A Note on Separation Number of Graphs

Abstract

In this paper, the concept of total domination is used to introduce a new graph parameter, called the *separation number* $S_n(G)$ of a graph G. We study the fundamental properties of this parameter and establish general lower and upper bounds for $S_n(G)$. Furthermore, the separation number is determined for various classes of graphs, including complete graphs and star graphs. We also examine the behavior of $S_n(G)$ under specific graph operations such as the corona of graphs and the conormal product of complete graphs.

Keywords: total domination set; separation number; complete graphs, complete graphs,star graphs

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1 Introduction

A *graph*[3],[1],[9] is an ordered pair $G = (V(G), E(G))$, where $V(G)$ is a finite non-empty set and $E(G)$ is a collection of unordered pairs of vertices called edges. A *total dominating set*[5], [2], [4] of a graph $G = (V, E)$ with no isolated vertex is set S of vertices of G such that every vertex of G is adjacent to a vertex in S. Latheesh Kumar[7] introduced a new separation axiom in graphs using the concept of total domination in analogues to the separation axioms in topology, [6],[8]. A simple connected graph G is said to satisfy the T_0 axiom [7] if for any two distinct vertices u and v of G, there exists a total dominating set S such that either $u \in S$ but not v or $v \in S$ but not u. Using this a new parameter called separation number of graphs is introduced.

2 Main Results

Definition 2.1. A separating set $S(G)$ of a graph G is a set of total dominating sets that separates all the vertices of G.

Definition 2.2. A separating set is called a minimal separating set if no proper subset of it is a separating set.

Definition 2.3. A minimum separating set is a separating set containing minimum number of sets.

Definition 2.4. The separation number $S_n(G)$ of a graph G is the number of sets in a minimum separating set of G.

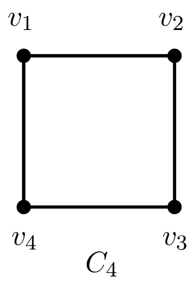

Figure 1: C_4

The separating sets of C_4 shown in Figure 1 are given below:

$$
\begin{aligned}
S_1 &= \{\{v_1, v_2\}, \{v_2, v_3\}\}, \\
S_2 &= \{\{v_2, v_3\}, \{v_3, v_4\}\}, \\
S_3 &= \{\{v_3, v_4\}, \{v_4, v_1\}\}, \\
S_4 &= \{\{v_1, v_2, v_3\}, \{v_2, v_3, v_4\}, \{v_3, v_4, v_1\}\}, \\
S_5 &= \{\{v_2, v_3, v_4\}, \{v_3, v_4, v_1\}, \{v_4, v_1, v_2\}\}, \\
S_6 &= \{\{v_3, v_4, v_1\}, \{v_4, v_1, v_2\}, \{v_1, v_2, v_3\}\}, \\
S_7 &= \{\{v_4, v_1, v_2\}, \{v_1, v_2, v_3\}, \{v_2, v_3, v_4\}, \}.
\end{aligned}
$$

Here S_4, S_5, S_6 and S_7 are minimal separating sets while S_1, S_2 and S_3 are minimum separating sets of C_4. Therefore $S_n(C_4) = 2$.

Theorem 2.1. *For a T_0 graph G with n vertices,* $\lceil \log_2 n \rceil \leq S_n(G) \leq n - 1$.

Proof. Let $V(G) = \{v_1, v_2, \ldots, v_n\}$. Let $k = \lceil \log_2 n \rceil$, so that $2^k \geq n$.

List the first *n* binary strings of length k (i.e., strings of the form $(b_1, b_2, \ldots, b_k)$ where $b_i \in \{0, 1\}$). Assign these strings to the vertices as follows: Let v_1 be the first binary string, v_2 be the second binary string, $\ldots, v_n$ be the n^{th} binary string.

Define $S = \{S_1, S_2, \ldots, S_k\}$, where $S_i = \{v_j \mid$ the i-th bit of v_j is $0\}$. In this construction, for any distinct vertices v_p and v_q, their assigned binary strings differ in at least one coordinate, say the i-th bit. Then, S_i contains exactly one of v_p and v_q, thus separating them. Hence, $k = \lceil \log_2 n \rceil \leq S_n(G)$.

Next we show that $S_n(G) \leq n - 1$.

Case (i) G has no vertices of degree one
$S(G) = \{V - \{v_i\}|\ i = 1, 2, \ldots, n-1\}$ is a separating set of G.

Case (ii) Let $v_1, v_2, \ldots, v_r$ be the pendant vertices of G. Since G is a T_0 graph all the pendant vertices are adjacent to a single vertex. Let it be v_n. Then $S(G) = \{V - \{v_i\}|\ i = 1, 2, \ldots, n-1\}$ is a separating set of G with $n-1$ sets. Therefore $S_n(G) \leq n-1$. This completes the proof.

□

Remark 2.1. The bounds of the above theorem are sharp. For the complete graphs K_3 and K_4, $S(K_3) = \{\{v_1, v_2\}, \{v_2, v_3\}\} = S(K_4)$. Therefore $S_n(K_3) = 2 = n-1$ and $S_n(K_4) = 2 = \lceil \log_2 4 \rceil$.

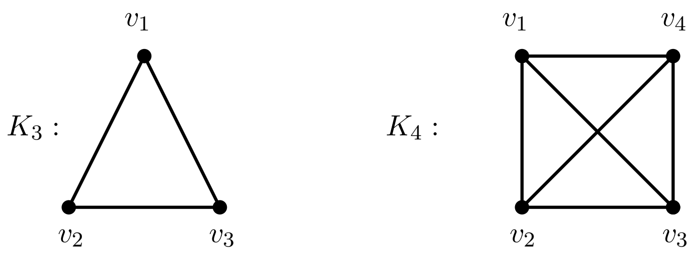

Figure 2: K_3 and K_4

Theorem 2.2. *For the path* P_3, $S_n(P_3) = 2$.

Proof. Let P_3 be as shown in the figure.

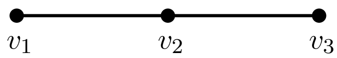

Figure 3: P_3

Then the set $S(P_3) = \{\{v_1, v_2\}, \{v_2, v_3\}\}$ separates all the vertices of P_3. Since $\lceil \log_2 3 \rceil = 2$, it follows that $S_n(P_3) = 2$. □

Theorem 2.3. *The separation number of the complete graph* K_n, *with* $n \geq 3$ *vertices is precisely* $S_n(K_n) = \lceil log_2\ n \rceil$.

Proof. Every two element subset of the complete graph K_n constitutes a total dominating set. Consequently, the statement follows from Theorem 2.1. □

Theorem 2.4. *The separation number of the star graph* $K_{1,n}$, *is* $S_n(K_{1,n}) = \lceil log_2\ (n+1) \rceil$.

Proof. In the proof of Theorem 2.1, the binary string $(0, 0, \ldots, 0)$ is assigned to the vertex v_1. The separating sets are defined as

$$S_i = \{v_j \mid \text{the } i\text{-th bit of } v_j \text{ is } 0\}.$$

Thus, $v_1 \in S_i$ for all i.

Let v_1 be the central vertex of degree n in the star graph $K_{1,n}$. By construction, each separating set S_i contains at least two vertices, ensuring that $|S_i| \geq 2$.

Consequently, each S_i forms a total dominating set of $K_{1,n}$. This completes the proof. □

Theorem 2.5. *The separation number of the wheel graph $W_{1,n}$, is $S_n(W_{1,n}) = \lceil log_2\ (n+1) \rceil$.*

Proof. Let v_1 be the vertex of degree n in $W_{1,n}$. Then proceeding as in Theorem 2.4 we can prove the result. □

Theorem 2.6. *The separation number of the complete bipartite graph $K_{m,n}$, is $\lceil log_2\ (m+n) \rceil$.*

Proof. The separation number of the star graph is already obtained. Also, $S_n(K_{2,2}) = 2$. So we exclude star graphs and $K_{2,2}$ in this discussion. Let $m + n \geq 5$. Let $X = \{v_1, v_2, \ldots, v_m\}$ and $Y = \{u_1, u_2, \ldots, u_n\}$ be the partite sets of $K_{m,n}$. We arrange the vertices as a sequence such that $P = (v_1, v_2, v_3, u_1, u_2, u_3, v_4, v_5, v_6, u_4, u_5, u_6, \ldots)$. Consider the first $m+n$ binary strings of length $k = \lceil log_2\ (m{+}n) \rceil$ of $0'$s and $1'$s. These binary strings serve as labels for the vertices in the sequence P. Assign the first binary string to v_1 , the second to v_2 and continue this process so that each vertex in the sequence P receives a unique binary string as its label corresponding to its position in the sequence. We form the separation set $S(G) = \{S_1, S_2, \ldots, S_k\}$ such that $S_i = \{w_r | i^{th}\ bit\ of\ w_r\ is\ 0\}$. This construction guarantees that each S_i includes at least one vertex from both X and Y, thereby ensuring that S_i forms a total dominating set of the complete bipartite graph $K_{m,n}$. In this construction, each vertex is assigned a unique binary string of length k. For any pair of distinct vertices, their corresponding binary strings differ in at least one position say, the i-th bit. This difference ensures that there exists a set which contains exactly one of the two vertices, thereby satisfying the separation property. This completes the proof. □

Theorem 2.7. *Let G be a graph with $n \geq 2$ vertices. Then the separation number of the corona of G with complete graph K_m is given by*

$$S_n(G \circ K_m) = \begin{cases} n\lceil log_2 m \rceil + n - 1 & \textit{if } m \equiv 0 \pmod 4 \\ n\lceil log_2(m+1) \rceil & \textit{otherwise} \end{cases}$$

Proof. **Case i:** When $m \not\equiv 0 \pmod 4$. For the complete graph K_m, $S_n(K_m) = \lceil log_2 m \rceil$. While building the separating set, the vertex v_1 is assigned as the first binary string, v_2 is assigned as the second binary string and so on. Since $m \not\equiv 0 \pmod 4$, the string $(1, 1, 1, \ldots, 1)$ is excluded from labelling the vertices. So all vertices of K_m are included in atleast one of the separating sets. In $G \circ K_m$, all vertices of the i^{th} copy of K_m are adjacent to the i^{th} vertex of G. So there are n copies of K_{m+1} adjoined to the n vertices of G. So the union of the separating families of each copy of K_{m+1} will be a minimal separating family of $K_m \circ G$. Therefore $S_n(G \circ K_m) = n\lceil log_2(m+1) \rceil$

Case ii: When $m \equiv 0 \pmod 4$. Proceeding as in Case (i), the vertex v_1 is assigned the first binary string, v_2 the second, and so on, with v_n corresponding to the binary string $(1, 1, \ldots, 1)$. For each i, define

$$S_j = \{\, u \mid \text{the } j^{\text{th}} \text{ bit of the binary string assigned to } u \text{ is } 0 \,\}.$$

It is evident that the vertex v_n does not belong to any of the sets S_j. As in Case (i) if we select the union of the separating sets then the excluded vertices of the n copies of K_m can not be separated. So for $i = 1, 2, \ldots, n-1$. we set the separating set of the i^{th} copy of K_m as $S^i = \{S_1, S_2, \ldots, S_{\lceil log_2 n \rceil}, V^i(K_m)\}$, where $V^i(K_m)$ is the vertex set of the i^{th} copy of K_m. For the n^{th} copy of K_m, the separating set is $S^i = \{S_1, S_2, \ldots, S_{\lceil log_2 n \rceil}$. Then the union of all these separating sets will form a separating set of $G \circ K_m$. Therefore $S_n(G \circ K_m) = (n-1)[\lceil log_2 m \rceil + 1] + \lceil log_2 m \rceil = n\lceil log_2 m \rceil + n - 1$. This completes the proof.

□

Theorem 2.8. *The separation number of the join of K_m with a graph G is given by $S_n(K_m \vee G) = \lceil log_2(m+k) \rceil$, where k is the number of vertices in G.*

Proof. Let $V(K_m) = \{v_1, v_2, \ldots, v_m\}$ and $V(G) = \{u_1, u_2, \ldots, u_k\}$. The first m binary strings are assigned to the vertices $v_1, v_2, \ldots, v_m$ and the next k binary strings are assigned to the vertices $u_1, u_2, \ldots, u_k$. Let $S = \{S_1, S_2, \ldots, S_{\lceil log_2(m+k) \rceil}\}$, where $S_j = \{\, w \mid \text{the } j^{\text{th}} \text{ bit of the binary string assigned to } w \text{ is } 0 \,\}$. Then S is a minimum separating set of $K_m \vee G$ consisting of total dominating sets. This completes the proof. □

Theorem 2.9. *The separation number of the conormal product[**?**] or disjunctive product of K_m with a graph G with k vertices is given by $S_n(K_m * G) = \lceil log_2(mk) \rceil$.*

Proof. Let x_1, x_2 be two vertices of K_m. Since they are adjacent in K_m, any two pair of vertices (x_1, y_1) and (x_2, y_2) in $K_m * G$, are adjacent. Therefore proceeding as in Theorem 2.8 the result can be proved. □

Theorem 2.10. *The separation number of the conormal product of two star graphs $K_{1,n}$ and $K_{1,m}$ is given by $S_n(K_{1,n} * K_{1,m}) = \lceil log_2(mn) \rceil$.*

Proof. Let $V(K_{1,n}) = \{u, x_1, x_2, \ldots, x_n\}$ and $V(K_{1,m}) = \{u, y_1, y_2, \ldots, y_m\}$, where $d(u) = m$ and $d(v) = m$. Consider the first mn binary strings of length k, where $k = \lceil log_2(mn) \rceil$. Assign the first string to the vertex (u, v) and the remaining strings to other vertices. Then if S is a subset of $V(K_{1,n} * K_{1,m})$ with $(u, v) \in S$ and $|S| \geq 2$, then S is a total domination set of $K_{1,n} * K_{1,m}$. Then the proof follows by proceeding as in Theorem 2.8. □

Conclusion

In this paper, we introduced the separation number $S_n(G)$ of a graph G based on the concept of total domination. We established general lower and upper bounds for $S_n(G)$ and determined its exact values for certain classes of graphs such as complete graphs, star graphs, the corona of graphs, and the conormal product of complete graphs. The results provide a new perspective on domination parameters and contribute to the structural understanding of graphs. Future work may focus on extending the concept to other graph products and exploring algorithmic methods for computing $S_n(G)$ in complex networks.

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