# EQUI-NEIGHBOR POLYNOMIAL EQUIVALENT CLASSES OF GRAPHS

***Authors' contributions***

*This work was carried out in collaboration among all authors. All authors read and approved the final manuscript.*

***Original Research Article***

## Abstract

Let $G(V,E)$ be a simple graph of order $n$ with vertex set $V$ and edge set $E$. Let $(u,v)$ denotes an unordered vertex pair of distinct vertices of $G$. For a vertex $u \in G$, let $N(u)$ be the set of all vertices of $G$ which are adjacent to $u$ in $G$. Then for $0 \leq i \leq n-1$, the $i$-equi neighbor set of $G$ is defined as: $N_e(G,i) = \{(u,v) : u,v \in V, u \neq v$ and $|N(u)| = |N(v)| = i\}$. The equi-neighbor polynomial $N_e[G;x]$ of $G$ is defined as $N_e[G;x] = \sum_{i=0}^{(n-1)} |N_e(G,i)|x^i$. Two graphs $G$ and $H$ are said to be $ENP-$ equivalent if and only if $N_e[G;x]$=$N_e[H;x]$. A graph $H$ is said to be $ENP$-unique if $H$ is $ENP$-equivalent to itself. A graph $G$ is said to be $ENP$-cardinal unique if there does not exist a graph $H$ with the same number of vertices as $G$ such that $H$ is equivalent to $G$. This paper identifies several ENP-unique graphs, explores ENP-equivalent graph classes, and characterizes specific ENP-ardinal unique graph classes.

*Keywords: $i-$ equi neighbor set; equi neighbor* polynomial, ENP-equivalent graph,ENP-ardinal unique graph

2010 Mathematics Subject Classification: 05C31; 05C39

# 1 Introduction

Graph theory provides powerful tools for modeling relationships, and within this field, graph polynomials are essential for characterizing structural properties(3). This paper focuses on the equi-neighbor polynomial (ENP), $N_e[G;x]$, which is used to classify graphs based on the distribution of vertices that share the same degree. Let $G(V,E)$ be a simple graph of order $n$ (number of vertices). For any vertex $u \in V$, let $N(u)$ be the set of its adjacent vertices (its neighbors).

The present authors derived the equi neighbour polynomial of some well known graphs(1), (**?** )r6) .The equi-neighbor polynomial for graphs formed through certain binary graph operations is investigated in (7). In (8) MN Husin etal examined the neighborhood polynomial of some interested standard graph networks. Detailed data on neighborhood polynomials are discussed in (9), (10), and (11). In (2), (5) and (4), Shikhi M.etal investigated common neighbor polynomial of graphs. polynomial approach to linear algebra is discussed in (6).

The core concept is the $i$-equi neighbor set of $G$, denoted $N_e(G,i)$, which is defined for $0 \leq i \leq n-1$ as the set of unordered pairs of distinct vertices $(u,v)$ where both vertices have the same degree $i$:

$$N_e(G,i) = \{(u,v) : u,v \in V, u \neq v \text{ and } |N(u)| = |N(v)| = i\} \quad (1.1)$$

$$= \Big\{(u,v) \in \binom{V}{2} : |N(u)| = |N(v)| = i\Big\}, \quad (1.2)$$

where $\binom{V}{2}$ denotes the set of all unordered pairs of vertices of $G$. The equi neighbor polynomial $N_e[G;x]$ is then defined as the generating function where the coefficient of $x^i$ is the cardinality of the $i-$ equineighbor set(1). That is,

$$N_e[G;x] = \sum_{i=0}^{(n-1)} |N_e(G,i)|x^i.$$

Based on this polynomial, we define graph equivalence relations and uniqueness properties: Two graphs $G$ and $H$ are $ENP$-equivalent (denoted $G \overset{\mathcal{N}_{\mathcal{E}}}{\sim} H$) if and only if their equi-neighbor polynomials are identical, that is, $N_e[G;x] = N_e[H;x]$. Obviously, the relation $\overset{\mathcal{N}_{\mathcal{E}}}{\sim}$ is an equivalence relation on the class $\mathcal{G}$ of all finite simple graphs. The set of all graphs $ENP$-equivalent to a graph $G$ is denoted as $[G]_{\mathcal{N}_{\mathcal{E}}}$ and is defined as $[G]_{\mathcal{N}_{\mathcal{E}}} = \{H \in \mathcal{G} : N_e[H;x] = N_e[G;x]\}$. A graph H is said to be $ENP-$unique if $[H]_{\mathcal{N}_{\mathcal{E}}} = \{H\}$. Observe that isomorphic graphs have same equi neighbour polynomial, that is, if $H \cong G$, then $H$ is $ENP$ - equivalent to $G$. But non isomorphic graphs may have same equi neighbor polynomial. For instane, the graphs shown in the figure are $ENP$ -equivalent.

Graph polynomials are crucial tools in graph theory. In this work, we focus on the equi-neighbor polynomial(ENP). Let $G(V,E)$ be a simple graph of order $n$. Let $(u,v)$ denotes an unordered vertex pair of distant vertices of $G$. The $i$-equi neighbour set of $G$ is defined as $N_e(G,i) = \{(u,v) : u,v \in V, u \neq v$ and $|N(u)| = |N(v)| = i\}$, for $0 \leq i \leq n-1$. The polynomial $N_e[G;x] = \sum_{i=0}^{(n-1)} |N_e(G,i)|x^i$ is defined as the equi neighbour polynomial of $G$. We say that two graphs $G$ and $H$ are $ENP$-equivalent ($G \overset{\mathcal{N}_\mathcal{E}}{\sim} H$) if and only if $N_e[G;x] = N_e[H;x]$.

Obviously, the relation $\overset{\mathcal{N}_\mathcal{E}}{\sim}$ is an equivalence relation on the class $\mathcal{G}$ of all simple finite graphs. The set of all graphs $ENP$-equivalent to a graph $G$ is denoted as $[G]_{\mathcal{N}_\rceil}$ and is defined as $[G]_{\mathcal{N}_\rceil} = \{H \in \mathcal{G} : N_e[H;x] = N_e[G;x]\}$. A graph H is said to be $ENP-$unique if $[H]_{\mathcal{N}_\rceil} = \{H\}$. Observe that isomorphic graphs have the same equi neighbour polynomial, that is, if $H \cong G$, then $H$ is $ENP$ - equivalent to $G$. But non-isomorphic graphs may have the same equi-neighbor polynomial. For instance, the graphs shown in the figure 1 have the same equi-neighbor polynomial $3x^2+x$. Hence $G$ and $H$ are $ENP$ -equivalent. In this paper, we aim to identify and characterize various graph classes

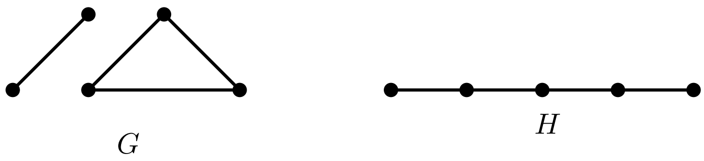


Figure 1: Two $ENP$-equivalent graphs $G$ and $H$

based on these definitions, establishing fundamental theorems relating the ENP of a graph to that of its complement and unions, and determining which well-known graph families possess ENP-unique or ENP-cardinal unique properties.we identify some $ENP-$ unique graphs, some $ENP$-equivalent graph classes, and some $ENP$-cardinal unique graph classes.

## 2 Properties of the Equi-neighbor Polynomial

We begin by establishing properties of the ENP, particularly its relationship with graph complements and components.

**Theorem 2.1.** *Let $G$ be a graph with $n$ vertices and let $\overline{G}$ denotes the complement of $G$. Then $N_e[\overline{G};x]$=$x^{n-1}N_e[G;\frac{1}{x}]$.*

*Proof.* Observe that the sum of the degrees of a vertex in $G$ and $\overline{G}$ equals $n-1$. Let $u$ and $v$ be two distinct verties in $V(G)$ such that $|N(u)| = |N(v)| = i$ in $G$. Then $u$ has $n-1-i$ neighbors in $\overline{G}$ which are the non-adjacent vertices of $u$ in $G$. The same is true for $v$. Hence, the pair $(u,v)$ has $(n-1-i)$-equi-neighbors in $\overline{G}$ and there are $|N_e(G,i)|$ such pairs. Thus,

$$N_e[\overline{G};x] = \sum_{i=0}^{n-1} |N_e(G,i)|x^{n-1-i} = x^{n-1}N_e\left[G;\frac{1}{x}\right].$$

This completes the proof. □

**Theorem 2.2.** *Let $G$ be a graph with $n$ vertices($n \geq 3$). Then $\overline{G} \in [G]_{\mathcal{N}_{\mathcal{E}}}$ iff $n(G,i) = n(G, n-1-i)$; $n(G,i) \geq 2$; $0 \leq i \leq n-1$, where $n(G,i)$ represents the number of vertices of $G$ with $i$ neighbors.*

*Proof.* Assume that $\overline{G} \in [G]_{\mathcal{N}_{\mathcal{E}}}$. From the definition of ENP it follows that $N_e[\overline{G}; x] = N_e[G; x]$.Therefore, $|N(G,i)| = |N_e(\overline{G}, i)|$ for $0 \leq i \leq n-1$. This tells us that $n(G,i) = n(\overline{G}, i)$; for $n(G,i) \geq 2$; $0 \leq i \leq n-1$. Observe that the necessary part of the proof follows from the fact that the sum of the degree of a vertex in $G$ and in $\overline{G}$ equals $(n-1)$. Conversely, assume that $n(G,i) = n(G, n-1-i)$ ; $n(G,i) \geq 2$; $0 \leq i \leq n-1$. This means that $n(G,i) = n(\overline{G}, i)$; $n(G,i) \geq 2$; $0 \leq i \leq n-1$. It follows that $N_e[G; x] = N_e[\overline{G}; x]$. This completes the proof. □

**Lemma 2.3.** *If $G_1$ and $G_2$ are two components of a graph $G$ with $n$ and $m$ vertices respectively, satisfying $n \geq m$, then*

$$N_e[G; x] = N_e[G_1; x] + N_e[G_2; x] + \sum_{i=0}^{m-1} n(G_1, i) n(G_2, i) x^i,$$

*where $n(G_k, i)$ represents the number of vertices of $G_k$ with $i$ neighbors, $k = 1, 2$.*

**Theorem 2.4.** *For a graph $G$, $G + K_1 \in [G]_{\mathcal{N}_{\mathcal{E}}}$ if and only if $G$ has no isolated vertex.*

*Proof.* Assume that $G + K_1 \in [G]_{\mathcal{N}_{\mathcal{E}}}$. This means that $N_e[G + K_1; x] = N_e[G; x]$.
Note that

$$N_e[G + K_1; x] = N_e[G; x] + N_e[K_1; x] + n(G, 0)(\text{ lemma } 2.3)$$

This implies that $N_e[G + K_1; x] = N_e[G; x] + n(G, 0) \Rightarrow n(G, 0) = 0$. Conversely, assume that $G$ has no isolated vertex. This means that $n(G, 0) = 0$. Now, from lemma 2.3 we have

$$N_e[G + K_1; x] = N_e[G; x] + N_e[K_1; x] = N_e[G; x]$$

Hence $G + K_1 \in [G]_{\mathcal{N}_{\mathcal{E}}}$. This completes the proof. □

**Theorem 2.5.** *Let $G$ be a graph other than the null graph with $n$ isolated vertices. Let $H$ be any graph with $m$ isolated vertices and $H \in [G]_{\mathcal{N}_{\mathcal{E}}}$. Then the following results are satisfied.*

*(i) If $n = 0$ or $1$, then $m = 0$ or $1$.*

*(ii) If $n \geq 2$, then $m = n$.*

*Proof.* Here, we consider two cases:

**Case(i)** Let $n = 0$ or $1$. Then $|N_e(G, 0)| = 0$. Since $H \in [G]_{\mathcal{N}_{\mathcal{E}}}$, then we have $|N_e(H, 0)| = 0$. Therefore, the number of isolated vertices in $H$ equals either $0$ or $1$.

**Case(ii)** Let $n \geq 2$. Since $H \in [G]_{\mathcal{N}_{\mathcal{E}}}$, then $|N_e(H, 0)| = \binom{n}{2}$. Hence, the result follows from the property $7$ of $N_e[G; x]$ (1).

This completes the proof.

□

# 3 ENP-Unique and ENP-Cardinal Unique Graphs

We now characterize specific families of graphs based on their ENP uniqueness properties.

**Lemma 3.1.** *A sequence $S$ such that $d_1, d_2, \cdots, d_n$ of non negative integers with $d_1 \geq d_2 \geq \cdots \geq d_n$ and $n \geq 2$, $d_1 \geq 1$ is graphical if and only if the sequence $d_2 - 1, d_3 - 1, \cdots, d_{d_1+1}, d_{d_1+2}, \cdots, d_n$ is graphical.*

**Theorem 3.2.** *The null graph is the only graph that is $ENP$ unique.*

*Proof.* Let $G$ be a null graph with $n$ vertices. Then $N_e[G;x] = \binom{n}{2}$. Let $H$ be any graph with $m$ vertices such that $H \in [G]_{\mathcal{N}_\mathcal{E}}$. Then obviously $m \geq n$. Also $N_e[H;x] = \binom{n}{2}$. It follows that the number of isolated vertices in $H$ is equal to $n$. Let $V(H) = \{u_1, u_2, \ldots, u_n, u_{n+1}, \ldots, u_m\}$; where $|N(u_i)| = 0$ for $i = 1, 2, \ldots, n$; $|N(u_i)| = p_i$ for $i = n+1, n+2, \ldots, m$. Note that $0 < p_i < m-1$ and $p_i \neq p_j$ for $i, j \in \{n+1, n+2, \ldots, m\}$; $i \neq j$. Let $H_1$ be the subgraph of $H$ induced by $u_1, u_2, \ldots, u_n$ and $H_2$ be the subgraph of $H$ induced by $u_{n+1}, \ldots, u_m$. Then $H = H_1 + H_2$. Now the degree sequence of the graph $H_2$ is $p_{n+1}, p_{n+2}, ..., p_m$; where $0 < p_i < m-1$ and $p_i \neq p_j$ for $i, j \in \{n+1, n+2, ..., m\}$; $i \neq j$. But this sequence is not graphical, by lemma 3.1. This means that $H$ can contain only the $n$ isolated vertices $u_1, u_2, ..., u_n$. Thus $H \cong G$. This completes the proof. □

## 3.1 Cycle graphs $C_n$

**Lemma 3.3.** *For a cycle graph $C_n$, we have*

$$N_e[C_n;x] = \binom{n}{2}x^2, n \geq 3.$$

**Theorem 3.4.** *The cycle graph $C_n$ is $ENP$ cardinal unique for $n = 3, 4, 5$.*

*Proof.* Let $H$ be any graph with $n$ vertices such that $H \in [C_n]_{\mathcal{N}_\mathcal{E}}$. Then by the lemma 3.3, $N_e[H;x] = \binom{n}{2}x^2$; $n = 3, 4, 5$. This means that all the $n$ vertices in $H$ have two neighbors. Then obviously $H \cong C_n$.This completes the proof. □

**Theorem 3.5.** *The cycle graph $C_n$ is not $ENP$ cardinal unique for $n \geq 6$.*

*Proof.* Let $H$ be the graph with $n$ vertices; $n \geq 6$ consists of precisely a vertex disjoint $m$-cycle; $3 \leq m \leq n-3$, and $(n-m)$-cycle. Then $C_n$ and $H$ have the same degree sequence. Hence $H \in [C_n]_{\mathcal{N}_\mathcal{E}}$. Observe that $C_n$ and $H$ are not isomorphic.

This completes the proof. □

## 3.2 Complete graphs $K_n$

**Lemma 3.6.** *For a complete graph $K_n$, we have*

$$N_e[K_n;x] = \binom{n}{2}x^{n-1}; n \geq 2.$$

**Theorem 3.7.** *For $n \geq 1$, the Complete graph $K_n$ is $ENP-$cardinal unique.*

*Proof.* Let $H$ be any graph with $n$ vertices such that $H \in [K_n]_{\mathcal{N}_\mathcal{E}}$; $n \geq 1$. Then by lemma 3.6 $N_e[H;x] = \binom{n}{2}x^{n-1}$ . This means that all the $n$ vertices in $H$ have $(n-1)$ neighbors.Then obviously $H \cong K_n$. This completes the proof. □

## 3.3 The Star graph $K_{1,n}$

**Lemma 3.8.** *For a star graph $K_{1,n}$, $N_e[K_{1,n};x] = \begin{cases} \binom{n}{2}x, & n \neq 1 \\ x, & n = 1. \end{cases}$*

**Theorem 3.9.** *The star graph $K_{1,n}$; $n \geq 1$ is $ENP-$ cardinal unique only if $n$ is odd.*

*Proof.* Here, we consider the following two cases.

**Case(i)** Assume that $n$ is odd. Let $H$ be any graph with $n+1$ vertices such that $H \in [K_{1,n}]_{\mathcal{N}_{\mathcal{E}}}$; $n \geq 1$.Then by lemma 3.8 we have $N_e[H;x] = \binom{n}{2}x$. This means that the $n$ vertices of $H$ have only one neighbor. Then $H \cong K_{1,n}$.

**Case(ii)** Assume that $n$ is even. Let $H$ be a graph on $n+1$ vertices that consists of precisely vertex disjoint, $\frac{n}{2}$ path graphs of length two and an isolated vertex. Then obviously $N_e[H;x] = \binom{n}{2}x$. Hence $H \in [K_{1,n}]_{\mathcal{N}_{\mathcal{E}}}$. But $K_{1,n}$ and $H$ are not isomorphic.

This completes the proof. □

## 3.4 The Complete Bipartite graph $K_{m,n}$

**Lemma 3.10.** *For a complete bipartite graph $K_{m,n}$; where $m, n \geq 2$, we have the following.*

$$N_e[K_{m,n};x] = \begin{cases} \binom{m}{2}x^n + \binom{n}{2}x^m, & m \neq n, \\ n(2n-1)x^n, & m = n. \end{cases}$$

**Theorem 3.11.** *The complete bipartite graph $K_{m,n}$ is not $ENP-$ cardinal unique; for every $m, n > 2$.*

*Proof.* Without loss of generality assume that $m \leq n$. First we assume that $m < n$.

**Case(i)** Assume that both $m$ and $n$ are odd.

**Subcase(i)** Let $m-2 > n-m$. Here we construct a graph $H$ on $m+n$ vertices as follows. Let $V(H) = \{u_1, u_2, ..., u_m, v_1, v_2, ..., v_n\}$. Draw the edges of $H$ as follows. Join $u_1$ with $v_2$ and $v_i$; $i \in \{1,2,...,n\}$; $i \neq 3$. Join $u_j$ with $u_{j+1}$ and $v_i$; $i \in \{1,2,...,n\}$; $i \neq j, n-(j-2)$ for $j = 2,3,4,5,...,m-1$. Join $u_m$ with $v_i$; $i \in \{1,2,...,n\}$; $i \neq m$. As a result of this the number of neighbors of $u_i$; $n(u_i)$ equals $n$; for $i = 1,2,...,m$.

Also $n(v_1) = m$, $n(v_3) = m-2$ and

$$n(v_i) = \begin{cases} m-1, & i = 2,4,5,6,...,n-m+2 \\ m-2, & i = n-m+3, n-m+4, ..., m \\ m-1, & i = m+1, m+2, ..., n. \end{cases}$$

It follows that the number of $v_i$'s with $(m-1)$ neighbors equals $2(n-m)$ and let us rename those $v_i$'s as $w_1, w_2, ..., w_{2(n-m)}$. The number of $v_i$'s with $(m-2)$ neighbors equals $2m-n-1$ and let us rename those $v_i$'s as $l_1, l_2, ..., l_{2m-n-1}$. Now complete $E(H)$ by including the edges $w_1l_1, l_1l_2, l_2l_3, ..., l_{2m-n-2}l_{2m-n-1}, l_{2m-n-1}w_{2(n-m)}$ and $w_2w_3, w_4w_5, ..., w_{2n-2m-2}w_{2n-2m-1}$. Thus the graph $H$ has the same degree sequence as that of $K_{m,n}$. But $H$ is not isomorphic to $K_{m,n}$, since $H$ has the odd cycle $u_1u_2v_1u_1$.

**Subcase(ii)** Let $m-2 < n-m$.

Here we construct a graph $H_1$ on $m+n$ vertices as follows. Let $V(H_1) = \{a_1, a_2, ..., a_m, b_1, b_2, ..., b_n\}$. Draw the edges of $H_1$ as follows. Join $a_1$ with $b_2$ and $b_i$; $i \in \{1,2,...,n\}$; $i \neq 3$. Join $a_j$ with $a_{j+1}$ and $b_i$; $i \in \{1,2,...,n\}$; $i \neq j, n-(j-2)$ for $j = 2,3,4,5,...,m-1$. Join $a_m$ with $v_i$; $i \in \{1,2,...,n\}$; $i \neq m$. As a result of this the number of neighbors of $a_i$; $n(a_i)$ equals $n$; for $i = 1,2,...,m$.

Also $n(b_1) = m$, $n(b_3) = m-2$ and

$$n(b_i) = \begin{cases} m-1, & i = 2,4,5,6,...,m \\ m, & i = m+1, m+2, ..., n-m+2 \\ m-1, & i = n-m+3, n-m+4, ..., n. \end{cases}$$

It follows that the number of $b_i$'s with $(m-1)$ neighbors equals $2m-4$ and let us rename those $b_i$'s as $p_1, p_2, ..., p_{2m-4}$. The number of $b_i$'s with $(m-2)$ neighbors equals one which is $b_3$. Now complete $E(H_1)$ by including the edges $p_1b_3, b_3p_{2m-4}, p_2p_3, p_4p_5, ..., p_{2m-6}p_{2m-5}$. Thus the graph $H_1$ has the same degree sequence as that of $K_{m,n}$. But $H_1$ is not isomorphic to $K_{m,n}$, since $H_1$ has the odd cycle $a_1a_2b_1a_1$.

**Case(ii)** Assume that both $m$ and $n$ are even.

**Subcase(i)** Let $m-2 < n-m$. The discussion under this case coincides with the subcase(ii) of case(i).

**Subcase(ii)** Let $m-2 > n-m$. Here we construct a graph $F$ on $m+n$ vertices as follows. Let $V(F) = \{q_1, q_2, ..., q_m, r_1, r_2, ...r_n\}$. Draw the edges of $F$ as follows. Join $q_1$ with $q_2$ and $r_i$; $i \in \{1, 2, ..., n\}$; $i \neq 2$. Join $q_j$ with $q_{j+1}$ and $r_i$; $i \in \{1, 2, ..., n\}$; $i \neq j-1, j$ for $j = 2, 3, 4, 5, ..., m-1$. Join $q_m$ with $r_i$; $i \in \{1, 2, ..., n\}$, $i \neq m$. As a result of this the number of neighbors of $q_i$; $n(q_i)$ equals $n$; for $i = 1, 2, ..., m$ and

$n(r_1) = m-1$, $n(r_2) = m-3$ and

$$n(r_i) = \begin{cases} m-2, & i = 3, 4, 5, ..., m-2 \\ m-1, & i = m-1, m \\ m, & i = m+1, m+2, ..., n. \end{cases}$$

It follows that the number of $r_i$'s with $(m-1)$ neighbors equals $3$ and let us rename those $r_i$'s as $s_1, s_2$ and $s_3$. The number of $r_i$'s with $(m-2)$ neighbors equals $m-4$ and let us rename those $r_i$'s as $t_1, t_2, ..., t_{m-4}$. Also the number of $r_i$'s with $(m-3)$ neighbors equals one which is $r_2$. Now complete $E(F)$ by including the edges $s_1r_2, r_2t_1, t_1t_2, t_2t_3, ..., t_{m-5}t_{m-4}, t_{m-4}r_2$ and $s_2s_3$. Thus the graph $F$ has the same degree sequence as that of $K_{m,n}$. But $F$ is not isomorphic to $K_{m,n}$, since $F$ has the odd cycle $q_1q_2q_3...q_mr_1q_1$.

**Subcase(iii)** Let $m-2 = n-m$. Here we construct a graph $H^*$ on $m+n$ vertices as follows. Let $V(H^*) = \{x_1, x_2, ..., x_m, y_1, y_2, ...y_n\}$. Draw the edges of $H^*$ as follows. Join $x_1$ with $x_2$ and $y_i$; $i \in \{1, 2, ..., n\}$; $i \neq 3$. Join $x_j$ with $x_{j+1}$ and $y_i$; $i \in \{1, 2, ..., n\}$; $i \neq j, n-(j-2)$ for $j = 2, 3, 4, 5, ..., m-1$. Join $x_m$ with $y_i$; $i \in \{1, 2, ..., n\}$; $i \neq m$. As a result of this the number of neighbors of $x_i$; $n(x_i)$ equals $n$; for $i = 1, 2, ..., m$.

Then $n(y_1) = m$, $n(y_3) = m-2$ and $n(y_i) = m-1$; $i = 2, 4, 5, ..., n$.

It follows that the number of $y_i$'s with $(m-2)$ neighbors equals one, which is $y_3$. The number of $y_i$'s with $(m-1)$ neighbors equals $n-2$ and let us rename those $y_i$'s as $k_1, k_2, ..., k_{n-2}$. Now complete $E(H^*)$ by including the edges $k_1y_3, y_3k_{n-2}$; $k_2k_3, k_4k_5, ..., k_{n-4}k_{n-3}$. Thus the graph $H^*$ has the same degree sequence as that of $K_{m,n}$. But $H^*$ is not isomorphic to $K_{m,n}$, since $H^*$ has the odd cycle $x_1x_2y_1x_1$.

**Case(iii)** Assume that $m$ is odd and $n$ is even.

**Subcase(i)** Let $m-2 > n-m$. The discussion under this case coincides with the subcase(ii) of case(ii).

**Subcase(ii)** Let $m-2 < n-m$. The discussion under this case coincides with the subcase(ii) of case(i).

**Subcase(iii)** Let $m-2 = n-m$. The discussion under this case coincides with the subcase(iii) of case(ii).

**Case(iv)** Assume that $m$ is even and $n$ is odd.

**Subcase(i)** Let $m-2 > n-m$. The discussion under this case coincides with the subcase(i) of case(i).

**Subcase(ii)** Let $m-2 < n-m$. The discussion under this case coincides with the subcase(ii) of case(ii).

Now we assume that $m = n$. Here we construct a graph $G$ with $2n$ vertices as follows. Let $V(G) = V_1 \cup V_2$; where $V_1 = \{c_1, c_2, ..., c_n\}$; $V_2 = \{d_1, d_2, ..., d_n\}$. Draw complete graphs with the $n$ vertices in $V_1$ and $V_2$. Then complete $E(G)$ by drawing the edges $c_1d_1, c_2d_2, ..., c_nd_n$. Thus the graph $G$ has the same degree sequence as that of $K_{m,n}$. But $G$ is not isomorphic to $K_{m,n}$, since $G$ has the odd cycle $c_1c_2c_3c_1$. This completes the proof.

□

### 3.5 The friendship graph $F_n$

A friendship graph $F_n$ is the one-point union of $n$ copies of the cycle $C_3$.

**Lemma 3.12.** *For $n \geq 2$, we have $N_e[F_n; x] = n(2n-1)x^2$ (1).*

**Theorem 3.13.** *Let $F_n$ be the Friendship graph with $P$ vertices. Then $F_n$ is not $ENP-$ cardinal unique; $n \geq 2$.*

*Proof.* Since $F_n$ is the Friendship graph, the number of vertices, $P = 2n + 1$. From lemma 3.12, we have $N_e[F_n; x] = n(2n-1)x^2$. Let $H$ be the graph on $2n + 1$ vertices that consists precisely of $n$ numbers of path graph of length two and an isolated vertex. Hence $N_e[H; x] = N_e[F_n; x]$. But $F_n$ and $H$ is not isomorphic. This completes the proof. □

**Theorem 3.14.** *The null graph on $n$ vertices is $ENP-$ cardinal unique; $n \geq 1$.*

*Proof.* Let $G$ be a null graph with $n$ vertices. Then $N_e[G; x] = \binom{n}{2}$. Let $H$ be any graph with $n$ vertices such that $H \in [G]_{\mathcal{N}_\mathcal{E}}$. Then $N_e[H; x] = \binom{n}{2}$. It follows that the number of isolated vertices in $H$ equals $n$. Hence $H \cong N_n$. This completes the proof. □

## 4 Equivalence of Graph Operations

This section investigates the ENP equivalence of graphs resulting from specific operations or structures.

**Theorem 4.1.** *If $G$ is $ENP-$ cardinal unique, then $G + N_n$ is also $ENP-$ cardinal unique; where $N_n$ is the null graph on $n$ vertices; $n \geq 1$.*

*Proof.* Let the number of vertices in $G$ be $m$.

**Case(i)** Assume that $n(G, 0) = 0$. From lemma 2.3, we have

$$N_e[G + N_n; x] = N_e[G; x] + N_e[N_n; x] = N_e[G; x] + \binom{n}{2} \tag{4.1}$$

Let $H$ be any graph on $m+n$ vertices such that $H \in [G+N_n]_{\mathcal{N}_\mathcal{E}}$. Then the number of isolated vertices in $H$ is $n$, by Theorem 5. Now

$$N_e[H; x] = N_e[G + N_n; x] = N_e[G; x] + \binom{n}{2}, \text{by lemma } 2.3 \tag{4.2}$$

Let $V(H) = X \cup Y$; where $X = \{u \in V(H) : |N(u)| \geq 1\}$ and $Y = \{v_1, v_2, ..., v_n\}$; where $v_i \in V(H)$ and $|N(v_i)| = 0$; $i = 1, 2, ..., n$. Let $H^*$ be a subgraph of $H$ induced by $X$. Then obviously $H \cong H^* + N_n$. Now

$$N_e[H; x] = N_e[H^*; x] + N_e[N_n; x]; \text{by lemma 2.3} \tag{4.3}$$

$$= N_e[H^*; x] + \binom{n}{2} \tag{4.4}$$

From (**2**) and (**3**) it follows that $N_e[G; x] = N_e[H^*; x]$. Also note that $|V(G)| = |V(H^*)| = m$. Since $G$ is $ENP-$ cardinal unique; $H^* \cong G$. Hence $H \cong G + N_n$.

Therefore $G + N_n$ is $ENP-$ cardinal unique.

**Case(ii)** Assume that $n(G, 0) \geq 1$. Let $n(G, 0) = p$; $p \geq 1$. Also let $V(G) = A \cup B$; where $A = \{u \in V(H) : N(u) \geq 1)\}$ and $B = \{w_1, w_2, ..., w_p\}$; where $w_i \in V(G)$ and $N(w_i) = 0$; $i = 1, 2, ..., p$. Let $G^*$ be a subgraph of $G$ induced by $B$. Then obviously $G \cong G^* + N_p$. Now

$$\begin{aligned} N_e[G; x] &= N_e[G^*; x] + N_e[N_p; x]; \text{by lemma } 2.3 \\ &= N_e[G^*; x] + \binom{p}{2} \end{aligned} \tag{4.5}$$

Let $K$ be any graph on $m + n$ vertices such that $K \in [G + N_n]_{\mathcal{N}_\mathcal{E}}$. Then the number of isolated vertices in $K$ is $p + n$, by Theorem 5. Now

$$\begin{aligned} N_e[K; x] &= N_e[G + N_n; x] \\ &= N_e[G^* + N_p + N_n; x]; \text{since} G \cong G^* + N_p \\ &= N_e[G^* + N_{p+n}; x] \\ &= N_e[G^*; x] + N_e[N_{p+n}; x]; \text{ by lemma } 2.3 \\ &= N_e[G^*; x] + \binom{p+n}{2} \end{aligned} \tag{4.6}$$

Let $V(K) = A^* \cup B^*$ where $A^* = \{u \in V(K) : |N(u)| \geq 1)\}$ and $B^* = \{m_1, m_2, ..., m_{p+n}\}$; where $m_i \in V(K)$ and $|N(m_i)| = 0$; $i = 1, 2, ..., p + n$. Let $K^*$ be the subgraph of $K$ induced by $A^*$. Then obviously $K = K^* + N_{p+n}$. Now

$$\begin{aligned} N_e[K; x] &= N_e[K^* + N_{p+n}; x] \\ &= N_e[K^*; x] + N_e[N_{p+n}; x]; \text{ by lemma } 2.3 \\ &= N_e[K^*; x] + \binom{p+n}{2} \end{aligned} \tag{4.7}$$

From (**5**) and (**6**), we have $N_e[G^*; x] = N_e[K^*; x]$. Note that $|V(G^*)| = |V(K^*)|$. Also $G^*$ is $ENP-$ cardinal unique; by case(i). Therefore

$$\begin{aligned} G^* \cong K^* &\Rightarrow G^* + N_{p+n} \cong K^* + N_{p+n} \\ &\Rightarrow G^* + N_p + N_n \cong K^* + N_{p+n} \\ &\Rightarrow G + N_n \cong K. \end{aligned}$$

This completes the proof. □

## 4.1 Path graphs $P_n$

**Lemma 4.2.** *For a path graph $P_n$, we have*

$$N_e[P_n; x] = x + \binom{n-2}{2} x^2; n \geq 2.$$

**Theorem 4.3.** *The path graph $P_n$; for $n = 2, 4$ is $ENP-$ cardinal unique.*

*Proof.* Let $H$ be any graph with $n$ vertices such that $H \in [P_n]_{\mathcal{N}_\mathcal{E}}$; $n = 2, 4$. Then by lemma 4.2, $N_e[H; x] = x + \binom{n-2}{2} x^2$; $n = 2, 4$.

**Case(i)** Let $n = 2$. Then $N_e[H; x] = x$. This means that both the vertices in $H$ are of degree $1$. Then $H \cong P_2$.

**Case(ii)** Let $n = 4$. Then $N_e[H; x] = x + x^2$. This means that the two vertices in $H$ are of degree $1$ and the remaining two vertices in $H$ are of degree $2$. Then obviously $H \cong P_4$.

This ompletes the proof. □

**Theorem 4.4.** *The path graph $P_n$; $n \in \{3, 5, 6, 7, 8, ...\}$ is not $ENP-$ cardinal unique.*

*Proof.* Let us consider the following two cases.

**Case(i)** Let $n = 3$. Then lemma 4.2, $N_e[P_3; x] = x$. Let $H$ be a graph on $3$ vertices that consists precisely of a path graph of length two and an isolated vertex. Then $N_e[H; x] = x$. But obviously $H$ is not isomorphic to $P_3$.

**Case(ii)** Let $n \geq 5$. From lemma 4.2,, we have $N_e[P_n; x] = x + \binom{n-2}{2} x^2; n \geq 5$. Let $H$ be a graph on $n$ vertices that consists precisely of a path graph of length two and a cycle graph on $n - 2$ vertices; $n \geq 5$. Then

$$\begin{aligned}
N_e[H; x] &= N_e[P_2 + C_{n-2}; x] \\
&= N_e[P_2; x] + N_e[C_{n-2}; x]; \text{by lemma } 2.3 \\
&= x + \binom{n-2}{2} x^2; \text{by lemma } 4.2, \text{ by lemma } 3.3 \\
&= N_e[P_n; x].
\end{aligned}$$

But $H$ is not isomorphic $P_n$. This completes the proof. □

**Theorem 4.5.** *If $G_1$ and $G_2$ are $ENP-$ cardinal unique $G_1 + G_2$ need not be $ENP-$ cardinal unique.*

*Proof.* Let $G_1 = K_4$ and $G_2 = C_3$. $G_1$ is $ENP-$ cardinal unique by theorem 3.7 and $G_2$ is $ENP-$ cardinal unique by theorem 3.4. Now consider the graph $G_1 + G_2$.

$$\begin{aligned}
N_e[G_1 + G_2; x] &= N_e[K_4 + C_3; x] \\
&= N_e[K_4; x] + N_e[C_3; x]; \text{by lemma } 2.3 \\
&= 6x^3 + 3x^2; \text{by lemma } 3.6, \text{ by lemma } 3.3
\end{aligned}$$

Let $G$ be the graph on $7$ vertices as shown in the figure. Then $N_e[G; x] = 6x^3 + 3x^2$. But clearly $G$ is not isomorphic to $G_1 + G_2$. This completes the proof.

□

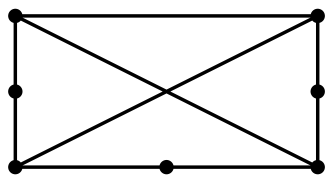

Figure 2: The graph G

## 4.2 The Lollipop graph $L_{m,n}$

The lollipop graph $L_{m,n}$ is a graph obtained by joining a complete graph $K_m$ to a path $P_n$ with a bridge.

**Lemma 4.6.** *For a Lollipop graph $L_{m,n}$,*

$$N_e[L_{m,n};x] = \begin{cases} \binom{m-1}{2}x^{m-1} + \binom{n-1}{2}x^2, & \textit{if } m \geq 4 \\ \binom{n-1}{2}x^2 + x, & \textit{if } m = 1 \\ \binom{n}{2}x^2 + x, & \textit{if } m = 2 \\ \binom{n+1}{2}x^2, & \textit{if } m = 3. \end{cases}$$

**Theorem 4.7.** $L_{n,1} \in [K_{n-1}]_{\mathcal{N}_\mathcal{E}}$

*Proof.* The result follows from lemma 4.6 and lemma 3.6. ☐

## 4.3 The Wheel graph $W_n$

A Wheel graph $W_n, n > 3$ is obtained by taking the join of the cycle $C_{n-1}$ and $K_1$.

**Lemma 4.8.**

$$N_e[W_n;x] = \begin{cases} \binom{n-1}{2}x^3, & \textit{if } n \neq 4 \\ 6x^3, & \textit{if } n = 4. \end{cases}$$

**Theorem 4.9.** $[F_n]_{\mathcal{N}_\mathcal{E}}$ *contains both* $C_{2n}$ *and* $W_{2n+1}$.

*Proof.* The result follows from lemma 3.3 and lemma 4.8. ☐

A Tadpole $T_{(n,l)}$ is a graph obtained by attaching a path $P_l$ to one of the vertices of the cycle $C_n$ by a bridge.

**Lemma 4.10.** *For a Tadpole graph $T_{(n,l)}$ with $n > 2$, we have the following.*

$$N_e[T_{(n,l)};x] = \begin{cases} N_e[C_n;x] + N_e[P_l;x] + n(l-2)x^2 - x, & \textit{if } l > 1 \\ N_e[C_n;x] + N_e[P_l;x] - (n-1)x^2, & \textit{if } l = 1. \end{cases}$$

**Theorem 4.11.** $T_{n,l} \in [C_{n+l-2}]_{\mathcal{N}_\mathcal{E}}$.

*Proof.* The result follows from lemma 4.10 and lemma 3.3. ☐

**Theorem 4.12.** $[L_{1,n}]_{\mathcal{N}_\mathcal{E}}$ *contains* $P_{n+1}$.

*Proof.* The result follows from lemma 4.6 and lemma 4.2. ☐

# 5 Conclusion

In this paper, we identify and characterize various graph classes based on the definition of ENP-equivalent class. Also, establishing fundamental theorems relating the ENP of a graph to that of its complement and unions, and determining which well-known graph families possess ENP-unique or ENP-cardinal unique properties. We identify some $ENP-$ unique graphs, some $ENP$-equivalent graph classes, and some $ENP$-cardinal unique graph classes.

The concept of equi-neighbour polynomial equivalent classes of graphs is useful in studying structural similarities between graphs. Two graphs are said to be equivalent if they have the same equi-neighbour polynomial, even though the graphs themselves may not be isomorphic. This concept helps in classifying graphs into families that share similar neighbourhood structures and simplifies the analysis of large graph collections. It also serves as a graph invariant that aids in distinguishing non-isomorphic graphs and supports investigations related to the Graph Isomorphism Problem. Moreover, such polynomial equivalence is helpful in applications like network analysis, pattern recognition, and studies in Chemical Graph Theory, where the structural similarity of graphs or molecular models plays an important role.

# Disclaimer (Artificial Intelligence)

Author(s) hereby declare that NO generative AI technologies such as Large Language Models (ChatGPT, COPILOT, etc) and text-to-image generators have been used during writing or editing of manuscripts.

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