EXISTENCE, UNIQUENESS, BOUNDEDNESS, AND CONTINUOUS DEPENDENCE OF SOLUTIONS FOR FRACTIONAL ORDER FREDHOLM DIFFERENCE EQUATIONS

Abstract. In this paper, we investigate the existence and uniqueness of solutions to certain fractional-order Fredholm-type difference equation involving an iterated sum. In addition, we examine the boundedness and continuous dependence of solutions under various assumptions imposed on the associated functions. The results are established using finite difference inequalities with explicit estimates, and offer fundamental insights that may serve as a valuable reference for future research.

Keywords and phrases: Difference equation, Fractional order, Initial value problem, Inequality.

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1. Introduction

The set of natural numbers, including zero, is denoted by $\mathbb{N}_0$, and $\mathbb{N}_a = \{a, a+1, a+2, \dots\}$ for $a \in \mathbb{Z}$. Let $u(n) : \mathbb{N}_0 \to \mathbb{R}$. Consider the following nonlinear Fredholm type difference equation with iterated sum and order $\alpha \in (0,1)$:

$$\nabla^{\alpha} u(n+1) = F\left(n, u(n), \sum_{\sigma=0}^{\beta} k(n, \sigma, u(\sigma))\right) \tag{1}$$

$$u(0) = u_0 \tag{2}$$

where u, k, F, are the elements of $\mathbb{R}^m$ an n dimensional Euclidean space with norm || and $k : E \times \mathbb{R}^m \to \mathbb{R}^m, F : \mathbb{N}^2_{\alpha,\beta} \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ in which $E = \Big\{(n, s) \in \mathbb{N}^2_{0,\beta} : 0 \le s \le n \le n \le \beta\Big\}$.

The study of fractional differential equations was initiated earlier, and it has recently been established that many classes of such equations admit unique solutions [12]. Although the theory of integro-differential equations has been almost fully developed in parallel with that of differential equations [11, 17],the literature on fractional integro-differential equations is still less developed.Moreover, the advancement in the theory of fractional-order difference equations has been relatively minimal.

By allowing the order of the difference in the usual n^{th} difference expression to be any real or complex number, Diaz and Osler [5] defined the fractional difference. Later, Hirota [10] used Taylor's series to define the fractional order difference operator ∇^{α}, where α is any real number.

By altering Hirota's definition, Nagai [14] selected a different definition for the fractional order difference operator. Deekshitulu and Mohan [2] recently modified Nagai's definition for $0 < \alpha < 1$ so that there is no difference operator in the formula for ∇^{α}.

In 2010, Deekshitulu and Mohan [2] studied the existence and other properties of special version of equation (1) (see [1, 3, 4, 7, 13, 18]) and some of references cited therein ([6, 8, 9, 19, 20]). Authors are motivated by the work of Deekshitulu and Mohan, [2, 4]. Hence, the equation (1) considered in this paper is in the general spirit of the investigations.

The main objective of this paper is to examine the boundedness, uniqueness, and continuous dependence of solutions to the given equations under various assumptions on the associated functions. The analysis primarily employs finite difference inequalities, with explicit estimates available in [2, 15, 16]. We believe that the results, obtained through elementary analysis, offer fundamental insights and may serve as a valuable reference for future research.

2. Preliminaries

For clarity and consistency, the following notations and definitions are employed throughout the paper (more information refer [2]). For all $n_1, n_2 \in \mathbb{N}_0$ and $n_1 > n_2$,

$$\sum_{j=n_1}^{n_2} u(j) = 0, \qquad \prod_{j=n_1}^{n_2} u(j) = 1.$$

In other words, the products and empty sums are taken to be 1 and 0, respectively. If n and $n-1$ are in $\mathbb{N}_0$, then the backward difference operator ∇ for the function $u(n)$, is defined as follows:

$$\nabla u(n) = u(n) - u(n-1).$$

We now present some fundamental definitions and findings related to Nabla discrete fractional calculus.

Definition 1. [2] The extended binomial coefficient $\binom{a}{n}$, where $a \in \mathbb{R}$ and $n \in \mathbb{Z}$, is defined by

$$\binom{a}{n} = \begin{cases} \frac{\Gamma(a+1)}{\Gamma(a-n+1)\Gamma(n+1)}, & \text{if } n > 0, \\ 1, & \text{if } n = 0, \\ 0, & \text{if } n < 0, \end{cases} \tag{3}$$

Definition 2. [7] For any complex numbers α and β, we define $\binom{\alpha}{\beta}$ as follows:

$$\binom{\alpha}{\beta} = \begin{cases} \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-\beta+1)}, & \text{when } \alpha \text{ and } \alpha-\beta \text{ are neither zero nor negative integers,} \\ 1, & \text{when } \alpha = \beta = 0, \\ 0, & \text{when } \alpha = 0, \beta \text{ is neither zero nor negative integer,} \\ \text{undefined}, & \text{otherwise.} \end{cases} \tag{4}$$

Remark: Let α and β be any two complex numbers. If α, β, and $\alpha-\beta$ are neither zero nor negative integers, then

$$(\alpha+\beta)_n = \sum_{k=0}^{n} \binom{n}{k} (\alpha)_{n-k} (\beta)_k, \tag{5}$$

for any positive integer n.

In 2003, Nagai [14] introduced the following definition for fractional order difference operator.

Definition 3. Let $\alpha \in \mathbb{R}$ and m be an integer such that $m-1 < \alpha \le m$. The difference operator ∇ of order α, with step length ϵ, is defined as

$$\nabla^{\alpha} u(n) = \begin{cases} \nabla^{\alpha-m}[\nabla^m u(n)] = \epsilon^{m-\alpha} \sum_{j=0}^{n-1} \binom{\alpha-m}{j} (-1)^j \nabla^m u(n-j), & \text{if } \alpha > 0, \\ u(n), & \text{if } \alpha = 0, \\ \epsilon^{-\alpha} \sum_{j=0}^{n-1} \binom{\alpha}{j} u(n-j), & \text{if } \alpha < 0. \end{cases} \tag{6}$$

Studying the properties of the solution becomes challenging because the definition of $\nabla^{\alpha} u(n)$ given by Nagai [14] includes an ∇ operator and the term $(-1)^j$ inside the summation index. To circumvent this, Deekshitulu and Mohan [2, 4] provided the following definition for $\varepsilon = m = 1$.

Definition 4. The fractional sum operator of order α is defined as.

$$\nabla^{-\alpha} u(n) = \sum_{j=0}^{n-1} \binom{j+\alpha-1}{j} u(n-j) = \sum_{j=1}^{n} \binom{n-j+\alpha-1}{n-j} u(j). \tag{7}$$

The following definition of the fractional order difference operator of order α.

$$\nabla^{\alpha} u(n) = \sum_{j=0}^{n-1} \binom{j-\alpha}{j} \nabla(n-j) = \sum_{j=1}^{n} \binom{n-j-\alpha-1}{n-j} u(j) - \binom{n-\alpha-1}{n-1} u(0). \tag{8}$$

Remark: Assume that u, $v : \mathbb{N}_0^+ \to \mathbb{R}$ and $\alpha, \beta \in \mathbb{R}$ are constants such that $0 < \alpha, \beta, \alpha+\beta < 1$, and c, d are also constants. Then

(1) $\nabla^{\alpha}\nabla^{\beta} u(n) = \nabla^{\alpha+\beta} u(n)$,
(2) $\nabla^{\alpha}[(cu(n) + dv(n)] = c\nabla^{\alpha} u(n) + d\nabla^{\alpha} v(n)$,
(3) $\nabla^{-\alpha}\nabla^{\alpha} u(n) = u(n) - u(0)$,
(4) $\nabla^{\alpha}\nabla^{-\alpha} u(n) = u(n)$,
(5) $\nabla^{\alpha} u(0) = 0$ and $\nabla^{\alpha} u(1) = u(1) - u(0) = \nabla u(1)$.

3. EXISTENCE OF SOLUTION

The following theorem establishes the existence of a solution to equations (1)–(2).

Theorem 1. *There exists a solution $u(n)$ of the initial value problem* (1)-(2).

Proof. The existence of a solution to the Fredholm-type difference equation with an iterated sum is straightforward, since the solution can be represented as a recurrence relation involving the values of the unknown function at earlier arguments (for more clarity refer ([19]). It follows from the definition of the fractional sum operator and the initial condition.Hence, considering equation (7) and replacing $u(n)$ by $\nabla^{\alpha}u(n)$, we obtain

$$\nabla^{-\alpha}[\nabla^{\alpha}u(n)] = \sum_{j=1}^{n} \binom{n-j+\alpha-1}{n-j} [\nabla^{\alpha}u(j)],$$

or

$$u(n) - u(0) = \sum_{j=1}^{n} \binom{n-j+\alpha-1}{n-j} [\nabla^{\alpha}u(j)],$$

or

$$u(n) = u(0) + \sum_{j=0}^{n-1} \binom{n-j+\alpha-2}{n-j-1} [\nabla^{\alpha}u(j+1)], \tag{9}$$

or

$$u(n) = u_0 + \sum_{j=0}^{n-1} B(n-1,\alpha;j) \left[F\Big(j, u(j), \sum_{\sigma=0}^{\beta} K(j,\sigma,u(\sigma))\Big) \right] \tag{10}$$

where $B(n,\alpha;j) = \binom{n-j+\alpha-1}{n-j}$ for $0 \le j \le n$. The recurrence relation above indicates that (1)-(2) has a solution. □

4. UNIQUENESS OF SOLUTION

We now prove that the solutions to the fractional order difference equations (1)-(2) unique. For this, we need the following results.

Lemma 1. [4] For $n \in \mathbb{N}_0$, $\displaystyle\sum_{j=0}^{n} B(n,\alpha;j) = \binom{n+\alpha}{n}$.

For more clarity, we present some basic finite difference inequalities which play crucial role to establish the fractional difference inequalities.

Theorem 2. [15]*: Let $u(n)$, $a(n)$, and $b(n)$ be real non-negative functions defined on $\mathbb{N}_0$ and*

$$\Delta u(n) \le a(n)u(n) + b(n)$$

for $n \in \mathbb{N}_0$. Then

$$u(n) \le u(0) \prod_{j=0}^{n-1} \Big[1 + a(j)\Big] + \sum_{j=0}^{n-1} b(j) \prod_{k=j+1}^{n-1} \Big[1 + a(k)\Big],$$

for $n \in \mathbb{N}_0$.

Theorem 3. [3] *Let $u(n)$, $a(n)$, and $b(n)$ be real non-negative functions defined on $\mathbb{N}_0$. If*

$$u(n) \leq u(0) + \sum_{j=0}^{n-1}[a(j)u(j) + b(j)]$$

for $n \in \mathbb{N}_0$, then

$$\begin{aligned} u(n) &\leq u(0) \prod_{j=0}^{n-1} \Big[1 + a(j)\Big] + \sum_{j=0}^{n-1} b(j) \prod_{k=j+1}^{n-1} \Big[1 + a(k)\Big] \\ &\leq u(0) \exp\big(\sum_{j=0}^{n-1} a(j)\big) + \sum_{j=0}^{n-1} b(j) \exp\big(\sum_{k=j+1}^{n-1} a(k)\big). \end{aligned}$$

The following corollary is proved by B. G. Pachpatte ([15], p.12).

Corollary 1. Let $u(n)$ and $b(n)$ be real non-negative functions defined on $\mathbb{N}_0$, and $c \geq 0$ (a constant). If

$$u(n) \leq c + \sum_{j=0}^{n-1} \Big[b(j)u(j)\Big]$$

for $n \in \mathbb{N}_0$, then

$$u(n) \leq c \prod_{j=0}^{n-1} \Big[1 + b(j)\Big] \leq c \exp\big(\sum_{j=0}^{n-1} b(j)\big).$$

Finite fractional difference inequalities which provides explicit bounds on the unknown functions and analysis of various problems in the theory of finite fractional difference equations. So, on similar line of discrete inequalities mentioned above, we present the finite fractional inequalities.

Theorem 4. [3] *Let $u(n)$, $a(n)$, and $b(n)$ be real valued non-negative functions defined on $\mathbb{N}_0$. If for $n \in \mathbb{N}_0, 0 < \alpha < 1$,*

$$\nabla^\alpha u(n+1) \leq a(n)u(n) + b(n),$$

then

$$u(n) \leq u(0) \prod_{j=0}^{n-1} \Big[1 + B(n-1, \alpha; j)a(j)\Big] + \sum_{j=0}^{n-1} B(n-1, \alpha; j)b(j) \prod_{k=j+1}^{n-1} \Big[1 + B(n-1, \alpha; k)a(k)\Big].$$

Corollary 2. Let $u(n)$, $a(n)$, and $b(n)$ be real valued non-negative functions defined on $\mathbb{N}_0$. If for $0 < \alpha < 1, n \in \mathbb{N}_0$,

$$u(n) \leq u(0) + \sum_{j=0}^{n-1} B(n-1, \alpha; j)\Big[a(j)u(j) + b(j)\Big],$$

then

$$u(n) \leq u(0) \prod_{j=0}^{n-1} \Big[1 + B(n-1, \alpha; j)a(j)\Big] + \sum_{j=0}^{n-1} B(n-1, \alpha; j)b(j) \prod_{k=j+1}^{n-1} \Big[1 + B(n-1, \alpha; k)a(k)\Big].$$

for $n \in \mathbb{N}_0$.

In the literature, some authors used Corollary 2 to study the various properties of solutions of finite fractional difference equations. But direct use of this corollary leads to some flaws. So, we present the following fractional inequality to address the issue raised due to the use of Corollary 2.

Theorem 5. *Let $u(n)$, $a(n)$, and $b(n)$ be real non-negative functions defined on $\mathbb{N}_0$, and $c \geq 0$ (a constant). If for $0 < \alpha < 1$ and $n \in \mathbb{N}_0$,*

$$u(n) \leq c + \sum_{j=0}^{n-1} B(n-1, \alpha; j)\Big[a(j)u(j) + b(j)\Big], \tag{11}$$

then

$$u(n) \leq c \prod_{j=0}^{n-1} \Big[1 + B(n-1, \alpha; j)a(j)\Big] + \sum_{j=0}^{n-1} B(n-1, \alpha; j)b(j) \prod_{k=j+1}^{n-1} \Big[1 + B(n-1, \alpha; k)a(k)\Big], \tag{12}$$

for $n \in \mathbb{N}_0$.

Proof. Define a function $z(n)$ by the right hand side of (12). That is

$$z(n) = c + \sum_{j=0}^{n-1} B(n-1, \alpha; j)[a(j)u(j) + b(j)], \text{ for } n \in \mathbb{N}_0. \tag{13}$$

Then $z(0) = c, u(n) \leq z(n)$ and

$$\nabla^\alpha z(n+1) = a(n)u(n) + b(n), \text{ for } n \in \mathbb{N}_0. \tag{14}$$

As $u(n) \leq z(n)$, the equation (14) becomes

$$\nabla^\alpha z(n+1) \leq a(n)z(n) + b(n), \text{ for } n \in \mathbb{N}_0 \tag{15}$$

with $z(0) = c$, and $0 < \alpha < 1$.

Now, application of Theorem (4) to (15) yields.

$$z(n) \leq z(0) \prod_{j=0}^{n-1} \Big[1 + B(n-1, \alpha; j)a(j)\Big] + \sum_{j=0}^{n-1} B(n-1, \alpha; j)b(j) \prod_{k=j+1}^{n-1} \Big[1 + B(n-1, \alpha; k)a(k)\Big],$$

which implies

$$z(n) \leq c \prod_{j=0}^{n-1} \Big[1 + B(n-1, \alpha; j)a(j)\Big] + \sum_{j=0}^{n-1} B(n-1, \alpha; j)b(j) \prod_{k=j+1}^{n-1} \Big[1 + B(n-1, \alpha; k)a(k)\Big], \tag{16}$$

for $n \in \mathbb{N}_0$. Hence, using (16) in $u(n) \leq z(n)$, we get

$$u(n) \leq c \prod_{j=0}^{n-1} \Big[1 + B(n-1, \alpha; j)a(j)\Big] + \sum_{j=0}^{n-1} B(n-1, \alpha; j)b(j) \prod_{k=j+1}^{n-1} \Big[1 + B(n-1, \alpha; k)a(k)\Big]. \tag{17}$$

for $n \in \mathbb{N}_0$. This is the required inequality. □

The following theorem deals with uniqueness of the solution to fractional order difference equations.

Theorem 6. *Suppose that the functions K, F in equation* (1)-(2) *satisfy the conditions*

$$|F(n,x,y) - F(n,\bar{x},\bar{y})| \leq L_1 |x-\bar{x}| + L_2 |y-\bar{y}| \tag{18}$$

$$\sum_{\sigma=\alpha}^{\beta} |K(n,\sigma,v(\sigma) - K(n,\sigma,w(\sigma)| \leq L_3 |v(n) - w(n)| \tag{19}$$

where L_1, L_2 and L_3 are non-negative constants. Then the initial value problem (1)-(2) *has a unique solution.*

Proof. Let $v(n)$ and $w(n)$ be any two solutions of (1)-(2) satisfying $v(0) = w(0) = u_0$.Then recalling recurrence relation for solution and hypotheses, we get

$$\begin{aligned}
|v(n) - w(n)| &\leq \sum_{j=0}^{n-1} B(n-1,\alpha;j) \left| F\left(j, v(j), \sum_{\sigma=\alpha}^{\beta} K(j,\sigma,v(\sigma)\right) - F\left(j, w(j),, \sum_{\sigma=\alpha}^{\beta} K(j,\sigma,w(\sigma))\right) \right| \\
&\leq \sum_{j=0}^{n-1} B(n-1,\alpha;j) \left[L_1 \Big(|v(j) - w(j)| \Big) + L_2 \left(\sum_{\sigma=\alpha}^{\beta} |K(j,\sigma,v(\sigma)) - K(j,\sigma,w(\sigma))| \right) \right] \\
&\leq \sum_{j=0}^{n-1} B(n-1,\alpha;j) \Big[L_1 \Big(|v(j) - w(j)| \Big) + L_2 L_3 \Big(|v(j) - w(j)| \Big) \Big] \\
&< \epsilon + \sum_{j=0}^{n-1} B(n-1,\alpha;j) \Big((L_1 + L_2 L_3) |v(j) - w(j)| \Big)
\end{aligned}$$

Let $u(n) = |v(n) - w(n)|$. Then the above inequality implies for every $\epsilon > 0$

$$u(n) < \epsilon + \sum_{j=0}^{n-1} B(n-1,\alpha,j)(L_1 + L_2 L_3) u(j) \tag{20}$$

Hence, by application of Theorem (5) to (20), we get

$$\begin{aligned}
u(n) &< \epsilon \prod_{j=0}^{n-1} \Big[1 + (L_1 + L_2 L_3) B(n-1,\alpha;j) \Big] \\
&< \epsilon \exp \Big[\sum_{j=0}^{n-1} B(n-1,\alpha;j)(L_1 + L_2 L_3) \Big] \qquad (21)
\end{aligned}$$

Using Lemma (1) in (21), we obtain

$$u(n) < \epsilon \exp \left[(L_1 + L_2 L_3) \binom{n+\alpha-1}{n-1} \right].$$

or one can write it as

$$0 \leq u(n) \exp \left[-(L_1 + L_2 L_3) \binom{n+\alpha-1}{n-1} \right] < \epsilon, \tag{22}$$

for every n. Since the arbitrary nature of ϵ, and inequality (22) shows that the middle term is less every positive real number ϵ. Hence, we conclude that $u(n) = 0$. Therefore, we have $v(n) = w(n)$. This proves the uniqueness of the solutions. □

5. BOUNDEDNESS OF SOLUTION

The following Theorem shows boundedness of solution to the problem (1)-(2).

Theorem 7. *Suppose that the functions F, K in equation* (1)-(2) *satisfy the conditions* (18) *and* (19) *respectively. If $u : \mathbb{N}_0 \to \mathbb{R}$ is a solution of the Fredholm difference equation with iterated sum* (1)-(2), *then*

$$|u(n)| \leq (|u_0| + L_3) \prod_{j=0}^{n-1} \Big[(1 + L_1 + L_2 L_3) B(n-1, \alpha; j)\Big] \tag{23}$$

for $n \in \mathbb{N}_0$.

Proof. From the equation (10) and hypotheses, we estimate

$$\begin{aligned} |u(n)| &\leq |u_0| + \sum_{j=0}^{n-1} B(n-1, \alpha; j) \left[\left| F\left(j, u(j), \sum_{\sigma=0}^{\beta} K(j, \sigma, u(\sigma))\right) - F\left(j, 0, \sum_{\sigma=0}^{\beta} K(j, \sigma, 0)\right) \right| \right. \\ &\quad \left. + \left| F\left(j, 0, \sum_{\sigma=0}^{\beta} K(j, \sigma, 0)\right) \right| \right] \\ &\leq |u_0| + \sum_{j=0}^{n-1} B(n-1, \alpha; j) \left[L_1 |u(j)| + L_2 \sum_{\sigma=0}^{\beta} |K(j, \sigma, u(\sigma) - K(j, \sigma, 0)| + L_4 \right] \\ &\leq |u_0| + \sum_{j=0}^{n-1} B(n-1, \alpha; j) \Big[(L_1 + L_2) L_3 |u(j)| + L_4 \Big] \end{aligned} \tag{24}$$

where

$$L_4 = \max_{0 \leq j \leq n-1} \left(\left| F(j, 0, \sum_{\sigma=0}^{\beta} K(j, \sigma, 0) \right| \right)$$

Using the Theorem (5) to the inequality (24), we get

$$\begin{aligned} |u(n)| &\leq |u_0| \prod_{j=0}^{n-1} \Big[(1 + B(n-1, \alpha; j)(L_1 + L_2 L_3)\Big] \\ &\quad + \sum_{j=0}^{n-1} B(n-1, \alpha; j) L_4 \prod_{k=j+1}^{n-1} \Big[1 + B(n-1, \alpha; k)(L_1 + L_2 L_3) \Big], \end{aligned} \tag{25}$$

which is the required result. □

6. CONTINUOUS DEPENDENCE

In this section , we shall deal with continuous dependence of the problem (1)-(2) on the initial data, function induced therein and also on parameters.

6.1. **Dependence on initial data.** We first discuss dependence of solution on given initial data.

Theorem 8. *Suppose that the* (18)-(19) *hold . If* $v(n)$ *and* $w(n)$ *are solutions of* (1)-(2) *with initial data* $v(0) = v_0$ *and* $w(0) = w_0$ *respectively, then*

$$|v(n) - w(n)| \leq |v_0 - w_0| \exp\left[(L_1 + L_2 L_3)\binom{n+\alpha-1}{n-1}\right].$$

Proof. By using the fact that $v(n)$ and $w(n)$ are solutions of (1)-(2). Hence, by hypotheses and looking at the proof of Theorem (6), we have

$$|v(n) - w(n)| \leq |v_0 - w_0| + \sum_{j=0}^{n-1} B(n-1, \alpha; j)(L_1 + L_2 L_3)\,|v(j) - w(j)|\,. \tag{26}$$

Using the Theorem (5) to the inequality (26), we obtain

$$\begin{aligned} |v(n) - w(n)| &\leq |v_0 - w_0| \prod_{j=0}^{n-1}\Big[1 + (L_1 + L_2 L_3) B(n-1, \alpha; j)\Big] \\ &\leq |v_0 - w_0| \exp\left[\sum_{j=0}^{n-1}(L_1 + L_2 L_3) B(n-1, \alpha; j)\right]. \end{aligned} \tag{27}$$

Using Lemma(1) in (27), we obtain

$$|v(n) - w(n)| \leq |v_0 - w_0| \exp\left[(L_1 + L_2 L_3)\binom{n+\alpha-1}{n-1}\right]. \tag{28}$$

This demonstrates how the equation's solutions rely continuously on the initial data . □

6.2. **Dependence on function.** Consider the equation (1)-(2) and the corresponding equation

$$\nabla^{\alpha} u(n+1) = \overline{F}\left(n, u(n), \sum_{\sigma=0}^{\beta} \overline{k}(n, \sigma, u(\sigma))\right) \tag{29}$$

with condition (2), where $\overline{F}$ and $\overline{K}$ are defined as F and K.

The following theorem present the closeness of solutions.

Theorem 9. *Suppose that the* (18)-(19) *hold. Furthermore, assume that there exist constants* $\epsilon > 0$, *for which*

$$\left| F\left(j, w(j), \sum_{\sigma=0}^{\beta} \overline{K}(j, \sigma, w(\sigma))\right) - \overline{F}\left(j, w(j), \sum_{\sigma=0}^{\beta} \overline{K}(j, \sigma, w(\sigma))\right)\right| < \epsilon$$

If $v(n)$ *and* $w(n)$ *are respectively solutions of* (29) *and* (1) *with* (2), *then*

$$|v(n) - w(n)| \leq \sum_{j=0}^{n-1} B(n-1, \alpha; j)\epsilon \prod_{k=j+1}^{n-1}\left[1 + B(n-1, \alpha; j)(L_1 + L_2 L_3\right] \tag{30}$$

for $n \in \mathbb{N}_0$.

Proof. Let $v(n)$ and $w(n)$ be the solutions of (1)-(2) and (29) with (2) respectively. Then by hypotheses, we have

$$
\begin{aligned}
|v(n)-w(n)| &\leq \sum_{j=0}^{n-1} B(n-1,\alpha;j)\left[\left|F\left(j,v(j),\sum_{\sigma=0}^{\beta}K(j,\sigma,u(\sigma))\right)-\overline{F}\left(j,w(j),\sum_{\sigma=0}^{\beta}\overline{k}(j,\sigma,w(\sigma))\right)\right|\right.\\
&\leq \sum_{j=0}^{n-1} B(n-1,\alpha;j)\left[\left|F\left(j,v(j),\sum_{\sigma=0}^{\beta}K(j,\sigma,u(\sigma))\right)-F\left(j,w(j),\sum_{\sigma=0}^{\beta}\overline{k}(j,\sigma,w(\sigma))\right)\right|\right.\\
&\quad \left.+\left|F\left(j,w(j),\sum_{\sigma=0}^{\beta}\overline{k}(j,\sigma,w(\sigma))\right)-\overline{F}\left(j,w(j),\sum_{\sigma=0}^{\beta}\overline{k}(j,\sigma,w(\sigma))\right)\right|\right]\\
&\leq \sum_{j=0}^{n-1} B(n-1,\alpha;j)\left[L_1\left|v(j)-w(j)\right|+L_2\sum_{\sigma=0}^{\beta}\left|k(j,\sigma,v(\sigma))-\overline{k}(j,\sigma,w(\sigma))\right|+\epsilon\right]\\
&\leq \sum_{j=0}^{n-1} B(n-1,\alpha;j)\left[(L_1+L_2L_3)\left|v(j)-w(j)\right|+\epsilon\right] \qquad (31)
\end{aligned}
$$

The subsequent equation (32) is the result of applying (5).

$$
|v(n)-w(n)| \leq \sum_{j=0}^{n-1} B(n-1,\alpha;j)\epsilon\prod_{k=j+1}^{n-1}\left[1+B(n-1,\alpha;j)(L_1+L_2L_3\right] \qquad (32)
$$

The solutions to problems (1) and (29) with condition (2) are close to one another, as can be inferred from (32), if $\overline{F}$ and $\overline{K}$ are, respectively, close to F and K. □

6.3. **Dependence on Parameters.** We next consider the following Fredholm difference equations

$$
\nabla^{\alpha}u(n+1) = F\left(n,u(n),\sum_{\sigma=0}^{\beta}k(n,\sigma,u(\sigma),\mu_1)\right) \qquad (33)
$$

and

$$
\nabla^{\alpha}u(n+1) = F\left(n,u(n),\sum_{\sigma=0}^{\beta}k(n,\sigma,u(\sigma),\mu_2)\right) \qquad (34)
$$

with initial condition (2), where u, k, F, are the elements of $\mathbb{R}^m$ an n dimensional Euclidean space with norm || and $k : E\times\mathbb{R}^m \to \mathbb{R}^m, F : \mathbb{N}^2_{0,\beta}\times\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}\to\mathbb{R}^n$ and $\mu_1,\ \mu_2$ are arbitrary constant. In which $E = \left\{(n,s)\in\mathbb{N}^2_{0,\beta} : 0\leq s\leq n\leq n\leq\beta\right\}$.

Theorem 10. *Suppose that the functions F satisfying the condition*

$$
\begin{aligned}
&\left|F\left(n,v(n),\sum_{\sigma=0}^{\beta}K(n,\sigma,v(\sigma),\mu_1)\right)-F\left(n,w(n),\sum_{\sigma=0}^{\beta}K(n,\sigma,w(\sigma),\mu_1)\right)\right|\\
&\qquad\leq L_1\left|v(n)-w(n)\right|+L_2\sum_{\sigma=0}^{\beta}\left|K(n,\sigma,v(\sigma))-K(n,\sigma,w(\sigma))\right|\\
&\left(\left|F\big(n,w(n),\sum_{\sigma=0}^{\beta}K(n,\sigma,w(\sigma)),\mu_1\big)-F\big(n,w(n),\sum_{\sigma=0}^{\beta}K(n,\sigma,w(\sigma)),\mu_2\big)\right|\right)\leq L_5\left|\mu_1-\mu_2\right|
\end{aligned}
$$

where L_1, L_2, L_5 are non negative constants. If $v(n)$ and $w(n)$ are respectively solutions of (33) *and* (34) *with condition* (2)*, then*

$$|v(n)-w(n)| \le \sum_{j=0}^{n-1} B(n-1,\alpha;j)L_5\,|\mu_1-\mu_2| \prod_{k=j+1}^{n-1}\Bigg[1+B(n-1,\alpha;j)(L_1+L_2L_3)\Bigg], \tag{35}$$

for $n \in \mathbb{N}_0$.

Proof. From the assumptions, it follows that

$$\begin{aligned}
&|v(n)-w(n)| \\
&\le \sum_{j=0}^{n-1} B(n-1,\alpha;j)\Bigg[\Bigg|F\Bigg(j,v(j),\sum_{\sigma=0}^{\beta}K(j,\sigma,v(\sigma),\mu_1)\Bigg)-F\Bigg(j,w(j),\sum_{\sigma=0}^{\beta}K(j,\sigma,w(\sigma),\mu_2)\Bigg)\Bigg|\Bigg] \\
&\le \sum_{j=0}^{n-1} B(n-1,\alpha;j)\Bigg[\Bigg|F\Bigg(j,v(j),\sum_{\sigma=0}^{\beta}K(j,\sigma,v(\sigma),\mu_1)\Bigg)-F\Bigg(j,w(j),\sum_{\sigma=0}^{\beta}K(j,\sigma,w(\sigma),\mu_1)\Bigg)\Bigg| \\
&\quad+\Bigg|F\Bigg(j,w(j),\sum_{\sigma=0}^{\beta}K(j,\sigma,w(\sigma),\mu_1)\Bigg)-F\Bigg(j,w(j),\sum_{\sigma=0}^{\beta}K(j,\sigma,w(\sigma),\mu_2)\Bigg)\Bigg|\Bigg] \\
&\le \sum_{j=0}^{n-1} B(n-1,\alpha;j)\Bigg[(L_1+L_2L_3)\,|v(j)-w(j)|+L_5(|\mu_1-\mu_2|\Bigg]
\end{aligned} \tag{36}$$

With the help of Theorem (5) and the inequality (36), we get

$$|v(n)-w(n)| \le \sum_{j=0}^{n-1} B(n-1,\alpha;j)L_5\,|\mu_1-\mu_2| \prod_{k=j+1}^{n-1}\Bigg[1+B(n-1,\alpha;j)(L_1+L_2L_3)\Bigg], \tag{37}$$

This demonstrates how the parameters μ_1 and μ_2 affect the solution of equations (33) and (34) with condition (2). □

7. Example

We consider the following problem:

$$\nabla^{\frac{1}{2}}u(n+1) = F\left(n, u(n), \sum_{\sigma=0}^{n} k(n,\sigma,u(\sigma))\right) \tag{38}$$

with initial condition

$$u(0) = 1 \tag{39}$$

Let the functions F and k be defined as:

$$F(n,x,y) = \frac{x}{n+2} + \frac{1}{2(n+1)}\,y + \frac{1}{n+1}, \tag{40}$$

$$k(n,\sigma,u(\sigma)) = \frac{u(\sigma)}{(n+1)(\sigma+2)} \tag{41}$$

Solution: From the above information , we can write problem

$$\nabla^{\frac{1}{2}}u(n+1) = \frac{u(n)}{n+2} + \frac{1}{2(n+1)}\sum_{\sigma=0}^{n}\frac{u(\sigma)}{(n+1)(\sigma+2)} + \frac{1}{n+1} \tag{42}$$

From the equation 10, we can write the corresponding solution of the given problem 38-39, as

$$u(n) = 1 + \sum_{j=0}^{n-1} B(n-1, \frac{1}{2}; j) \left[\frac{u(j)}{j+2} + \frac{1}{2(j+1)} \sum_{\sigma=0}^{3} \frac{u(\sigma)}{(j+1)(\sigma+2)} + \frac{1}{j+1} \right] \tag{43}$$

Then $F(n, x, y)$ and $K(n, \sigma, u(\sigma))$ satisfy the conditions

$$\begin{aligned} |F(n,x,y) - F(n,\bar{x},\bar{y})| &= \left| \frac{x-\bar{x}}{n+2} + \frac{1}{2(n+1)}(y-\bar{y}) \right| \\ &\leq \frac{1}{n+2}|x-\bar{x}| + \frac{1}{2(n+1)}|y-\bar{y}| \\ &\leq |x-\bar{x}| + \frac{1}{2}|y-\bar{y}| \\ &\leq L_1 |x-\bar{x}| + L_2 |y-\bar{y}|, \end{aligned} \tag{44}$$

where $L_1 = 1$ and $L_2 = \frac{1}{2}$

$$\begin{aligned} \sum_{\sigma=0}^{3} \left| k\big(n,\sigma,v(\sigma)\big) - k\big(n,\sigma,w(\sigma)\big) \right| &= \sum_{\sigma=0}^{3} \left| \frac{v(\sigma)-w(\sigma)}{(n+1)(\sigma+2)} \right| \\ &= \frac{1}{n+1} \sum_{\sigma=0}^{3} \frac{|v(\sigma)-w(\sigma)|}{\sigma+2}. \end{aligned} \tag{45}$$

Let $|v - w| = \max_{0 \leq \sigma \leq n} |v(\sigma) - w(\sigma)|$. Then we have

$$\sum_{\sigma=0}^{3} \left| k(n,\sigma,v(\sigma)) - k(n,\sigma,w(\sigma)) \right| \leq \frac{|v-w|}{n+1} \sum_{\sigma=0}^{3} \frac{1}{\sigma+2} \leq 4\,|v-w|, \tag{46}$$

where $L_3 = 4$. Hence, in view of Theorem (6), we observe that

$$\begin{aligned} |u(n) - w(n)| &< \epsilon \exp \left[(L_1 + L_2 L_3) \binom{n+\alpha-1}{n-1} \right] \\ &< \epsilon \exp \left[(1 + 4\frac{1}{2}) \binom{n+\alpha-1}{n-1} \right] \\ &< \epsilon \exp \left[3 \binom{n+\alpha-1}{n-1} \right]. \end{aligned} \tag{47}$$

Since the arbitrary nature of ϵ, inequality (47) conclude that $u(n) \to 0$ as $n \to \infty$ and hence, we have $v(n) = w(n)$. This proves the uniqueness of the solutions.
In particular, for $n = 10$ and $\alpha = \dfrac{1}{2}$, the inequality (47) becomes

$$\begin{aligned} |u(n) - w(n)| &< \epsilon \exp \left[3 \binom{10 + \frac{1}{2} - 1}{10-1} \right] \\ &= \epsilon \exp \left[(3 \binom{9 + \frac{1}{2}}{9} \right] \\ &= \epsilon \exp \left[3 \binom{\frac{19}{2}}{9} \right]. \end{aligned} \tag{48}$$

Finally, referring the definition as in (3), one can have

$$|u(n) - w(n)| < \epsilon \exp \left[3 \frac{\Gamma(\frac{19}{2} + 1)}{\Gamma(\frac{19}{2} - 9 + 1)\Gamma(9+1)} \right]$$

$$
\begin{aligned}
&= \epsilon \exp\left[3 \times \frac{19}{2} \frac{\Gamma(\frac{19}{2})}{\Gamma(\frac{1}{2}+1)\Gamma(10)}\right] \\
&= \epsilon \exp\left[3 \times \frac{19}{2} \frac{\Gamma(\frac{19}{2})}{\frac{1}{2}\sqrt{\pi}10!}\right] \\
&= \epsilon \exp\left[3 \times 19 \frac{\frac{34459425\sqrt{\pi}}{512}}{\sqrt{\pi}10!}\right] \\
&= \epsilon \exp\left[\frac{57}{512} \times \frac{34459425}{10!}\right] \\
&\cong \epsilon \exp(0.2643). \qquad (49)
\end{aligned}
$$

Or equivalently, one can seen that

$$
0 \le |u(n) - w(n)| \exp(-0.2643) < \epsilon, \qquad (50)
$$

for every ϵ and n. Therefore, looking at the definition as in (3) and $n \to \infty$, we conclude that $u(n) = w(n)$. This proves the our required.

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