STUDY OF π- INCLINE STRUCTURE ON INCLINE ALGEBRA

Abstract: Incline algebra is an important algebraic framework used to study ordered algebraic systems with applications in decision theory, graph theory, and optimization. The present paper introduces and investigates the concept of the π-incline structure within the context of incline algebra, aiming to extend the structural and operational understanding of inclines. A π-incline is defined through specific algebraic properties that generalize conventional incline operations and provide a refined approach to analyzing order-preserving mappings and idempotent elements. This study explores the fundamental characteristics of π-incline structures, including their closure properties, homomorphic behavior, and interaction with existing incline operations. Various theoretical results are established to demonstrate how π-inclines contribute to the structural classification of incline algebras and help in identifying new relationships among algebraic elements. The paper also discusses conditions under which π-incline structures preserve regularity and stability within algebraic systems.

By presenting definitions, propositions, and illustrative examples, the work highlights the significance of π-incline structures in strengthening the theoretical foundation of incline algebra. The findings open pathways for further research in abstract algebra and its applications, particularly in areas involving ordered systems, optimization models, and algebraic representations of decision processes.

Keywords: Incline algebra, π-incline structure, ordered algebraic systems, idempotent elements, homomorphism, algebraic operations, structural properties, order-preserving mappings, abstract algebra, optimization models.

1.INTRODUCTION

Incline algebra has emerged as an important area within abstract algebra due to its ability to model ordered and idempotent algebraic systems. Since its development, it has attracted attention for its theoretical richness and practical relevance in areas such as decision-making, graph theory, fuzzy systems, and optimization. An incline is generally characterized by a set equipped with two binary operations satisfying order-preserving and idempotent properties, making it suitable for studying structured algebraic relationships.

In recent years, researchers have focused on extending the foundational aspects of incline algebra by introducing new structures and generalizations that enhance its applicability and theoretical depth in [1], to [15]. Among these developments, the concept of π-incline has gained significance as a means of examining refined operational behavior and structural variations within incline systems. The π-incline structure aims to capture additional algebraic features that are not fully addressed by traditional incline frameworks, thereby offering a broader perspective for analysis. The motivation behind studying π-incline structures lies in understanding how these structures interact with existing incline operations and how they influence properties such as closure, regularity, and homomorphic mappings. By investigating these aspects, it becomes possible to classify incline algebras more effectively and identify new relationships between their elements. This, in turn, contributes to the development of a more comprehensive algebraic theory.

This paper is devoted to the systematic study of π-incline structures on incline algebra. It presents fundamental definitions, examines essential properties, and establishes theoretical results that highlight the role of π-inclines in extending the framework of incline algebra. The work also aims to provide illustrative examples to clarify the concepts and demonstrate their relevance. Through this study, a foundation is laid for further research in advanced algebraic systems and their potential applications in mathematical modeling and computational structures.

2. STUDY OF π- INCLINE STRUCTURE

Definition 2.1. An incline is an algebraic structure $(\mathfrak{I},+,*)$ having a non-empty set $\mathfrak{I}$ and two binary operations + and * such that for all x,y,z in $\mathfrak{I}$, if the following laws hold

[K1] *Associative laws*

(i) $x + (y + z) = (x + y) + z$,

(ii) $x * (y * z) = (x * y) * z$.

[K2] *Commutative laws*

(i) $x + y = y + x$,

(ii) $x * y = y * x$.

[K3] *Distributive laws*

(i) $x * (y + z) = (x * y) + (x * z)$,

(ii) $(y + z) * x = (y * x) + (z * x)$.

[K4] *Idempotent law:* $x + x = x$.

[K5] *Incline law*

(i) $x + (x * y) = x$,

(ii) $y + (x * y) = y$.

In brief an incline is an algebraic structure with two operations, addition and multiplication. It generalizes distributive lattices in that multiplication need not be idempotent.

We have Boolean algebras ⊂ fuzzy algebras ⊂ distributive lattices ⊂ inclines ⊂ semi-rings. Therefore, it may have applications to new models in various sciences.

Definition 2.2. Let $x,y \in \Im$. The incline order relation denoted as *"$\leq$"* and is defined as $x \leq y \leftrightarrow x + y = y$.
From the incline axiom (K5) obiviously, we have
I. $x + y \geq x$ and $x + y \geq y$ for $x,y \in \Im$,
II. $xy \leq x$ and $xy \leq y$ for $x,y \in \Im$.
which are known as incline properties.

Here it is clear from the above structure that these operations make quantities to decrease "slide downhill." Hence, we decided to name it as incline and let $\Im$ denote an arbitrary incline where $\Im$ is the first letter of the Korean alphabet and is pronounced as "Gee-Uck" and it look like an incline.

Example 2.3: The Boolean algebra {0, 1} is trivial example of an incline under Boolean operations meet and join.

Example 2.4: The fuzzy algebra [0, 1] is also have an incline structure under the operations maximum and minimum.

Proposition 2.5: Every distributive lattice is an incline. An incline is a distributive lattice (semi-ring) if and only if $x^2 = x$; $\forall x$.

Proof. The first part of the statement is obvious.

Again, If $y \leq z$ then $y + z = z, \Longrightarrow x(y + z) = xz$ *i.e..* $xy \leq xz$. For any distributive lattice $x \wedge x = x$.

Conversely, we want to show that if $x^2 = x$ then $xy = x \wedge y$ true Here we have $xy \leq x$from incline properties.

Now from commutatively $xy \leq y$.

Suppose $u \leq x$ and $u \leq y$. Then $u = u^2 \leq xy$.

This proves that $xy = x \wedge y$.

All semi-lattice with additive identity 0 has an incline structure which is given by $xy = 0$.

Proposition 2.6: On any finite lattice there exists a unique maximal binary operation xy such that $xy < x$ and $xy < y$ and $x(y+z) = xy + xz$ such that $(y+z)\,x = yx + zx$. This opration is commutative.

Proof. If $x \bullet y$ and $x * y$ are operations of this kind, so are $x \bullet y + x * y$ and $x \bullet y + y * x$. Hence we can take the sum of all such oprations of this kind which will be the unique maximal one .if it were not commutative $xy + yx$ would be greater.

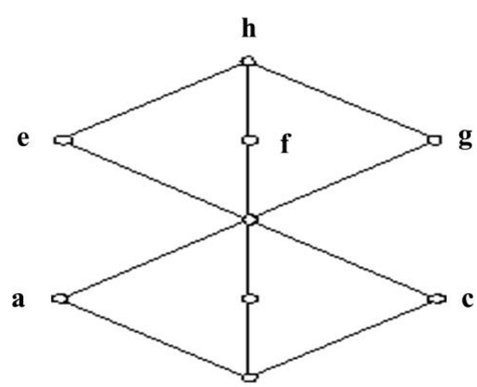
We do not whether this operation is always associative. It does give an upper bound on incline structures. Moreover, we will later show that if it is nonzero there exists a nonzero incline structure.

There exists an effective way to compute this maximal distributive operations on any finite lattice. Start with $xy = x \wedge y$. Then whenever $x(y+z) > xy + xz$ redefine :

$x(y+z) = xy + zx,\ uv = inf\,\{uv\,, xy + xz\}$ for $u < x\,, v < y + z$.

This process must terminate in finitely many steps since it always decreases the operation. The product must be greater than or equal to the maximal product at each stage. But since the result will be distributive it must equal the maximal distributive product. The product remains monotone at each step.

Example 2.7:



We have $h^2 = (e+f)\,(e+g)\,(e+g) < e + fg = e$ .so redefine $h^2 = e, f.f = g\,.g = d, h\,.f = h.g = d$. Also $h^2 = (f+g)\,(f+g) = d$. Redefine

$x\,.\,y\ = 0$ for $y\ < d$. This gives a distributive operation, Which is maximal. It is associative and is therefore an incline structure.

Proposition 2.8: In any incline, the set $\{x\colon x\ =\ x^2\}$ is a subincline which in itself is a distributive lattice.

Proof: Here It suffices to prove that it is closed under addition and multiplication ,by proposition 3.2.1 If $x^2 = x$, $y^2 = y$,

then $(xy)^2 = x^2y^2 = xy$. And $xy = x^2y^2 < x^2$.

Therefore $(x + y)^2 = x^2 + xy + y^2 = x^2 + y^2 = x + y$.

Example 2.9: For the fuzzy structure on [0 ,1] every element satisfies $x^2 =\ x$. For the structure xy, only . 0 , 1 do.

Proposition 2.10: Let π be a function from a lattice to a sublattice which is distributive and such that $\pi(x+y)=\ \pi\ (x) +\ \pi(y),\ \pi(xy) = \pi(x)\,\pi(y),$

$\pi\pi(x) = \pi\ (x)$ and $\pi(x) < x$. Then $xy\ =\ \pi(x)\,\pi\ (y)$ gives an incline structure in which $xy\ = (xy)^2$ Conversely every incline structure in which $xy = (xy)^2$ has this form.

Proof. The first statement follows from the definition .suppose that $xy\ = x^2y^2$. Then by the result above the set of elements of the form xy is a distributive sub-lattice. Let $\pi(x) = x^2$ Then we have, $\pi(x) < x$, and $\pi(xy) = x^2y^2 = xy$, $\pi\ (x + y) = x^2 + xy + y^2 = x^2 + x^2y^2 + y^2 = \pi\ (x) + \pi(y),\ \pi\ (\pi\ x) = x^4 = x^2x^2 = xx = \pi(x)$.

Definition 2.11: If $xy\ =\ x^2y^2$ in an incline ,we call it a π- incline.

Example 2.12: Boolean vectors have a π- incline structure in which $(a, b)\ (c, d)\ =\ (\,0\,, bd)$.

Theorem 2.13: There semi-lattice of $n-$ dimensional Boolean vectors v_n has exactly 2^n incline structures which are all π-inclines

Proof. As we have there exist 2^n π-inclines defineb by $\pi\ (x) =\ xv$ for any Boolean vector v. Let xy be any incline structure.

Let e_1 ,e_2…..,e_n be the basis for v_n .Then $e_i\,e_j < e_i\ \wedge\ e_j = 0$, for $i\ \neq j$ and so $e_i\,e_j = 0$ for $i\ \neq j$. and $e_i^2 \leq\ e_i$ so $e_i^2 =\ e_i$ or $e_i^2 = 0$. define v by $v_i = 1$ if and only if $e_i^2 = e_i$. Let $\pi(x) = xv$. This gives a π-incline.

Definition 2.14: An incline structure is nil if $x^3 = 0$ for any x. since $(x + y + z)^3 = 0$, this implies $xyz = 0$ for all x, y, z since $xyz < (x + y + z)^2$.

Theorem 2.15: Any finite semi-lattice admits a nontrivial incline structure if and only if it admits either a nontrivial π- incline structure or a nontrivial nil incline structure.

Proof: Suppose we have a nontrivial incline structure.

suppose that $a^2 = a$ for some $a \neq 0$. Choose n large enough that $x^n = x^{n+1}$ for all x in the semilattice. Then the product $x^n y^n$ satisfies.

$x^n y^n \leq xy \leq x$, $(x^n y^n)^n z^n = x^n y^n z^n = x^n (x^n y^n)^n$, $x^n y^n = y^n x^n$, $x^n(y + z)^n = x^n(y + z)^{2n} \leq x^n y^n + x^n z^n$ by expansion of $(y + z)^{2n}$. But $x^n (y + z)^n < x^n y^n + x^n z^n$ in also true.

This gives a nontrivial, π- incline structure.

Suppose $a^2 < a$ for all $a \neq 0$. Choose the least n such that $x^n = 0$ for all x .

Then $x_1 x_2 \ldots\ldots\ldots x_n = 0$ also since $x_1 x_2 \ldots\ldots\ldots x_n \leq (x_1 + x_2 + \cdots\ldots\ldots + x_n)^n$. We have $n \geq 3$. Let m be the maximal element of the semilattice.

Define $x * y = xy\, m^{n-3}$. Then this is a nil incline .

Associativity holds by $(x * y) * z = 0$, since there are at least n factors .

Theorem 2.16. A finite lattice has a nontrivial π- incline structure if and only if there exists a nonzero element v such that if $y + z \geq v$ then $y \geq v$ or $z \geq v$

Proof: Suppose v exists . Define a product by $xy = v$ if $x \geq v$ and $y \geq v$ and $xy = 0$ otherwise. Then $xy \leq x$. and $xy = yx$.

suppose $x(y + z) > xy + xz$.

Then, $x\,(y + z) = v, xy = 0, xz = 0$.

Then, $x \geq v, y + z \geq v$. So $y \geq v\ or\ z \geq v$. so $xy = v$ or $xz = v$. This is false. So distributive law holds.

suppose $x\,(y\,z) \neq (x\,y)\,z$ .Then By symmetry ,let $x\,(y\,z) = v, (x\,y)\,z = 0$ .Then $x \geq v$, $yz \geq v$so $y \geq v$, $z \geq v$ so $(xy)z = v$ This is false . so this gives a π- incline structure.

Conversely ,suppose we have a π- incline structure. Let v be a basis element for the image of π, which is bistributive. suppose $x + y \geq v$. Then, $\pi(x) + \pi(y) \geq v$. so $v = v\pi(x) + v\pi(y)$. so $v = v\pi(x)$ or $v\pi(y)$. so $x \geq v$ or $y \geq v$.

Example 2.17: Suppose a lattice has unique minimal nonzero element . Then it has a π- incline structure .

Theorem 2.18: A finite lattice admits a nonzero incline structure if and only if there exists a nonzero distributive product such that $xy \leq x$ and $xy \leq y$.

Proof: If an incline structure exists, this gives a nonzero distributive product.

Suppose the maximal distributive product xy satisfying $xy \leq x$ and $xy \leq y$ is nonzero. If there exists v such that if $x + y \geq v$ then $x \geq v$ or $y \geq v$ then we have a π- incline .

Suppose this is false. we next show that $x^2 < x$ for all $x > 0$ in the lattice.

First suppose x is a lattice basis element .Write $x \leq y + z$ where $y \ngeq x$, $x \nleq z$. Then $x^2 \leq x \wedge (y + z) = x \wedge y + x \wedge z$. Both $x \wedge y < x$ and $x \wedge z < x$.

Since x is a basis element it cannot be a sum of strictly smaller vectors . so $x \wedge y + x \wedge z \neq x$, so $x^2 \leq x \wedge y + x \wedge z < x$.

Next suppose x is not a basis element . write it as a sum $x_1 + x_2 + ... + x_n$ of basis elements which is minimal in the sense that if $y_i \leq x_i$ and $\sum y_i = x_i$ then $y_i = x_i$ for each i.

Such a minimal expansion exists since if any expression is not minimal we can replace it by a smaller one. Then,

$$x^2 = (x_1 + x_2 + \ldots + x_n)^2 \leq \sum x_i^2 + \sum_{i<j} x_i \wedge x_j \leq x_1^2 + x_2^2 + \ldots + x_n^2.$$

Here $x_i \wedge x_j \leq x_j$. since $x_1^2 < x_1$ we have a lesser expression for x, This is a contradiction .

So $x^2 < x$ for all x . So some iterated square $m^{2^{n+1}}$ of m is zero, where m is the maximal element.

Suppose $m^{2^{n+1}} = 0$ but $m^{2^n} \neq 0$. we will construct a nonzero commutative distributive product such that $(x * y) * z = 0$ and $x * (x * z) = 0$ always which will therefore be associative . If $m^2 = 0$ then $xy = 0$. so assume $n > 0$.

Case 1: $m^{2^n} m^{2^{n-1}}$ or $m^{2^{n-1}} m^{2^n} > 0$. Suppose the former Let the product $x * y$ be $m^{2^n}(xy)$ which is nonzero for $x = y = m$. Then $(x * y) * z = m^{2^n}((x * y)z) = m^{2^n}((m^{2^n}xy)z)$ which is zero since $m^{2^n} m^{2^n}$ is an upper bound for it . Likewise $x * (y * z) = m^{2^n}(x(y * z)) = m^{2^n}(x(m^{2^n}(yz)) = 0$.

This product is distributive, commutative and $x * y \leq x$.

Case 2: Both those are zero. Let $x * y$ be $x_1 y_1,$ where x_1, y_1 are obtained by substituting x, y for some two factors of $m^{2^{n-1}}$. (The case $n = 1$ is handled as before).

Then $x * (y * z)$ is bounded by $x_1 (y_1 z_1),$ which in turn is bounded by $m^{2^{n-1}} m^{2^n} = 0$. Likewise $(x * y) * z = 0$.

Proposition 2.19: Let w be the semilattice of subspace of v where v is a vector space of dimension at least 2 over a field . Then w has no incline structure except 0 .

Proof: Any one dimensional subspace x is contained in the span of two distinct one dimensional subspaces y, z . so $x + y = y + z = x z,$ and $xy = yz = xz = 0$.
Then$y^2 = (y + z)(y + x) = (x + y)(x + z) = x^2$.

So $x^2 \leq inf\{x, y\} = 0$. So any product of one dimension subspaces is zero.

The same proof shows that the lattice of partitions of a set of at least three elements has no incline structure since any basis element is contained in join of two others having meet zero.

Let z_2 denote the two element field {0 ,1 } where

$$0 + 0 = 1 + 1 = 1 . 0 = 0 . 1 = 0 . 0 = 0 , 1 . 1 = 1 + 0 = 0 + 1 = 1$$

Example 2.20: The lattice of equivalence relations on a three element set is isomorphic to the previous example and has no nontrivial incline structure.

3.Conclusion

In this paper, the concept of the π-incline structure on incline algebra has been systematically studied to enrich the theoretical framework of incline systems. By introducing and analyzing the defining properties of π-inclines, we have extended the structural understanding of classical incline algebra and highlighted the role of additional operational conditions in shaping algebraic behavior.

The investigation demonstrates that π-incline structures preserve essential incline properties such as idempotency and order compatibility while offering a refined perspective on homomorphisms, substructures, and regular elements. The results obtained provide a clearer classification of incline-based systems and reveal meaningful relationships between π-incline properties and existing algebraic characteristics.

Through definitions, propositions, and supporting discussions, this study establishes a foundational basis for further exploration of π-incline structures. The framework developed here may serve as a stepping stone for future research in abstract algebra, particularly in the study of ordered algebraic systems, algebraic modeling, and theoretical computer science applications.

Overall, the study of π-incline structures contributes to the advancement of incline algebra by deepening its structural theory and opening new directions for mathematical investigation.

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