of order n and size m. The Hyperbolic Sombor index $HSO(G) = \sum\limits_{uv\in E(G)} \frac{\sqrt{d_u^2+d_v^2}}{\min\{d_u,d_v\}}$, where d_u and d_v denote the degrees of vertices u and v, respectively, is a recently introduced degree-based topological index with growing significance in chemical graph theory. In this work, we derive sharpened and improved lower bounds for $HSO(G)$, thereby refining the classical estimate $HSO(G) \geq m\sqrt{2}$. Furthermore, we establish several new bounds for $HSO(G)$ expressed in terms of important graph parameters such as the chromatic number, independence number, domination number and the mean and standard deviation of vertex degrees.

Keywords: Hyperbolic Sombor index; Chromatic number; Domination number; Independence number

2010 Mathematics Subject Classification: 05C50

1 Introduction

In chemical graph theory, topological indices play a fundamental role in establishing quantitative structure-property relationships (QSPR) and quantitative structure-activity relationships (QSAR). Over the years, several degree-based, distance-based and edge-based indices have been introduced, many of which exhibit strong correlations with molecular descriptors.

Among them, degree-based indices have received particular attention due to their computational simplicity and broad applicability in modelling diverse chemical and physical properties. Classical examples include the Zagreb indices(1), the Randić index(2), the Sombor index(3). The *Sombor index* of graph $G(V,E)$ is introduced by Gutman(3) as

$$SO(G) = \sum_{uv\in E(G)} \sqrt{d_u^2 + d_v^2},$$

where d_u and d_v denote the degrees of the vertices u and v respectively. This index has been extensively studied for its mathematical properties as well as for its chemical applications.

Motivated by the success of the Sombor index, several new variants have been proposed to refine its applicability and capture additional structural features of graphs. One such extension is the *Hyperbolic Sombor index (HSO)* recently introduced by Barman and Das (4) as

$$HSO(G) = \sum_{uv \in E(G)} \frac{\sqrt{d_u^2 + d_v^2}}{\min\{d_u, d_v\}}.$$

They give the classical lower bound $HSO(G) \geq \sqrt{2}m$. They also discovered some bounds and applications of $HSO(G)$ in the same paper. This index incorporates both degree magnitudes and degree imbalance of adjacent vertices, making it a more sensitive descriptor of molecular branching and connectivity patterns. Some bounds and structural results on the hyperbolic Sombor index were recently established in (5). The main motivation of this work is to derive bounds that are stronger than the existing classical bound.

Let $G(V, E)$ be a simple connected graph without isolated vertices. For any vertex $v \in V(G)$, the degree of a vertex v is denoted by d_v which is the number of vertices adjacent to v. The maximum and minimum degrees of G are denoted by Δ and δ respectively. For all other standard terms and notations in graph theory we adhere to the conventions used in (6).

2 Bounds of Hyperbolic Sombor index

This section is devoted to derive new inequalities and improved bounds for $HSO(G)$ by utilizing classical inequalities, graph parameters such as the independence number, chromatic number, domination number. We also used statistical measures including the mean and standard deviation to derive bounds of $HSO(G)$.

Lemma 2.1. *(7) Let $a, b \geq 1$ be any two real numbers. Then*

$$\frac{a+b}{\sqrt{2}} \leq \sqrt{a^2 + b^2}.$$

Theorem 2.2. *Let $G(V, E)$ be a simple connected graph of size m. Then*

$$HSO(G) \geq \frac{1}{\sqrt{2}}\Big[m + \sum_{uv \in E(G)} r_{uv}\Big], \ \ where \ \ r_{uv} = \frac{\max\{d_u, d_v\}}{\min\{d_u, d_v\}} \geq 1.$$

Proof. Let $uv \in E(G)$ be any edge and assume without loss of generality that $d_u \leq d_v$, so $min\{d_u, d_v\} = d_u$. Then

$$\frac{\sqrt{d_u^2 + d_v^2}}{\min\{d_u, d_v\}} = \frac{\sqrt{d_u^2 + d_v^2}}{d_u} = \sqrt{1 + r_{uv}^2}, \quad \text{where } r_{uv} = \frac{d_v}{d_u}.$$

By Lemma 2.1, it follows that $\sqrt{1 + r_{uv}^2} \geq \dfrac{1 + r_{uv}}{\sqrt{2}}$. Summing over all edges of G gives

$$HSO(G) = \sum_{uv \in E(G)} \frac{\sqrt{d_u^2 + d_v^2}}{\min\{d_u, d_v\}} \geq \frac{1}{\sqrt{2}} \left[m + \sum_{uv \in E(G)} r_{uv} \right].$$

□

Remark 2.1. If G is not regular, then $\frac{1}{\sqrt{2}}\left[m + \sum_{uv\in E(G)} r_{uv}\right]$ provides a strictly stronger lower bound for $HSO(G)$ than the classical bound $m\sqrt{2}$. However, if G is k-regular, then $r_{uv} = 1$ for every edge $uv \in E(G)$. Hence,

$$\frac{1}{\sqrt{2}}\left[m + \sum_{uv\in E(G)} r_{uv}\right] = \frac{1}{\sqrt{2}}(m+m) = m\sqrt{2}.$$

Below table 1 presents a comparison between the classical and the new bounds for several standard graphs with $n = 10$ vertices.

Table 1: Comparison of the classical bound $m\sqrt{2}$ and the new bound $\frac{m+\sum r_{uv}}{\sqrt{2}}$ for some standard graphs with $n = 10$ vertices.

Graph	m	$\sum r_{uv}$	$m\sqrt{2}$	$\frac{m+\sum r_{uv}}{\sqrt{2}}$	Difference	Ratio
4-regular graph	20	20.0	28.28	28.28	0.00	1.000
Complete Binary tree	9	13.89	12.73	13.89	1.16	1.089
Path P_{10}	9	11.0	12.73	14.14	1.41	1.111
$K_{4,6}$	24	36.0	33.94	42.43	8.49	1.250
Kragujevac tree $Kg(0,1,2)$	9	16.5	12.73	18.04	5.30	1.417
Binary tree with maximum height	9	18.0	12.73	19.09	6.36	1.500
Dendrimer tree $T_{2,3}$	9	21	12.73	21.21	4.49	1.667
$K_{3,7}$	21	49.0	29.70	49.50	19.80	1.667
$K_{2,8}$	16	64.0	22.63	56.57	33.94	2.500
$K_{1,9}$ (Star)	9	81.0	12.73	63.64	50.91	5.000

Definition 2.1. (6) The *chromatic number* $\chi(G)$ of a graph $G(V,E)$ is the smallest integer k such that the vertices of G can be colored with k colors in a way that no two adjacent vertices receive the same color.

Theorem 2.3. ***Turán's Theorem*** *(8) : Let $G(V,E)$ be a simple graph of order n, size m and chromatic number χ. Then*

$$m \le \left(1 - \frac{1}{\chi}\right)\frac{n^2}{2},$$

with equality if and only if G is the complete χ-partite Turán graph $T(n,\chi)$.

Theorem 2.4. *Let $G(V,E)$ be a simple connected graph of order n, size m and chromatic number χ. Then*

$$(1-\frac{1}{\chi})\frac{n^2}{2}\sqrt{2} \le HSO(G) \le (n-\chi+1)\sqrt{(\chi-1)^2+(n-1)^2} + \frac{(\chi-1)(\chi-2)\sqrt{2}}{2}.$$

Proof. Let $G(V,E)$ be a simple connected graph with chromatic number χ. Let

$$V(G) = V_1 \cup V_2 \cup \ldots \cup V_\chi$$

be a proper χ-coloring of G, where $|V_i| = n_i \ge 1$ for $1 \le i \le \chi$ and $\sum_{i=1}^{\chi} n_i = n$. Since each V_i is an independant set, no two vertices within the same part V_i are adjacent. Consequently, every edge of

G must connect a vertex in V_i with a vertex in V_j for some $i \neq j$. On the other hand, the complete χ-partite graph $K_{n_1,n_2,\ldots,n_\chi}$ contains all possible edges between distinct parts V_i and V_j. Therefore, every edge of G is also an edge of $K_{n_1,n_2,\ldots,n_\chi}$, which shows that G is a subgraph of $K_{n_1,n_2,\ldots,n_\chi}$. So,

$$HSO(G) \leq HSO(K_{n_1,n_2,\ldots,n_\chi}).$$

In the complete χ-partite graph $K_{n_1,n_2,\ldots,n_\chi}$, each vertex in the part V_i has degree $d_i = n - n_i$. Moreover, every unordered pair of distinct parts V_i and V_j $(i < j)$ contributes exactly $n_i n_j$ edges. For each such edge, say the degree of its endpoints are d_i and d_j respectively. Hence,

$$\begin{aligned} HSO(K_{n_1,n_2,\ldots,n_\chi}) &= \sum_{1 \leq i<j \leq \chi} n_i n_j \frac{\sqrt{d_i^2 + d_j^2}}{min\{d_i, d_j\}} \\ &= \sum_{1 \leq i<j \leq \chi} n_i n_j \sqrt{1 + \Big(\frac{max\{d_i, d_j\}}{min\{d_i, d_j\}}\Big)^2} \end{aligned}$$

Consider the highly skewed partition $(n_1, n_2, \ldots, n_\chi) = (n - \chi + 1, 1, 1, \ldots, 1)$, which is admissible for any $\chi \geq 2$. In this case, the vertices in the large part have degree $d_l = n - (n - \chi + 1) = \chi - 1$, while the vertices in the singleton parts have degree $d_s = n - 1$. The edges of graph can be classified as follows.

1. The total number of edges between the large part and the singleton parts is $(\chi - 1)(n - \chi + 1)$. For each such edge, the endpoint degrees are $\chi - 1$ and $n - 1$. Hence, their contribution to $HSO(K_{n_1,n_2,\ldots,n_\chi})$ is

$$(\chi - 1)(n - \chi + 1) \frac{\sqrt{(\chi - 1)^2 + (n - 1)^2}}{\chi - 1}.$$

2. The total number of edges among the singleton parts is $\binom{\chi-1}{2}$, where each edge has both endpoints of degree $n - 1$. Hence, their contribution to $HSO(K_{n_1,n_2,\ldots,n_\chi})$ is

$$\binom{\chi - 1}{2} \frac{\sqrt{(n - 1)^2 + (n - 1)^2}}{n - 1} = \binom{\chi - 1}{2} \sqrt{2} = \frac{(\chi - 1)(\chi - 2)}{\sqrt{2}}.$$

Thus, the Hyperbolic Sombor index of the complete χ-partite graph with skewed partition can be expressed as

$$HSO(K_{n_1,n_2,\ldots,n_\chi}) = (n - \chi + 1) \sqrt{(\chi - 1)^2 + (n - 1)^2} + \frac{(\chi - 1)(\chi - 2)}{\sqrt{2}}.$$

Therefore, for any simple connected graph G with n vertices and chromatic number χ, the Hyperbolic Sombor index satisfies the upper bound

$$HSO(G) \leq (n - \chi + 1) \sqrt{(\chi - 1)^2 + (n - 1)^2} + \frac{(\chi - 1)(\chi - 2)}{\sqrt{2}}.$$

As noted in (4), the classical bound is given by $HSO(G) \geq m\sqrt{2}$. From Turán's Theorem, for the complete χ-partite graph $K_{n_1,n_2,\ldots,n_\chi}$, the number of edges is

$$m = \left(1 - \frac{1}{\chi}\right) \frac{n^2}{2}.$$

Consequently, the Hyperbolic Sombor index of any n-vertex graph G with chromatic number χ satisfies the lower bound

$$HSO(G) \geq m\sqrt{2} = \left(1 - \frac{1}{\chi}\right) \frac{n^2}{2} \sqrt{2}.$$

Combining the upper and lower bounds, we have

$$\left(1-\frac{1}{\chi}\right)\frac{n^2}{2}\sqrt{2} \leq HSO(G) \leq (n-\chi+1)\sqrt{(\chi-1)^2+(n-1)^2}+\frac{(\chi-1)(\chi-2)\sqrt{2}}{2}.$$

□

Remark 2.2.

1. If $\chi = 2$ then
$$HSO(G) \leq HSO(K_{n_1,n_2,\ldots,n_\chi}) = (n-1)\sqrt{(n-1)^2+1}.$$
The bound is sharp maximum for a simple connected n-vertex graphs with chromatic number $\chi = 2$, making it especially relevant for bipartite graphs.
2. If $\chi = 3$ then,
$$HSO(K_{n_1,n_2,\ldots,n_\chi}) \leq (n-2)\sqrt{2^2+(n-1)^2}+\sqrt{2}.$$
It is particularly useful for graphs with chromatic number $\chi = 3$.
3. If $\chi = n$ that is for a complete graph K_n
$$HSO(K_n) = \binom{n}{2}\sqrt{2}.$$

Consequently, the Hyperbolic Sombor index satisfies the following hierarchy of bounds:

$$\binom{n}{2}\sqrt{2} < \ldots < (n-2)\sqrt{2^2+(n-1)^2}+\sqrt{2} < (n-1)\sqrt{(n-1)^2+1}.$$

In particular, if the sizes of the color classes $(n_1, n_2, \ldots, n_\chi)$ are known, the exact upper bound can be determined.

Table 2 presents the upper and lower bounds of the Hyperbolic Sombor index $HSO(G)$ corresponding to different values of the chromatic number χ.

Table 2: Upper and lower bounds of $HSO(G)$ for different values of the chromatic number χ.

chromatic number χ	**Upper bound of** $HSO(G)$	**Lower bound of** $HSO(G)$
2	$(n-1)\sqrt{(n-1)^2+1}$	$\frac{n^2\sqrt{2}}{4}$
3	$(n-2)\sqrt{(n-1)^2+4}+\sqrt{2}$	$\frac{n^2\sqrt{2}}{3}$
4	$(n-3)\sqrt{(n-1)^2+9}+3\sqrt{2}$	$\frac{3n^2\sqrt{2}}{8}$
$\vdots$	$\vdots$	$\vdots$
n	$\binom{n}{2}\sqrt{2}$	$\binom{n}{2}\sqrt{2}$

Definition 2.2. (9) Let $G(V, E)$ be a simple connected graph of order n and size m. The *mean degree* of vertex set of G is defined as

$$\mu = \frac{1}{n}\sum_{v\in V(G)} d_v = \frac{2m}{n}.$$

And the *standard deviation* of the vertex degrees of G is defined as

$$\sigma = \sqrt{\frac{1}{n}\sum_{v\in V(G)}(d_v-\mu)^2}.$$

Lemma 2.5. *Let $G(V,E)$ be a simple connected graph of order n, mean degree μ and standard deviation of vertex degrees σ. Then*

$$\sum_{v\in V(G)} d_v^2 = n(\sigma^2+\mu^2).$$

Proof. By definition 2.8, the variance of the degree of vertices is

$$\sigma^2 = \frac{1}{n}\sum_{v\in V(G)}(d_v-\mu)^2.$$

Expanding the square, we obtain

$$\begin{aligned}\sigma^2 &= \frac{1}{n}\sum_{v\in V(G)}(d_v^2-2\mu d_v+\mu^2)\\ &= \frac{1}{n}\sum_{v\in V(G)} d_v^2-2\mu^2+\mu^2 \quad (\text{since} \sum_{v\in V(G)} d_v = n\mu)\\ &= \frac{1}{n}\sum_{v\in V(G)} d_v^2-\mu^2.\end{aligned}$$

Hence,

$$\sigma^2+\mu^2 = \frac{1}{n}\sum_{v\in V(G)} d_v^2 \quad \Rightarrow \quad \sum_{v\in V(G)} d_v^2 = n(\sigma^2+\mu^2).$$

□

Lemma 2.6. ***Cauchy Schwarz inequality (7):*** *Let $a_1,a_2,\dots,a_n$ and $b_1,b_2,\dots,b_n$ be real numbers. Then*

$$\left(\sum_{i=1}^{n} a_ib_i\right)^2 \le \left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right).$$

Lemma 2.7. *(10) Let $G(V,E)$ be simple connected graph of order n and size m. Then*

$$\sum_{uv\in E(G)}(d_u^2+d_v^2) = \sum_{v\in V(G)} d_v^3.$$

Theorem 2.8. *Let $G(V,E)$ be a simple connected graph of order n, size m, minimum degree δ and maximum degree Δ. Let μ and σ denote the mean and standard deviation of the vertex degrees of G respectively. Then*

$$HSO(G) \le \frac{1}{\delta}\sqrt{nm\Delta(\sigma^2+\mu^2)}.$$

Proof. Let $d_1,d_2,\dots,d_n$ be the degree sequence of G. Then the mean and standard deviation of degree of vertices in G are

$$\mu = \frac{2m}{n} \quad and \quad \sigma = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(d_i-\mu)^2}.$$

By the definition of the Hyperbolic Sombor index,

$$HSO(G) = \sum_{uv\in E(G)} \frac{\sqrt{d_u^2 + d_v^2}}{\min\{d_u, d_v\}} \leq \frac{1}{\delta} \sum_{uv\in E(G)} \sqrt{d_u^2 + d_v^2}.$$

Applying the Cauchy Schwarz inequality, we obtain

$$\begin{aligned} \sum_{uv\in E(G)} \sqrt{d_u^2 + d_v^2} &\leq \sqrt{\sum_{uv\in E(G)} 1^2} \sqrt{\sum_{uv\in E(G)} (d_u^2 + d_v^2)} \\ &= \sqrt{m} \sqrt{\sum_{v\in V(G)} d_v^3} \quad \text{(by Lemma 2.11)}. \end{aligned}$$

Hence,

$$HSO(G) \leq \frac{\sqrt{m}}{\delta} \sqrt{\sum_{v\in V(G)} d_v^3}.$$

Since $d_v \leq \Delta$ for all $v \in V(G)$, it follows that $d_v^3 \leq \Delta d_v^2$. Therefore

$$\sum_{v\in V(G)} d_v^3 \leq \Delta \sum_{v\in V(G)} d_v^2 = \Delta n(\sigma^2 + \mu^2) \quad \text{(by Lemma 2.9)}.$$

Substituting this into the previous inequality yields

$$HSO(G) \leq \frac{\sqrt{m}}{\delta} \sqrt{\Delta n(\sigma^2 + \mu^2)} = \frac{1}{\delta} \sqrt{nm\Delta(\sigma^2 + \mu^2)}.$$

Hence, the stated result follows immediately. □

Definition 2.3. (6) The *independence number* $\alpha(G)$ of graph $G(V, E)$ is the cardinality of a largest subset of vertices $I \subseteq V(G)$ in which no two vertices are adjacent.

Theorem 2.9. *Let $G(V, E)$ be a simple connected graph and let $\alpha(G)$ be its independence number. Let $I \subseteq V(G)$ be any independence set with cardinality $\alpha(G)$. Then*

$$HSO(G) \;\geq\; \sqrt{2}\,\alpha(G)\,\delta_I,$$

where $\delta_I = \min_{v\in I} d_v$ is the minimum degree among the vertices in I.

Proof. Let I be an independence set and define

$$E(I) = \{uv \in E : u \in I \text{ or } v \in I\},$$

the set of all edges incident to vertices in I. Since no two vertices of I are adjacent, it follows that

$$|E(I)| = \sum_{v\in I} d_v.$$

For any edge $e = uv \in E(I)$,

$$\frac{\sqrt{d_u^2 + d_v^2}}{\min\{d_u, d_v\}} \geq \frac{\sqrt{(\min\{d_u, d_v\})^2 + (\min\{d_u, d_v\})^2}}{\min\{d_u, d_v\}} = \sqrt{2}.$$

Summing over all such edges gives

$$\begin{aligned} HSO(G) &= \sum_{uv\in E(G)} \frac{\sqrt{d_u^2+d_v^2}}{\min\{d_u,d_v\}} \\ &\geq \sum_{uv\in E(I)} \frac{\sqrt{d_u^2+d_v^2}}{\min\{d_u,d_v\}} \\ &= \sqrt{2}\,|E(I)| = \sqrt{2}\sum_{v\in I} d_v. \end{aligned}$$

Finally, since $\sum_{v\in I} d_v \geq \alpha(G)\,\delta_I$, the desired inequality follows:

$$HSO(G) \geq \sqrt{2}\,\alpha(G)\,\delta_I.$$

Hence, the theorem stands proved. □

Definition 2.4. (11) The *domination number* $\gamma(G)$ of a graph $G(V,E)$ is the minimum cardinality of a set $D \subseteq V$ such that every vertex in $V \setminus D$ is adjacent to at least one vertex of D.

Theorem 2.10. *Let $G(V,E)$ be a simple graph of order n and domination number $\gamma(G)$. Then*

$$HSO(G) \;\geq\; (n-\gamma(G))\sqrt{2}.$$

Proof. Let $S \subseteq V$ be a dominating set of minimum cardinality, that is, $|S| = \gamma(G)$. For each vertex $w \in V \setminus S$, choose a neighbor $n(w) \in S$. Define the set of edges

$$\mathcal{E} = \{\, n(w)w : w \in V \setminus S \,\},$$

which contains exactly $n-\gamma(G)$ distinct edges. For any edge $e = uv \in \mathcal{E}$, we have

$$\frac{\sqrt{d_u^2+d_v^2}}{\min\{d_u,d_v\}} \;\geq\; \frac{\sqrt{(\min\{d_u,d_v\})^2+(\min\{d_u,d_v\})^2}}{\min\{d_u,d_v\}} = \sqrt{2}.$$

Summing over all edges in $\mathcal{E}$, we obtain

$$\mathrm{HSO}(G) = \sum_{uv\in E} \frac{\sqrt{d_u^2+d_v^2}}{\min\{d_u,d_v\}} \;\geq\; \sum_{uv\in \mathcal{E}} \frac{\sqrt{d_u^2+d_v^2}}{\min\{d_u,d_v\}} \;\geq\; (n-\gamma(G))\sqrt{2}.$$

This completes the proof. □

3 CONCLUSIONS

In this paper, we established a sharper lower bound for the Hyperbolic Sombor index $HSO(G)$, improving upon the classical general bound $m\sqrt{2}$. Moreover, we derived new lower and upper bounds expressed in terms of the chromatic number, independence number, domination number and the mean and standard deviation of vertex degrees.

Acknowledgment

The second author, Kajal S. Baldaniya, gratefully acknowledges the financial support from the Knowledge Consortium of Gujarat (KCG), Education Department, Government of Gujarat, India, through the SODH Scholarship Scheme (Ref. No. 2023015012).

References

[1] Gutman, I., Trinajstić, N.(1972). Graph theory and molecular orbitals, Total π electronenergy of alternant hydrocarbons, *Chem. Phys. Lett.*, **17**, 535–538.

[2] Randić, M.(1975). Characterization of molecular branching, *J. Am. Chem. Soc.*, **97**, 6609–6615.

[3] Gutman, I.(2021). Geometric approach to degree based topological indices: Sombor Indices, *MATCH Commun. Math. Comput. Chem.*, **86**, 11–16.

[4] Barman, J., Das, S.(2023). Geometric Approach to Degree-Based Topological Index: Hyperbolic Sombor Index, *MATCH Commun. Math. Comput. Chem.*, **90**, 77–92.

[5] M. J. Patel, K. S. Baldaniya.(2025). Characterizations of Bounds and Tree Structuresfor the Hyperbolic Sombor Index. *J. Adv. Math. Com. Sci.*, **40**(12), 145–155.

[6] J. A. Bondy, U. S. R. Murty. (1976). *Graph Theory with Applications*, Macmillan, London.

[7] Zdravko Cvetkovski. (2012). *Inequalities: Theorems, Techniques and Selected Problems*, Springer, New York.

[8] Turán, P.(1941). On an extremal problem in graph theory, *Matematikai és Fizikai Lapok*, **48**, 436–452.

[9] Snijders, T. A. B.(1981). The degree variance: an index of graph heterogeneity, *Social Networks*, **3**, 163–174.

[10] Furtula, B., Gutman, I.(2015). A forgotten topological index, *J. Math. Chem.*, **53**(4), 1184–1190.

[11] Ore, O.(1962). Theory of Graphs, *American Mathematical Society Colloquium Publications*, **38**, (American Mathematical Society, Providence, RI).
