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Rationality of Fuzzy Choice Functions under an S-Implication–S-Residuum Framework

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Abstract

Classically, formal individual choice and rationality are framed using precise preference relations. Individual preference relations and hence the rationality of choice are generally not precise, and this necessitates the studies under vague preference relations. Until now, available studies of rationality have done by utilizing R-implications in the associated continuous t-norm. However, another class of fuzzy implications, S-implications, which represent behavioral interpretation, i.e., lenience and non-satisfaction of criteria, have not yet been used for the study of rationality. We validated that S-implication with continuous t-norm fails to satisfy key results of rationality and hence introduced an operator called S-residuum, which logically represents the minimal degree required to hold the implication against the joint degree of satisfaction of the implication by t-norm. This happens due to failure of adjointness, the law of identity and the boundary condition in this structure. This S-implication-S-residuum framework is introduced and exploited to develop logical interrelations among rationality axioms built on vague preferences for the first time. This leads to reformulating the well-known rationality axioms, namely S-FWARP, S-FSARP, S-FWCA and S-FSCA. The main results are : S-FSCA $\Rightarrow$ S-FWCA $\Rightarrow$ S-FSWCA and S-FSARP $\Rightarrow$ S-FWARP $\Rightarrow$ S-SFWARP. Furthermore, we characterize regularity of fuzzy preference relation in terms of FCF satisfying S-FWCA and hence S-SFCA. We concluded that this new framework offers a behaviorally meaningful and logically coherent foundation for the study of vague choice and rationality

Keywords: S-implication, Fuzzy choice function, S-residuum operator, Rationality axioms.

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1 Introduction

The focus point of classical choice theory is the individual behavior of rationality from the inception of Samuelson's axiomatization of revealed preference (21), which provided a behavioral interpretation of rational choice. Uzawa (26) showed that the rationality of a choice function can be categorized with a preference relation, and Houthakker (14) exhibited that Samuelson's WAPR is not enough and hence he introduced the SAFRP. Further, Arrow (1) Richter (16) Sen (22),(23) Suzumura (24) and subsequent many others introduced preference based and consistency based classifications of rationality of choice functions. All these attempts use classical preferences for studying choice functions and their rationality. However, individual preferences are vague, and as a result, decision-making in real-life tends to be approximate rather than precise. This inspired the expansion of fuzzy choice theory, in which vagueness and partial satisfaction are clearly defined. Several researchers have extended rationality axioms using fuzzy set–theoretic concepts (11) (18) (17) (13) (20). To date, studies of fuzzy choice and rationality uses R-implications with associated continuous t-norms for interpretations. This R-implication-t-norm structure guarantees an algebraic rightness for formal confirmations of results. These studies are effective in analyzing rationality from an algebraic viewpoint. Even though S-implications provide a natural, behavior-oriented explanation of implication in fuzzy conditions by integrating non-acceptance and remedial reasoning, they have not been methodically utilized in fuzzy choice and rationality investigations. Since results based on adjointness of R-implications associated with continuous t-norms fail to satisfy, we point out that the S-implication-t-norm structure is insufficient for the characterization of the rational behavior of an individual. Hence, we present the S-implication-S-residuum structure by introducing S-residuum operators, which gives the minimal degree to hold the implication. But the non-commutative and non-associative nature of S-residuum operators, demands the reformulations of S-implication-based rationality axioms, including S-FWARP, S-FSARP, and S-SFWARP. In this paper, we derive fundamental properties of S-implications, S-residuum operators and examine the logical interrelationships among the rationality axioms. In our opinion, this new approach provides a logically balanced footing and meaningful insight for fuzzy rationality, allowing one to improve the theoretical environment of the development of fuzzy choice theory from different angles.

2 preliminaries

This section outlines fundamental concepts in fuzzy set theory that are needed for the present study, which include fuzzy sets, triangular norms, triangular conorms, fuzzy complements, fuzzy relations.
Recall that, a fuzzy set A in X is an element of I, where I is the set of memberships of γ in I. We will use $\mathscr{F}(X)$ to represent the set of all fuzzy subsets of X. For any two elements $A, B \in \mathscr{F}(X)$, we say that $B \supseteq A$ whenever $B(\gamma) \geq A(\gamma)$ holds for every $\gamma \in X$. A fuzzy subset A of X is non-zero if it takes a positive membership value for at least one element of X, and it is normal if its membership degree reaches 1, for some $\gamma \in X$.
Notation: Let $\gamma_1, \gamma_2, \ldots, \gamma_n \in X$. We denote by $[\gamma_1, \gamma_2, \ldots, \gamma_n]$ the characteristic function of the set $\{\gamma_1, \gamma_2, \ldots, \gamma_n\}$.

Definition 2.1. A fuzzy set $* \in \mathscr{F}(I^2)$ is a triangular norm, or t-norm in short, if * is associative, commutative, monotonic in both arguments and satisfies $\sigma * 1 = \sigma, \forall \sigma \in I$ (15)(8)(3).

Following standard examples frequently quoted in many references. For $\sigma, \omega, \tau \in I$:
Standard fuzzy intersection : $\sigma * \omega = \min(\sigma, \omega)$. (8)
Algebraic product : $\sigma * \omega = \sigma \cdot \omega$. (8)
Bounded difference : $\sigma * \omega = \max(0, \sigma + \omega - 1)$. (8)

Drastic intersection : (8)

$$\sigma * \omega = \begin{cases} \min\{\sigma, \omega\}, & \text{if } \max\{\sigma, \omega\} = 1, \\ 0, & \text{otherwise.} \end{cases}$$

Definition 2.2. A fuzzy set $\circledast \in \mathscr{F}(I^2)$ is a triangular conorm, or t-conorm in short, if $\circledast$ is associativite, commutativite, monotonic in both arguments and satisfies $\sigma \circledast 0 = \sigma, \forall \sigma \in I$ (15)(8)(3).

Following examples of t-conorms are reported in literature and used frequently. $\sigma, \omega, \tau \in I$
Standard union: $\sigma \circledast \omega = \max(\sigma, \omega)$.(8)
Algebraic sum: $\sigma \circledast \omega = \sigma + \omega - \sigma\omega$.(8)
Bounded sum: $\sigma \circledast \omega = \min(1, \sigma + \omega)$.(8)
Drastic union:(8) $\sigma \circledast \omega = \begin{cases} \max\{\sigma, \omega\}, & \text{if } \min\{\sigma, \omega\} = 0, \\ 1, & \text{otherwise.} \end{cases}$

Definition 2.3. A decreasing fuzzy set $c \in \mathscr{F}(I)$ is known as a fuzzy complement if it satisfies boundary conditions viz. $c(0) = 1$ and $c(1) = 0$ (8).

Following standard examples are given in (8) and in many other references. For $\sigma \in I$

1. Standard fuzzy complement: $c(\sigma) = 1 - \sigma$(8).
2. $c(\sigma) = \frac{1}{2}(1 + \cos(\pi\sigma))$(8).
3. For fixed $t \in I$, $c_t(\sigma) = \chi_{\{\sigma | \sigma \leq t\}(\sigma)}$

A fuzzy complement c is said to be continuous, if it is a continuous as a function. Further, the involutive condition of c is given as $c(c(\sigma)) = \sigma$, $\forall \sigma \in I$.

Note that the Examples (1) & (2) are continuous fuzzy complements and the Example (1) is an involutive fuzzy complements.

Definition 2.4. (5), (9) A fuzzy set $R \in \mathscr{F}(X^2)$ is known as a binary fuzzy relation on X.

1. The reflexivity of R is $\gamma_1 R \gamma_1 = 1$, for all $\gamma_1 \in X$,
2. The $*$-transitivity of R is $\gamma_1 R \gamma_2 * \gamma_2 R \gamma_3 \leq \gamma_1 R \gamma_3$, for all $\gamma_1, \gamma_2, \gamma_3 \in X$ (5),
3. R is total, if $\gamma_1 R \gamma_2 > 0$ or $\gamma_2 R \gamma_1 > 0$, for all distinct $\gamma_1, \gamma_2 \in X$ (9),
4. R is strongly total, if either of the $\gamma_1 R \gamma_2$ or $\gamma_2 R \gamma_1$ is 1, for all distinct $\gamma_1, \gamma_2 \in X$ (9).

3 Fuzzy S-implications and Negations

The non classical fuzzy version of the classical implication known as R-implications is used in different studies because of their rich algebraic properties and residual structure. Formally, an R-implication is a fuzzy relation on I defined by $\sigma \delta_r \omega = \sup\{\gamma \in I \mid \sigma * \gamma \leq \omega\}$, where $*$ denotes a continuous t-norm (8). The variety of properties of R-implications makes them appropriate for testing the rational behavior of an individual. Until now, they are therefore broadly utilized for the investigations of rationality under a fuzzy environment (5; 6; 7; 11; 18; 17). Despite the class of S-implications exhibiting generalization of the classical implication, i.e. richer logical interpretation of the classical implication, they have received less consideration in the investigation of fuzzy choice and rationality. In the following discussion, we give an illustration to visualize the distinction of R-implication and S-implication. Consider a task that needs high mental effort (σ) and low performance (ω). The R-implication $\sigma \delta_r \omega$ decreases greatly, as poor outcomes harm the consistency necessity, whereas the S-implication strength $\sigma \delta_s \omega$ allows low performance to remain logically admissible when effort is inadequate. This shows S-implications are suitable in non-classical situations. Here, we evaluate their properties and apply them to investigate the fuzzy choice and the rationality.

Definition 3.1. (8) Any fuzzy binary relation δ on I is a fuzzy implication, if $1\delta 1 = 0\delta 0 = 0\delta 1 = 1$ and $1\delta 0 = 0.$

Example 3.1. *The binary fuzzy relation δ_s on I defined as : $\sigma\delta_s\omega = c(\sigma) \circledast \omega$,for $\sigma, \omega \in I$ is a fuzzy implication, where $\circledast$ is a t-conorm and c is a fuzzy complement. This class of fuzzy implications is known as the class of S-implications(8).*

This implication is actually (S,N)-implication given in (19). A particular pair of t-conorms and fuzzy complements and gives different $S-$implications few of them are:

1. **Kleene-Dienes implication**(KDI): $\sigma\delta_s\omega = \max(1-\sigma, \omega)$, where c and $\circledast$ are standard fuzzy complement and t-conorm respectively (8).
2. **Reichenbach implication**(RI) : $\sigma\delta_s\omega = 1 - \sigma + \sigma\omega$, where c and $\circledast$ are standard fuzzy complement and algebraic product respectively (8).
3. **Łukasiewicz implication** (LI): $\sigma\delta_s\omega = \min(1, 1-\sigma+\omega)$, where c is a standard fuzzy complement and $\circledast$ is a bounded sum (8).
4. **Drastic implication**(DI): $\sigma\delta_s\omega = \begin{cases} \omega, & \text{if } \sigma = 0 \\ 1-\sigma, & \text{if } \omega = 0 \\ 1, & \text{otherwise,} \end{cases}$

   where c and $\circledast$ are standard fuzzy complement and drastic union respectively (8).

One can easily verify that these S-implications satisfies the following relation: $DI \geq LI \geq RI \geq KDI.$

The following comparison table provides structural difference of S-implications and R-implications

Property	**S-implication**	**R-implication**
Constructibility from negation and disjunction	Yes	No
Explicit negation sensitivity	Strong	Weak
Behavioral interpretability	High	Low
Non-residuated flexibility	Yes	No
Decision-oriented monotonicity	Yes	Not intended

Table 3.1

Throughout the paper we shall denote the complement of $\sigma \in I$ by σ^c instead of $c(\sigma)$.

Theorem 3.2. *For $\sigma, \omega, \tau \in I$, we have*

(i) $\sigma\delta_s\omega \geq \omega$.

(ii) $1\delta_s\sigma = \sigma$.

(iii) $0\delta_s\sigma = \sigma\delta_s 1 = 1$.

(iv) $\sigma\delta_s\tau \geq \omega\delta_s\tau$ *and* $\tau\delta_s\omega \geq \tau\delta_s\sigma$ *if* $\omega \geq \sigma$.

(v) $\sigma\delta_s(\omega\delta_s\tau) = \omega\delta_s(\sigma\delta_s\tau)$.

(vi) $\sigma\delta_s(\omega\delta_s\tau) = \omega\delta_s(\sigma\delta_s\tau) = (\sigma * \tau)\delta_s\tau$, *if* $(*, \circledast, c)$ *is dual triple.*

(vii) *If* $\tau \geq \sigma \circledast \omega$, *then* $\omega\delta_s\tau \geq \min\{\sigma, \omega\}$.

(viii) $\sigma\delta_s\omega = \omega^c\delta_s\sigma^c$, *if* c *is idempotent.*

(ix) $\omega\delta_s\tau \geq (\sigma * \omega)$,*if* $\tau \geq (\sigma * \omega)$.

(x) $\sigma\delta_s\omega = \omega^c\delta_s\sigma^c$, *if c is involutive.*

(xi) $(\sigma * \omega)^c = \sigma\delta_s\omega^c$, *if* $(*, \circledast, c)$ *is dual triplet.*

Proof. (i) By definition of S-implication, we have $\sigma\delta_s\omega = c(\sigma) \circledast \omega \geq \omega$.

(ii) $1\delta_s\sigma = 1^c \circledast \sigma = 0 \circledast \sigma = \sigma$.

(iii) $0\delta_s\sigma = 0^c \circledast \sigma = 1 \circledast \sigma = 1$ and $\sigma\delta_s 1 = \sigma^c \circledast 1 \geq 0 \circledast 1 = 1$. But $\sigma^c \circledast 1 \leq 1$ always . So, $\sigma\delta_s 1 = 1$.

(iv) Let $\sigma \leq \omega$. Then $\omega^c \leq \sigma^c$. Thus, $\omega\delta_s\tau = \omega^c \circledast \tau \leq \sigma^c \circledast \tau = \sigma\delta_s\tau$. i.e. $\omega\delta_s\tau \leq \sigma\delta_s\tau$.
Also, $\tau\delta_s\sigma = \tau^c \circledast \sigma \leq \tau^c \circledast \omega = \tau\delta_s\omega$.

(v) The commutativity and associativity of $\circledast$, gives

$$\begin{aligned}\sigma\delta_s(\omega\delta_s\tau) &= \sigma^c \circledast [\omega^c \circledast \tau]\\ &= [\sigma^c \circledast \omega^c] \circledast \tau,\\ &= \omega\delta_s(\sigma\delta_s\tau).\end{aligned}$$

(vi) We have,

$$\begin{aligned}\sigma\delta_s(\omega\delta_s\tau) &= \sigma^c \circledast [\omega\delta_s\tau]\\ &= \sigma^c \circledast [\omega^c \circledast \tau]\\ &= [\sigma^c \circledast \omega^c] \circledast \tau\\ &= [(\sigma * \omega)^c] \circledast \tau, \text{since } (*, \circledast, c) \text{ is a dual triplet}\\ &= (\sigma * \omega)\delta_s\tau.\end{aligned}$$

(vii) Let $\sigma \circledast \omega \leq \tau$. Then $\omega\delta_s\tau = \omega^c \circledast \tau \geq \omega^c \circledast (\sigma \circledast \omega) = (\omega^c \circledast \omega) \circledast \sigma \geq \sigma$. Similarly, $\omega\delta_s\tau \geq \omega$.

(viii) Let c be an idempotent fuzzy complement. Then

$$\begin{aligned}\sigma\delta_s\omega &= \sigma^c \circledast \omega\\ &= \omega \circledast \sigma^c\\ &= [\omega^c]^c \circledast \sigma^c, \text{ since c is idempotent}\\ &= \omega^c\delta_s\sigma^c.\end{aligned}$$

(ix) Suppose that $\tau \geq \sigma * \omega$. Then $\omega\delta_s\tau = \omega^c \circledast \tau \geq \omega^c \circledast (\sigma * \omega) \geq (\sigma * \omega)$.

(x) Suppose c is involutive. Then $\sigma\delta_s\omega = \sigma^c \circledast \omega = \sigma^c \circledast [\omega^c]^c = [\omega^c]^c \circledast \sigma^c = \omega^c\delta_s\sigma^c$.

(xi) Since$(*, \circledast, c)$ is dual, $(\sigma * \omega)^c = \sigma^c \circledast \omega^c = \sigma\delta_s\omega^c$.

□

Definition 3.2. A binary fuzzy relation $\delta_s^{\leftrightarrow}$ on I defined as $\sigma\delta_s^{\leftrightarrow}\omega$ =$[\sigma\delta_s\omega] \wedge [\omega\delta_s\sigma]$ is called S-bi-implication, where δ_s is a fuzzy S- implication.

Theorem 3.3. *For* $\sigma, \omega, c \in I$ *the following properties hold:*

(i) $\sigma\delta_s\omega$ *and* $\omega\delta_s\sigma \geq \sigma\delta_s^{\leftrightarrow}\omega$

(ii) $\sigma\delta_s^{\leftrightarrow}\omega = \omega\delta_s^{\leftrightarrow}\sigma$,

(iii) $(\sigma\delta_s^{\leftrightarrow}\tau) \circledast (\omega\delta_s\omega) \geq (\sigma\delta_s^{\leftrightarrow}\omega) \circledast (\omega\delta_s^{\leftrightarrow}\sigma)$, *if* $\wedge$ *and* $\circledast$ *are distributive.*

Proof. (i) We have, $\sigma\delta_s^{\leftrightarrow}\omega = (\sigma\delta_s\omega) \wedge (\omega\delta_s\sigma) \leq \omega\delta_s\sigma$ also $\sigma\delta_s^{\leftrightarrow}\omega \leq \sigma\delta_s\omega$.

(ii) We have, $\sigma\delta_s^{\leftrightarrow}\omega = (\sigma^c \circledast \omega) \wedge (\omega^c \circledast \sigma) = (\omega^c \circledast \sigma) \wedge (\sigma^c \circledast \omega = \omega\delta_s^{\leftrightarrow}\sigma$.

(iii) Let $\wedge$ and $\circledast$ be distributive. Then, $(\sigma\delta_s^{\leftrightarrow}\omega)\circledast(\omega\delta_s^{\leftrightarrow}\tau) = [(\sigma^c\circledast\omega)\wedge(\omega^c\circledast\sigma)]\circledast[(\omega^c\circledast\tau)\wedge(c^c\circledast\omega)] = [(\sigma^c \circledast \omega) \circledast (\omega^c \circledast \tau)] \wedge [(\omega^c \circledast \sigma) \circledast (\omega^c \circledast \tau)] \wedge [(\sigma^c \circledast \omega) \circledast (\tau^c \circledast \omega)] \wedge [(\omega^c \circledast \sigma) \circledast (\tau^c \circledast \omega)] = [(\sigma\delta_s\omega) \circledast (\omega\delta_s\tau)] \wedge [(\omega\delta_s\sigma) \circledast (\omega\delta_s\tau)] \wedge [(\sigma\delta_s\omega) \circledast (\tau\delta_s\omega)] \wedge [(\omega\delta_s\sigma) \circledast (\tau\delta_s\omega)] \leq [(\sigma\delta_s\tau) \circledast (\omega\delta_s\omega)] \wedge [(\delta_s\sigma) \circledast (\omega\delta_s\omega)] = [(\sigma\delta_s\tau) \wedge (\tau\delta_s\sigma)] \circledast (\omega\delta_s\omega) = (\sigma\delta_s^{\leftrightarrow}\tau) \circledast (\omega\delta_s\omega)$.
Therefore, $(\sigma\delta_s^{\leftrightarrow}\tau) \circledast (\omega\delta_s\omega) \geq (\sigma\delta_s^{\leftrightarrow}\omega) \circledast (\omega\delta_s^{\leftrightarrow}\sigma)$.

□

We now illustrate the properties of (δ_r). that will not hold for δ_s. Let $\sigma, \omega, \tau \in I$. Then

(i) $\tau \geq \sigma * \omega \iff \omega\delta_s\tau \geq \sigma$ (adjointness)

(ii) $\omega \geq \sigma \implies \sigma\delta_s\omega = 1$ (boundry conditions)

(iii) $\sigma\delta_s\sigma = 1$ (identity law)

(iv) $\sigma\delta_s(\omega\delta_s\tau) = \omega\delta_s(\sigma\delta_s\tau) = (\sigma * \omega)\delta_s\tau$

(v) $\sigma\delta_s\tau \geq (\sigma\delta_s\omega) * (\omega\delta_s\tau)$.

(vi) $\omega\delta_s\tau \geq \sigma \implies \tau \geq \sigma \circledast \omega$

(i) Suppose $* =$ min, $\circledast =$ max and $\sigma^c = 1 - \sigma$. Then for $\sigma = 0.95, \omega = 0.1$ and $\tau = 0.2$, we have $\sigma * \omega =$min$\{0.9, 0.1\} = 0.1 \leq 0.2 = \tau$. But $\omega\delta_s\tau = \omega^c \circledast \tau =$ max $\{0.9, 0.2\} = 0.9 < 0.95 = \sigma$. Also, if $\sigma = 0.4, \omega = 0.5$ and $\tau = 0.2$, then $\omega\delta_s\tau = \omega^c \circledast \tau =$ max $\{0.5, 0.2\} = 0.5 \geq 0.4 = \sigma$. But $\sigma \circledast \omega =$ min $\{0.4, 0.5\} = 0.4 > 0.2 = \tau$. Thus, $\sigma * \omega \leq \tau \Longleftrightarrow \sigma \leq \omega\delta_s\tau$ does not holds.

(ii) Suppose $* =$ min, $\circledast =$ max and $\sigma^c = 1 - \sigma$. Then for $\sigma = 0.9$ and $\omega = 0.9$, we have $\sigma \leq \omega$. But $\sigma\delta_s\omega = \sigma^c \circledast \omega =$ max$\{0.1, 0.9\} = 0.9 \neq 1$. Thus, $\sigma \leq \omega \implies \sigma\delta_s\omega = 1$ does not holds.

(iii) Suppose $* =$ min, $\circledast =$ max and $\sigma^c = 1 - \sigma$. If $\sigma = 0.7$, then $\sigma\delta_s\sigma =$ max $\{0.3, 0.7\} = 0.7 \neq 1$. Thus, $\sigma\delta_s\sigma \neq 1$.

(iv) Suppose $\sigma * \omega = \begin{cases} \sigma, & \text{if } \omega = 1, \\ \omega, & \text{if } \sigma = 1 \\ 0, & \text{otherwise,} \end{cases}$ and $\sigma \circledast \omega = \min\{1, \sigma + \omega\}$.
Let $\sigma = 0.9, \omega = 0.9$ and $\tau = 0.7$. Then $\sigma\delta_s(\omega\delta_s\tau) = 0.9$ But $(\sigma * \omega)\delta_s\tau = 1$ Hence, $\sigma\delta_s(\omega\delta_s\tau) \neq (\sigma * \omega)\delta_s\tau$.

(v) Suppose $* =$ min, $\circledast =$ max and $\sigma^c = 1 - \sigma$. Let $\sigma = 0.6, \omega = 0.5$ and $\tau = 0.4$. Then $(\sigma\delta_s\omega) * (\omega\delta_s\tau) = [\sigma^c \circledast \omega] * [\omega^c \circledast \tau] = [0.4 \vee 0.5] * [0.5 \vee 0.4] = 0.5 \wedge 0.5 = 0.5$. and $(\sigma\delta_s\tau) = [\sigma^c \circledast \tau] = [0.4 \vee 0.4] = 0.4$. Hence, $(\sigma\delta_s\omega) * (\omega\delta_s\tau) \nleq \sigma\delta_s\tau$.

(vi) Suppose $* =$ min, $\circledast =$ max and $\sigma^c = 1 - \sigma$. Let $\sigma = 0.5, \omega = 0.5, \tau = 0.1$. Then $\sigma = 0.5 \leq 0.5 =$ max$\{0.5, 0.1\} = \omega^c \circledast \tau$ but $\sigma \circledast \omega = \sigma^c \circledast \omega =$ max$\{0.5, 0.5\} = 0.5 \geq 0.1 = \tau$. Thus $\sigma \leq \omega\delta_s\tau \implies \sigma \circledast \omega \leq \tau$ does not holds.

The following adjoint property of R-implications with t-norm: $\sigma * \omega \leq \tau \iff \sigma \leq \omega\delta_r\tau$ is crucial in proving many results in fuzzy choice theory (6),(9),(10),(11),(12). As seen above this property will not hold for S-implications with t-norm. This makes us think of some other binary operation on I that allows us to obtain it for S-implications. For this purpose, we define a special operator Ⓢ on I as follows:

$$\sigma Ⓢ \omega = inf\{\gamma_1 \in I \mid \sigma \leq \omega\delta_s\gamma_1\}.$$

We shall call this operator as the S-residuum operator .
For the pair of t-conorm $\circledast$ and fuzzy complement c, the corresponding S-residuum operator are given as follows:

1) For the standerd t-conorm $\circledast$ and standard fuzzy complement c, $\sigma Ⓢ \omega = \begin{cases} 0, & \text{if } \sigma \leq \omega^c, \\ \sigma, & \text{otherwise,} \end{cases}$

2) $\sigma Ⓢ \omega = max\{0, \sigma - \omega^c\}$, where $\circledast$, c are bounded sum and standard fuzzy complement respectively.

Here, we give the logical meaning of the S-residuum operator against the well-known operator namely t-norm. We expect that in choice theory the use of an operator is to measure the admissibility of alternatives instead of imposing joint fulfillment of criteria. For example, a skilled person, who lacks formal qualifications, will generally hire for a particular job to be done accurately. This can be done using S-implication-S-residuum structure.
The conceptual distinction between t-norms, t-conorms, and the residuum Ⓢ is summarized in the following table:

Operator	Role	Interpretation
t-norm	Conjunction (and)	joint satisfaction
t-conorm	Disjunction (or)	alternative satisfaction
Residuum Ⓢ	Inverse implication	how much is minimally required

Thus, t-norms and t-conorms gives aggregation-based degrees, whereas $\sigma Ⓢ \omega$ gives the minimum value of γ required so that the S-implication $\omega \delta_s \gamma$ reaches the σ. In sum,Ⓢ determines the minimal needs, least effort, and thresholds of justification.
Note that this S-residuum operator has many useful properties, which we have listed bellow.

Theorem 3.4. *For $\sigma, \omega, c \in I$, the following holds*

(i) $\sigma \geq \sigma Ⓢ \omega$,

(ii) $0 Ⓢ \sigma = \sigma Ⓢ 0 = 0$.

(iii) $\sigma Ⓢ 1 = \sigma$.

(iv) $\tau \geq \sigma Ⓢ \omega \iff \omega \delta_s \tau \geq \sigma$.

(v) $\omega Ⓢ \tau \geq \sigma Ⓢ \tau$ *and* $\tau Ⓢ \omega \leq \tau Ⓢ \sigma$, *if* $\omega \geq \sigma$

Proof. (i) $\sigma Ⓢ \omega =$ inf $\{\gamma_1 \in I \mid \sigma \leq \omega^c \circledast \gamma_1\}$. We consider the cases
case 1)If $\sigma \leq \omega^c$, then $\sigma Ⓢ \omega = 0$, since $0 \in \{\gamma_1 \in I | \sigma \leq \sigma \circledast \gamma_1\}$.
case 2)If $\sigma > \omega^c$. Then $\sigma Ⓢ \omega = \sigma$, since $\sigma \leq \omega^c \circledast \sigma$ implies $\sigma \in \{\gamma_1 \in I | \sigma \leq \omega^c \circledast \gamma_1\}$. Hence, $\sigma \geq \sigma Ⓢ \omega$.

(ii) $0 Ⓢ \sigma = inf\ \{\gamma_1 \in I \mid 0 \leq \sigma^c \circledast \gamma_1\} = inf\ \{\gamma_1 \in I\} = 0$ and $\sigma Ⓢ 0 = inf\ \{\gamma_1 \in I \mid \sigma \leq 0^c \circledast \gamma_1\} = inf\ \{\gamma_1 \in I\} = 0$.

(iii) $\sigma Ⓢ 1 = inf\ \{\gamma_1 \in I \mid \sigma \leq 1^c \circledast \gamma_1\} = inf\ \{\gamma_1 \in I \mid \sigma \leq 0 \circledast \gamma_1\} = \sigma$.

(iv) Let $\sigma Ⓢ \omega \leq \tau$. Then $inf\{\gamma_1 \in I \mid \sigma \leq \omega^c \circledast \gamma_1\} \leq \tau$. Thus, for any $\gamma_1 \in I$ such that $\sigma \leq \omega^c \circledast \gamma_1$. We have $\gamma_1 \leq \tau$. But then, by monotonicity of Ⓢ, we have $\sigma \leq \omega^c \circledast \tau$. Therefore, $\sigma \leq \omega \delta_s \tau$.
Now, suppose that $\sigma \leq \omega \delta_s \tau$ i .e. $\sigma \leq \omega^c \circledast \tau \implies \tau \in \{\gamma_1 \in I \mid \sigma \leq \omega^c \circledast \gamma_1\} \implies \tau \geq$ inf $\{\gamma_1 \in I \mid \sigma \leq \omega^c \circledast \gamma_1\} \implies \tau \geq \sigma Ⓢ \omega$.

(v) Let $\sigma \leq \omega$. Then $\{\gamma_1 \in I \mid \sigma \leq \tau^c \circledast \gamma_1\} \supseteq \{\gamma_1 \in I \mid \omega \leq \tau^c \circledast \gamma_1\}$. Therefore $\sigma Ⓢ \tau = inf\ \{\gamma_1 \in I \mid \sigma \leq \tau^c \circledast \gamma_1\} \leq inf\ \{\gamma_1 \in I \mid \omega \leq \tau^c \circledast \gamma_1\} = \omega Ⓢ \tau$. Hence $\sigma Ⓢ \tau \leq \omega Ⓢ \tau$.
Similarly, Let $\sigma \leq \omega$. Then $\sigma \leq \omega \implies \sigma^c \geq \omega^c \implies \sigma^c \circledast \gamma_1 \geq \omega^c \circledast \gamma_1$. Therefore $\{\gamma_1 \in I \mid \tau \leq \sigma^c \circledast \gamma_1\} \supseteq \{\gamma_1 \in I \mid \tau \leq \omega^c \circledast \gamma_1\}$. Hence inf $\{\gamma_1 \in I \mid \tau \leq \sigma^c \circledast \gamma_1\} \leq$ inf $\{\gamma_1 \in I \mid \tau \leq \omega^c \circledast \gamma_1\}$ i.e. $\tau Ⓢ \sigma \geq \tau Ⓢ \omega$.

□

The following examples shows that the S-residuum operator Ⓢ is neither commutative nor associative.

Example 3.5. *The S-residuum operator* Ⓢ *is not commutative, as for the standard complement and standard t-conorm, if* $\sigma = 0.8, \omega = 1$ *then* 0.8Ⓢ1= *inf* $\{\gamma_1 \in [0,1] | 0.8 \leq 1^c \vee \gamma_1\}$= *inf* $\{\gamma_1 \in [0,1] | 0.8 \leq 0 \vee \gamma_1\} = 0.8$ *and* 1Ⓢ0.8= *inf* $\{\gamma_1 \in [0,1] | 1 \leq 0.8^c \vee \gamma_1\}$= *inf* $\{\gamma_1 \in [0,1] | 1 \leq 0.2 \vee \gamma_1\} = 1$. *Therefore,* 0.8Ⓢ$1 \neq 1$Ⓢ0.8. *Hence,* Ⓢ *is not commutative.*

Example 3.6. *The S-residuum operator* Ⓢ *is not associative, as for the standard complement and standard t-conorm, if* $\sigma = 0.2, \omega = 1, \tau = 0.5$ *then* $[0.2$Ⓢ$1]$Ⓢ0.5= *[inf* $\{\gamma_1 \in [0,1] | 0.2 \leq 1^c \vee x\}]$Ⓢ$0.5$= *[inf* $\{\gamma_1 \in [0,1] | 0.2 \leq 0 \vee \gamma_1\}]$Ⓢ$0.5 = 0.2$Ⓢ$0.5$=0 *and* 0.2Ⓢ$[1$Ⓢ$0.5] = 0.2$Ⓢ$1 = 0.2$ *Therefore,* $[0.2$Ⓢ$1]$Ⓢ$0.5 \neq 0.2$Ⓢ$[1$Ⓢ$0.5]$ *Hence,* Ⓢ *is not associative.*

The negation of $\sigma \in I$ based on δ_s is $\neg_s 0 = \sigma \delta_s 0$. As $\sigma \delta_s 0 = \sigma^c$, it is the usual fuzzy complement of $\sigma \in I$. Here, we have investigated the properties of the negation based on the S implication, in analogy to that of the negation based on the R implication, as follows:

Theorem 3.7. *For* $\sigma, \omega \in I$ *, the following holds*

(i) $\neg_s 0 = 1$ *and* $\neg_s 1 = 0$

(ii) $\sigma \leq \omega \implies \neg_s \omega \leq \neg_s \sigma$

(iii) σⓈ$\neg_s \sigma = 0$*, if the fuzzy complement* c *is involutive.*

(iv) $\sigma \leq \neg_s \omega \iff \sigma$Ⓢ$\omega = 0$

(v) $\sigma = \neg_s \neg_s \sigma$*, if the fuzzy complement* c *is involutive,*

(vi) $\neg_s \sigma = \neg_s \neg_s \neg_s \sigma$*, if* c *is involutive.*

Proof. (i) and (ii) are trivial.

(iii) Let the fuzzy complement c be involutive. Then σⓈ$\neg_s \sigma = \sigma$Ⓢ$\sigma^c = inf\ \{\gamma_1 \in I \mid \sigma \leq [\sigma^c]^c \circledast \gamma_1\} = inf\ \{\gamma_1 \in I \mid \sigma \leq \sigma \circledast \gamma_1\} = 0$.

(iv) Let $\sigma \leq \neg_s \omega$. Then σⓈ$\omega = inf\ \{\gamma_1 \in I \mid \sigma \leq \omega^c \circledast \gamma_1\} = 0$. Hence $\sigma \leq \neg_s \omega \implies \sigma$Ⓢ$\omega = 0$. Now suppose σⓈ$\omega = 0$ i.e. σⓈ$\omega = inf\ \{\gamma_1 \in I \mid \sigma \leq \omega^c \circledast \gamma_1\} = 0$. This implies that $\gamma_1 = 0$. Therefore, $\sigma \leq \omega^c \circledast \gamma_1 = \omega^c \circledast 0 = \omega^c$. Hence $\sigma \leq \omega^c$.

(v) If the fuzzy complement c is involutive, then $\neg_s \neg_s \sigma = [\sigma^c]^c = \sigma$ i.e. $\sigma = [\sigma^c]^c$.

(vi) If the fuzzy complement is involutive, then $\neg_s \neg_s \neg_s \sigma = [[\sigma^c]^c]^c = [\sigma]^c = \neg_s \sigma$.

□

We verify that the involutiveness of c is necessary for the properties (iii), (v) and (vi). For this, consider the fuzzy complement defined as $c(\sigma) = \frac{1}{2}(1 + \cos(\pi\sigma))$. If $\sigma = 0.33$ and $\sigma \circledast \omega = \max\{\sigma, \omega\}$, then $\sigma^c = (0.33)^c = 0.75, \quad \sigma$Ⓢ$\neg_s \sigma = 0.33 \neq 0$ Hence, property (iii) does not hold. Also, $\sigma^c = (0.33)^c = 0.75, \quad [\sigma^c]^c = (0.75)^c = 0.15 < 0.33 = \sigma$. Hence, property (v) is not true. Lastly, for the same value of σ, $\sigma^c = (0.33)^c = 0.75, \quad [\sigma^c]^c = (0.75)^c = 0.15, \quad [[\sigma^c]^c]^c = \neg_s \neg_s \neg_s \sigma = (0.15)^c = 0.99$. Thus, property (vi) does not hold.

4 S-Implication-Based Fuzzy Choice Functions

In this section we outlined a thorough formulation of fuzzy choice functions (FCFs) and investigated their rationality axioms using δ_s and the S-residuum operators. For the classical concepts and results, we refer to Uzawa (26), Arrow (1), Richter (16), and Sen (22; 23) etc. Also, for basic concepts in fuzzy choice theory we refer to Georguscu (10; 11; 12), Mordeson et al. (18), Desai (5), Desai and Chaudhari(6; 7) Suzumura (25).

Definition 4.1. (11) Let $\mathscr{B}$ be a nonempty collection of $\mathscr{F}(\Gamma)$, where Γ denotes the set of all available alternatives. A function $\psi \in \mathscr{F}(\Gamma)^{\mathscr{B}}$ is called a FCF, if for any $A \in \mathscr{B}$, $\psi(A) \neq \phi$ and $A \supseteq \psi(A)$.

Georgescu extends the Uzawa–Arrow–Sen classical framework of choice to fuzzy choice, providing a rigorous formulation that accounts for uncertainty in preferences under the following hypotheses (11)

(H1) For every A in $\mathscr{B}$, both A and $\psi(A)$ are normal fuzzy subsets of Γ.

(H2) Characteristics functions of all finite sets in Γ are in $\mathscr{B}$.

Hypotheses (H1) and (H2) are necessary to ensure that fuzzy choices are non-trivial and that the framework remains consistent with classical finite crisp choice scenarios. So, we continue to assume these hypotheses for the present study.
We now use the S-residuum operator to define a fuzzy preference, strict preference and fuzzy equivalent relations defined by the FCF ψ as follows:

Definition 4.2. Let $\psi \in \mathscr{F}(\Gamma)^{\mathscr{B}}$ be a FCF. We define the fuzzy relations $\mathscr{R}$, $\mathscr{P}$, $\mathscr{I}$ on $\Gamma \in \mathscr{F}(\Gamma)$ for any $\gamma_1, \gamma_2 \in \Gamma$ as:

(i) $\gamma_1 \mathscr{R} \gamma_2 = \bigvee_{S \in \mathscr{B}} [\psi(A)(\gamma_1) Ⓢ S(\gamma_2)]$

(ii) $\gamma_1 \mathscr{P} \gamma_2 = (\gamma_1 \mathscr{R} \gamma_2) Ⓢ (\gamma_2 \mathscr{R} \gamma_1^c)$

(iii) $\gamma_1 \mathscr{I} \gamma_2 = (\gamma_1 \mathscr{R} \gamma_2) Ⓢ (\gamma_2 \mathscr{R} \gamma_1)$

These definitions of $\mathscr{R}$, $\mathscr{P}$, $\mathscr{I}$ are different from given in (1) as they are based on more general operator Ⓢ.

Theorem 4.1. *Let ψ be a FCF on Γ. Then $\mathscr{R}$ is reflexive and strongly total.*

Proof. Since $\psi(A)$ is normal fuzzy set, $\psi(A)(\gamma_3) = 1$, for some $\gamma_3 \in \Gamma$. Hence, $A(\gamma_3) = 1$. Then for any $\gamma_1 \in \Gamma$, $\gamma_1 \mathscr{R} \gamma_2 = \bigvee_{s \in B} [\psi(A)(\gamma_1) Ⓢ \psi(\gamma_1)] \geq \psi(A)(\gamma_3) Ⓢ \psi(\gamma_3) = \psi(A)(\gamma_3) = 1$. Hence, $\gamma_1 \mathscr{R} \gamma_2$ is reflexive. To prove $\mathscr{R}$ is total, let $\gamma_1, \gamma_2 \in \Gamma$. Since $\psi(A)$ is normal fuzzy set, we have $\gamma_1 \mathscr{R} \gamma_2 = \bigvee_{\mathscr{B}} [\psi(A)(\gamma) Ⓢ A(\gamma_1)] \geq \psi([\gamma_1, \gamma_2])(\gamma_1) Ⓢ [\gamma_1, \gamma_2](\gamma_2) = 1 > 0$ or $\gamma_2 \mathscr{R} \gamma_1 = \bigvee_{A \in \mathscr{B}} [\psi(A)(\gamma_2) Ⓢ A(\gamma_1)] \geq \psi([\gamma_1, \gamma_2])(\gamma_2) Ⓢ [\gamma_1, \gamma_2](\gamma_1) = 1 > 0$. Hence, $\mathscr{R}$ is total. □

remark 4.1. Observe that if $\mathscr{R}$ is strongly total, then $\mathscr{R}$ is total; however, the converse need not hold.

It is well established in the literature that the rational behavior of an individual is described by the class of all R-greatest elements of a given fuzzy preference relation. Here we use the S-residuum operator based fuzzy preference relation $\mathscr{R}$ generated from the given fuzzy choice function ψ to establish the relation between FCFs ψ and its image $\bar{\psi}$ with respect to δ_s.

Definition 4.3. Let ψ be a FCF. Then $\hat{\psi} \in \mathscr{F}(\Gamma)^{\mathscr{B}}$ defined for any $A \in \mathscr{B}$ and $\gamma_1 \in \Gamma$ as

$$\hat{\psi}(A)(\gamma_1) = A(\gamma_1) * \left(\bigwedge_{\gamma_2 \in \Gamma} [A(\gamma_2) \delta_s (\gamma_1 \mathscr{R} \gamma_2)] \right).$$

Thus $\bar{\psi}$ is called the image of ψ with respect to δ_s.

Theorem 4.2. *For any fuzzy choice function ψ and for any $A \in \mathscr{B}$, $\gamma_1 \in \Gamma$, we have $\psi(A)(\gamma_1) * \psi(A)(\gamma_1) \leq \hat{\psi}(A)(\gamma_1)$.*

Proof. Let $\gamma_1 \in \Gamma$. Then by definition of $\mathscr{R}$, we have $\psi(A)(\gamma_1) Ⓢ A(\gamma_2) \leq \mathscr{R}(\gamma_1, \gamma_2), \forall \gamma_2 \in \Gamma$. Thus, $\psi(A)(\gamma_1) \leq A(\gamma_2) \delta_s (\gamma_1 \mathscr{R}, \gamma_2), \forall \gamma_2 \in \Gamma$. Therefore, $\psi(A)(\gamma_1) \leq \bigwedge_{\gamma_2 \in \Gamma} [A(\gamma_2) \delta_s (\gamma_1 \mathscr{R} \gamma_2)]$. But $\psi(A)(\gamma_1) \leq A(\gamma_1)$, gives $\psi(A)(\gamma_1) * \psi(A)(\gamma_1) \leq A(\gamma_1) * \bigwedge_{\gamma_2 \in \Gamma} [A(\gamma_2) \delta_s (\gamma_1 \mathscr{R} \gamma_2)] = \hat{\psi}(A)(\gamma_1)$. □

This example shows that when S-implication is used with t-norm, the above theorem did not holds in general.

Example 4.3. *Let $\Gamma = \{\sigma, \omega\}$ be the set of alternatives, where σ indicate the solar power (clean but capacity-limited) and ω indicate the grid electricity (reliable and fully accessible). Let $\mathscr{B} = \{A\}$, where the fuzzy availability set $A(\sigma) = 0.6, A(\omega) = 1$ is indicates that solar power is available only to degree 0.6 due to limited sunlight, roof capacity, and weather uncertainty, while grid electricity is fully available. Define the fuzzy choice function $\psi(A)(\sigma) = 0.4, \psi(A)(\omega) = 1$, meaning that the household uses solar power only to degree 0.4, despite higher availability, due to intermittency risk, storage limitations, and adjustment costs, whereas grid electricity is chosen with full certainty. Let us take the operators $\gamma_1 * \gamma_2 = \gamma_1.\gamma_2$, $c(\gamma_1) = 1 - \gamma_1$, $\gamma_1 Ⓢ \gamma_2 = \max\{\gamma_1, \gamma_2\}$. The fuzzy revealed preference relation induced by ψ is $\mathscr{R}(\gamma_1, \gamma_2) = V_{A\in\mathcal{B}}[\psi(A)(\gamma_1) * A(\gamma_2)], \forall \gamma_1, \gamma_2 \in \Gamma$. A direct computation yields $\mathscr{R} = \begin{pmatrix} 0.24 & 0.4 \\ 0.6 & 1.0 \end{pmatrix}$, where $\mathscr{R}(\sigma, \sigma) = (0.4).(0.6) = 0.24$, $\mathscr{R}(\sigma, \omega) = (0.4).(1) = 0.4$, $\mathscr{R}(\omega, \sigma) = (1).(0.6) = 0.6$, $\mathscr{R}(\omega, \omega) = (1).(1) = 1$. Now, the fuzzy choice function, called the image of ψ, derived by the fuzzy preference relation $\mathscr{R}$ generated by ψ is given by $\hat{\psi}(A)(\gamma_1) = A(\gamma_1) * \bigwedge_{\gamma_1 \in \Gamma}[A(\gamma_3) \rightsquigarrow \mathscr{R}((\gamma_1, \gamma_3))], \forall \gamma_1, \gamma_2 \in \Gamma$. For $\gamma_1 = \omega$, $\hat{\psi}(A)(\omega) = 1.[[0.6 \rightsquigarrow 0.6] \wedge [1 \rightsquigarrow 1]] = (0.6) \wedge 1 = 0.6$. On the other hand, $\psi(A)(\omega) * \psi(A)(\omega) = (1).(1) = 1$. Hence, $\psi(A)(\omega) * \psi(A)(\omega) > \hat{\psi}(A)(\omega)$ showing that the observed choice of ω exceeds the degree justified by preference-consistent rationalization. This indicates excessive or biased choice behavior that cannot be explained by the revealed preference relation. This shows that the Proposition 4.5 [9] not hold for S-implication and t-norm.*

corollary 4.1. If $*$ is the standard t-norm, then for any $A \in \mathscr{B}$, $\psi(A) \subseteq \hat{\psi}(A)$.

Proof. Since $*$ is standard t-norm $\psi(A)(\gamma_1) * \psi(A)(\gamma_1) = \psi(A)(\gamma_1)$. □

Definition 4.4. Let ψ be a FCF on Γ. Then ψ is called a normal FCF if $\psi(A) = \hat{\psi}(A)$, for all $A \in \mathscr{B}$.

Definition 4.5. Let ψ be a FCF on Γ. We define the fuzzy relations $\bar{\mathscr{R}}$, $\bar{\mathscr{P}}$ and $\bar{\mathscr{I}}$ on Γ, for any $\gamma_1, \gamma_2 \in \Gamma$ as follows:

(i) $\gamma_1 \bar{\mathscr{R}} \gamma_2 = \psi([\gamma_1, \gamma_2])(\gamma_1)$

(ii) $\gamma_1 \bar{\mathscr{P}} \gamma_2 = (\gamma_1 \bar{\mathscr{R}} \gamma_2) Ⓢ ((\gamma_2 \bar{\mathscr{R}} \gamma_1))^c$

(iii) $\gamma_1 \bar{\mathscr{I}} \gamma_2 = (\gamma_1 \bar{\mathscr{R}} \gamma_2) Ⓢ (\gamma_2 \bar{\mathscr{R}} \gamma_1)$

Theorem 4.4. *For any FCFs ψ on Γ, we have $\bar{\mathscr{R}} \subseteq \mathscr{R}$.*

Proof. We have $\gamma_1 \bar{\mathscr{R}} \gamma_2 = \psi([\gamma_1, \gamma_2])(\gamma_1) = \psi([\gamma_1, \gamma_2])(\gamma_1) Ⓢ [\gamma_1, \gamma_2](\gamma_2) \leq \bigvee_{A \in \mathscr{B}}[\psi(A)(\gamma_1) Ⓢ A(\gamma_2)] = \gamma_1 \mathscr{R} \gamma_2$. □

Definition 4.6. Let ψ be a FCF. Then $\bar{\psi} \in \mathscr{F}(\Gamma)^{\mathscr{B}}$ is defined for any $A \in \mathscr{B}$ and $\gamma_1 \in \Gamma$ as

$$\bar{\psi}(A)(\gamma_1) = A(\gamma_1) * \left(\bigwedge_{\gamma_2 \in \Gamma} [A(\gamma_2) \delta_s (\gamma_1 \bar{\mathscr{R}} \gamma_2)] \right).$$

remark 4.2. Let ψ be a FCF. Then $\bar{\psi}(A) \subseteq \hat{\psi}(A)$, since $\bar{\mathscr{R}} \subseteq \mathscr{R}$ and by 3.2 (iv).

Theorem 4.5. *If ψ is a normal FCF, then $\gamma_1 \bar{R} \gamma_2 = \gamma_1 \mathscr{R} \gamma_2$, for any $\gamma_1, \gamma_2 \in \Gamma$. Further, $\bar{\psi} = \hat{\psi}$.*

Proof. For any $\gamma_1, \gamma_2 \in \Gamma$, we have

$$\begin{aligned}
\gamma_1 \bar{\mathscr{R}} \gamma_2 &= \psi([\gamma_1, \gamma_2])(\gamma_1) \\
&= \hat{\psi}([\gamma_1, \gamma_2])(\gamma_1) \\
&= [\gamma_1, \gamma_2](\gamma_1) * \bigwedge_{\gamma_3 \in \Gamma} \big([[\gamma_1, \gamma_2](\gamma_3)\delta_s(\gamma_1 \mathscr{R} \gamma_3)]\big) \\
&= 1 * \{([\gamma_1, \gamma_2](\gamma_1)\delta_s(\gamma_1 \mathscr{R} \gamma_1)) \wedge ([\gamma_1, \gamma_2](\gamma_2)\delta_s(\gamma_1 \mathscr{R} \gamma_2))\} \\
&\quad \wedge \bigwedge_{\gamma_3 \in \Gamma - \{\gamma_1, \gamma_2\}} [[\gamma_1, \gamma_2](\gamma_3)\delta_s(\gamma_1 \mathscr{R} \gamma_3)] \\
&= \{1\delta_s(\gamma_1 \mathscr{R} \gamma_1) \wedge 1\delta_s(\gamma_1 \mathscr{R} \gamma_2)\} \wedge \bigwedge_{\gamma_3 \in \Gamma - \{\gamma_1, \gamma_2\}} [0\delta_s(\gamma_1 \mathscr{R} \gamma_3)] \\
&= \{(\gamma_1 \mathscr{R} \gamma_1) \wedge (\gamma_1 \mathscr{R} \gamma_2)\} \wedge \bigwedge_{\gamma_3 \in \Gamma - \{\gamma_1, \gamma_2\}} [1] \\
&= (\gamma_1 \mathscr{R} \gamma_1) \wedge (\gamma_1 \mathscr{R} \gamma_2) \\
&= (\gamma_1 \mathscr{R} \gamma_2), \text{ since } \mathscr{R} \text{ is reflexive.}
\end{aligned}$$

Therefore, $\gamma_1 \bar{\mathscr{R}} \gamma_2 = \gamma_1 \mathscr{R} \gamma_2$ and hence $\bar{\psi} = \hat{\psi}$. □

remark 4.3. Let ψ be a FCF. If $\bar{\mathscr{R}} = \mathscr{R}$, then $\bar{\mathscr{P}} = \mathscr{P}$ and $\bar{\mathscr{I}} = \mathscr{I}$.

5 Axioms of Fuzzy Revealed Preferences Based on S-implications

Despite the lack of associativity, the rationality axiom in the S-implication-S-residuum structures is reformulated in this section and the interrelationships among these axioms are verified. Also, regularity of the fuzzy preference generated by the fuzzy choice function under the condition of S-FWCA is proved.

Definition 5.1. Let $\mathscr{R}$ be a fuzzy relation on Γ. Then the Ⓢ-transitive closure of $\mathscr{R}$ is defined as $\gamma_1 W \gamma_2 = (\gamma_1 \mathscr{R} \gamma_2) \vee \bigvee_{k \in \mathbb{I}} \bigvee_{\gamma_3, \gamma_4, \ldots, \gamma_{k+2} \in \Gamma} \Big(\ldots\Big((\gamma_1 \mathscr{R} \gamma_3) \circledS (\gamma_3 \mathscr{R} \gamma_4)\Big) \circledS \ldots \circledS (\gamma_k \mathscr{R} \gamma_2)\Big)$.

Clearly, $\mathscr{R} \subseteq W$.

Example 5.1. *Let $\Gamma = \{\sigma, \omega, \tau\}$ and $\mathscr{R}$ be a fuzzy relation given as*

$$\mathscr{R} = \begin{bmatrix} 1 & 0 & 0.2 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}.$$

Then for the standard fuzzy complement c and t-conorm $\circledast$, the Ⓢ-transitive closure of $\mathscr{R}$ is

$$W = \begin{bmatrix} 1 & 0 & 0.2 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}.$$

Definition 5.2. Let ψ $\psi \in \mathscr{F}(\Gamma)^{\mathscr{B}}$ be a We define the fuzzy relation $\widetilde{P}$ on Γ by

$$\gamma_1 \widetilde{P} \gamma_2 = \bigvee_{A \in \mathscr{B}} \{\Big(\psi(A)(\gamma_1) \circledS A(\gamma_2)\Big) \circledS [\psi(A)(\gamma_2)]^c\}, \quad \text{for any } \gamma_1, \gamma_2 \in \Gamma.$$

If ψ is normal, then $\bar{\mathscr{P}} = \widetilde{P} = \mathscr{P}$.
We shall denote the transitive closure of $\widetilde{P}$ by $P^{\circledast\prime}$. Then $\widetilde{P} \subseteq P^{\circledast\prime}$.

Example 5.2. *Let $\Gamma = \{\sigma, \omega\}$ and the FCF is defined by $\psi([\sigma]) = 1$, for all $\sigma \in \Gamma$, $\psi([\sigma,\omega])(\sigma) = 1$, $\psi([\sigma,\omega])(\omega) = 0.3$ and let c and $\circledast$ be the standard fuzzy complement and t-conorm respectively. Then $\sigma\widetilde{P}\sigma = \bigvee_{A\in\mathscr{B}}\{\Big(\psi(A)(\sigma)\circledS A(\sigma)\Big)\circledS[\psi(A)(\sigma)]^c\} = 0$, $\omega\widetilde{P}\omega = \bigvee_{A\in\mathscr{B}}\{\Big(\psi(A)(\omega))\circledS A(\omega)\Big)\circledS[\psi(A)(\omega)]^c\} = 0$, $\sigma\widetilde{P}\omega = \{\Big(\psi([\sigma,\omega])(\sigma)\circledS[\sigma,\omega](\omega)\Big)\circledS[\psi([\sigma,\omega])(\omega)]^c\} = 0$, $\omega\widetilde{P}\sigma = \{\Big(\psi([\sigma,\omega])(\omega)\circledS[\sigma,\omega](\sigma)\Big)\circledS[\psi([\sigma,\omega])(\sigma)]^c\} = 1$. Therefore, $\widetilde{P}$ and the transitive closure $P^{\circledast\prime}$ are $\gamma_1\widetilde{P}\gamma_2 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, $\gamma_1 P^{\circledast\prime}\gamma_2 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$.*

Definition 5.3. $A \in \psi \in \mathscr{F}(\Gamma)^{\mathscr{B}}$ is said to satisfy

(i) S-residium Fuzzy Weak Axiom of Revealed Preference (S-FWARP), if
$$\neg_s(\gamma_2\mathscr{R}\gamma_1) \geq \gamma_1\widetilde{P}\gamma_2, \text{ for all } \gamma_1,\gamma_2 \in \Gamma.$$

(ii) S-residium Fuzzy Strongly Weak Axiom of Revealed Preference (S-FSWARP), if
$$\neg_s(\gamma_2\mathscr{R}\gamma_1) \geq \gamma_1\bar{P}\gamma_2 \text{ for all } \gamma_1,\gamma_2 \in \Gamma.$$

(iii) S-residium Fuzzy Strong Axiom of Revealed Preference (S-FSARP), if
$$\neg_s(\gamma_2\mathscr{R}\gamma_1) \geq \gamma_1 P^{\circledast\prime}\gamma_2 \text{ for all } \gamma_1,\gamma_2 \in \Gamma.$$

(iv) S-residium Fuzzy Weak Congruence Axiom (S-FWCA), if
$\psi(A)(\gamma_1) \geq (\gamma_1\mathscr{R}\gamma_2)\circledS\Big(\psi(A)(\gamma_2)\circledS A(\gamma_1)\Big)$ for any $\gamma_1,\gamma_2 \in \Gamma$ and $A \in B$.

(v) S-residium Fuzzy Strongly Weak Congruence Axiom (S-FSWCA), if
$\psi(A)(\gamma_1) \geq (\gamma_1\bar{R}\gamma_2)\circledS\Big(\psi(A)(\gamma_2)\circledS A(\gamma_1)\Big)$, for any $\gamma_1,\gamma_2 \in \Gamma$ and $A \in B$.

(vi) S-residium Fuzzy Strong Congruence Axiom (S-FSCA), if
$\psi(A)(\gamma_1) \geq (\gamma_1 W\gamma_2)\circledS\Big(\psi(A)(\gamma_2)\circledS A(\gamma_1)\Big)$, for any $\gamma_1,\gamma_2 \in \Gamma$ and $A \in \mathscr{B}$,
where $(\gamma_1 W\gamma_2)$ denotes the transitive closure of $(\gamma_1\mathscr{R}\gamma_2)$.

Theorem 5.3. *Let $\psi \in \mathscr{F}(\Gamma)^{\mathscr{B}}$ be a FCF. Then*

(i) S-FSCA $\implies$ S-FWCA $\implies$ S-FSWCA

(ii) S-FSARP $\implies$ S-FWARP $\implies$ S-SFWARP.

Proof. (i) Since $\gamma_1\mathscr{R}\gamma_2 \leq \gamma_1 W\gamma_2$ and $\gamma_1\bar{\mathscr{R}}\gamma_2 \leq \gamma_1\mathscr{R}\gamma_2$, $\forall\gamma_1,\gamma_2 \in \Gamma$, we have S-FSCA $\implies$ S-FWCA and S-FWCA $\implies$ S-FSWCA.

(ii) Since $\gamma_1\widetilde{P}\gamma_2 \leq \gamma_1 P^{\circledast\prime}\gamma_2$ and $\gamma_1\bar{\mathscr{P}}\gamma_2 \leq \gamma_1\widetilde{P}\gamma_2$, $\forall\gamma_1,\gamma_2 \in \Gamma$, we have S-FSARP $\implies$ S-FWARP and S-FWARP $\implies$ S-SFWARP. ☐

Definition 5.4. A fuzzy relation $\mathscr{R}$ is called S-transitive if $\gamma_1\mathscr{R}\gamma_3 \geq (\gamma_1\mathscr{R}\gamma_2)\circledS(\gamma_2\mathscr{R}\gamma_3)$, for all $\gamma_1,\gamma_2,\gamma_3 \in X$.

Example 5.4. *Let $\Gamma = \{\sigma,\omega,\tau\}$ be the set of alternatives and the fuzzy relation is*
$$\mathscr{R} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

Then, we check that R is S-transitive. In the following table

γ_1	γ_2	γ_3	$\gamma_1\mathscr{R}\gamma_2$	$\gamma_2\mathscr{R}\gamma_3$	$(\gamma_1\mathscr{R}\gamma_2)Ⓢ(\gamma_2\mathscr{R}\gamma_3)$	$\gamma_1\mathscr{R}\gamma_3$	$\gamma_1\mathscr{R}\gamma_3 \geq (\gamma_1\mathscr{R}\gamma_2)Ⓢ(\gamma_2\mathscr{R}\gamma_3)$
σ	σ	σ	1	1	1	1	Yes
σ	σ	ω	1	1	1	1	Yes
σ	σ	τ	1	1	1	1	Yes
σ	ω	σ	1	0	0	1	Yes
σ	ω	ω	1	1	1	1	Yes
σ	ω	τ	1	1	1	1	Yes
σ	τ	σ	1	0	0	1	Yes
σ	τ	ω	1	0	0	1	Yes
σ	τ	τ	1	1	1	1	Yes
ω	σ	σ	0	1	0	0	Yes
ω	σ	ω	0	1	0	1	Yes
ω	σ	τ	0	1	0	1	Yes
ω	ω	σ	1	0	0	0	Yes
ω	ω	ω	1	1	1	1	Yes
ω	ω	τ	1	1	1	1	Yes
ω	τ	σ	1	0	0	0	Yes
ω	τ	ω	1	0	0	1	Yes
ω	τ	τ	1	1	1	1	Yes
τ	σ	σ	0	1	0	0	Yes
τ	σ	ω	0	1	0	0	Yes
τ	σ	τ	0	1	0	1	Yes
τ	ω	σ	0	0	0	0	Yes
τ	ω	ω	0	1	0	0	Yes
τ	ω	τ	0	1	0	1	Yes
τ	τ	σ	1	0	0	0	Yes
τ	τ	ω	1	0	0	0	Yes
τ	τ	τ	1	1	1	1	Yes

Since for all possible combinations of $\{\gamma_1, \gamma_2, \gamma_3\}$, $\gamma_1\mathscr{R}\gamma_3 \geq (\gamma_1\mathscr{R}\gamma_2)Ⓢ(\gamma_2\mathscr{R}\gamma_3)$ is true the given fuzzy relation $\mathscr{R}$ is S-transitive.

Theorem 5.5. *If $\widetilde{P}$ and $\mathscr{R}$ are S-transitive, then S-FWCA implies S-FSCA and S-FWARP implies S-FSARP.*

Proof. We have, $P^{\circledast\prime} = (\gamma_1\widetilde{P}\gamma_2) \vee \bigvee_{k\in\mathbb{N}} \bigvee_{\gamma_1',\gamma_2',\ldots,\gamma_k'\in\Gamma} \left(\ldots \left((\gamma_1\widetilde{P}\gamma_1')Ⓢ(\gamma_1'\widetilde{P}\gamma_2') \right) Ⓢ \ldots Ⓢ(\gamma_k'\widetilde{P}\gamma_2) \right)$.
Since $\widetilde{P}$ is transitive, we have

$$P^{\circledast\prime} = \gamma_1\widetilde{P}\gamma_2 \vee \gamma_1\widetilde{P}\gamma_2 = \gamma_1\widetilde{P}\gamma_2. \tag{5.1}$$

$\gamma_1 P^{\circledast\prime}\gamma_2 \leq \neg_s(\gamma_2\mathscr{R}\gamma_1) = [\gamma_2\mathscr{R}\gamma_1]^c$, for all $\gamma_1, \gamma_2 \in \Gamma$. By equation 5.1, $\gamma_1\widetilde{P}\gamma_2 \leq \neg_s\gamma_2\mathscr{R}\gamma_1 = [\gamma_2\mathscr{R}\gamma_1]^c$, for all $\gamma_1, \gamma_2 \in \Gamma$. Hence S-FWARP implies S-FSARP.
We have, $\gamma_1 W\gamma_2 = (\gamma_1\mathscr{R}\gamma_2) \vee \bigvee_{k\in\mathbb{N}} \bigvee_{\gamma_1',\gamma_2',\ldots,\gamma_k'\in\Gamma} \left(\ldots \left((\gamma_1\mathscr{R}\gamma_1')Ⓢ(\gamma_1'\mathscr{R}\gamma_2') \right) Ⓢ \ldots Ⓢ(\gamma_k'\mathscr{R}\gamma_2) \right)$. Since we have $\mathscr{R}$ is transitive,

$$\gamma_1 W\gamma_2 = \gamma_1\mathscr{R}\gamma_2 \vee \gamma_1\mathscr{R}\gamma_2 = \gamma_1\mathscr{R}\gamma_2. \tag{5.2}$$

$(\gamma_1 \mathscr{R} \gamma_2) \circledS \Big(\psi(A)(\gamma_2) \circledS A(\gamma_1) \Big) \leq \psi(A)(\gamma_1),$ for any $\gamma_1, \gamma_2 \in \Gamma$ and $A \in \mathscr{B}$.

By using equation 5.2, $(\gamma_1 W \gamma_2) \circledS \Big(\psi(A)(\gamma_2) \circledS A(\gamma_1) \Big) \leq \psi(A)(\gamma_1),$ for any $\gamma_1, \gamma_2 \in \Gamma$ and $A \in \mathscr{B}$. Hence, S-FWCA implies S-FSCA. □

Definition 5.5. A fuzzy preference relation $\mathscr{R}$ is called a regular, if it is reflexive, S-transitive and strongly total.

Theorem 5.6. *If the FCF ψ satisfies S-FWCA, then the fuzzy preference relation $\mathscr{R}$ generated by ψ is a regular on Γ.*

Proof. We shall prove the following conditions : The reflexivity and strongly total of $\mathscr{R}$ is already established in the Theorem 4.1. Finally, we show that $\mathscr{R}$ is S-transitive. Let $\gamma_1, \gamma_2, \gamma_3 \in \Gamma$. We shall prove that

$$\gamma_1 \mathscr{R} \gamma_3 \leq (\gamma_1 \mathscr{R} \gamma_2) \circledS (\gamma_2 \mathscr{R} \gamma_3).$$

Denote $B = [\gamma_1, \gamma_2, \gamma_3]$. Then $\psi(B)(\gamma_1) = 1$ or $\psi(B)(\gamma_2) = 1$ or $\psi(B)(\gamma_3) = 1$. We studies these three cases separately. **Case.1:** $\psi(B)(\gamma_1) = 1$**.**

Then $\gamma_1 \mathscr{R} \gamma_3 = \bigvee_{A \in \mathscr{B}} \big(\psi(A)(\gamma_1) \circledS A(\gamma_3)\big) \geq \psi(B)(\gamma_1) \circledS B(\gamma_3) = 1 \circledS 1 = 1$. Thus, $\gamma_1 \mathscr{R} \gamma_3 = 1$, and the inequality $\gamma_1 \mathscr{R} \gamma_3 \leq (\gamma_1 \mathscr{R} \gamma_2) \circledS (\gamma_2 \mathscr{R} \gamma_3)$ is trivially satisfied.

Case.2: $\psi(B)(\gamma_2) = 1$**.**

Now by S-FWCA, we have:

$$(\gamma_2 \mathscr{R} \gamma_3) \circledS \Big(\psi(B)(\gamma_3) \circledS B(\gamma_2) \Big) \leq \psi(B)(\gamma_2) \tag{5.3}$$

and $(\gamma_1 \mathscr{R} \gamma_2) \circledS \Big(\psi(B)(\gamma_2) \circledS B(\gamma_1) \Big) \leq \psi(B)(\gamma_1)$ But since $B(\gamma_1) = B(\gamma_2) = 1$, consequently, $(\gamma_2 \mathscr{R} \gamma_3) \circledS \psi(B)(\gamma_3) \leq \psi(B)(\gamma_2)$ and

$$(\gamma_1 \mathscr{R} \gamma_2) \circledS \psi(B)(\gamma_2) \leq \psi(B)(\gamma_1). \tag{5.4}$$

Since $\psi(B)(\gamma_2) = 1$, above inequality gives $(\gamma_2 \mathscr{R} \gamma_3) \circledS \psi(B)(\gamma_3) \leq 1 \implies \gamma_2 \mathscr{R} \gamma_3 \leq \psi(B)(\gamma_3) \delta_s 1 = 1$. Thus $(\gamma_1 \mathscr{R} \gamma_2) \circledS 1 \leq \psi(B)(\gamma_1) \implies \gamma_1 \mathscr{R} \gamma_2 \leq \psi(B)(\gamma_1)$. Therefore, $(\gamma_1 \mathscr{R} \gamma_2) \circledS (\gamma_2 \mathscr{R} \gamma_3) \leq \gamma_1 \mathscr{R} \gamma_2 \leq \psi(B)(\gamma_1) \leq \psi(B)(\gamma_1) \circledS B(\gamma_3) = \gamma_1 \mathscr{R} \gamma_3$. Hence, $(\gamma_1 \mathscr{R} \gamma_2) \circledS (\gamma_2 \mathscr{R} \gamma_3) \leq \gamma_1 \mathscr{R} \gamma_3$.

Case.3: $\psi(B)(\gamma_3) = 1$**.**

In this case, the inequality by using equation 5.4, $(\gamma_2 \mathscr{R} \gamma_3) \circledS \psi(B)(\gamma_3) \leq \psi(B)(\gamma_2)$ becomes $\gamma_2 \mathscr{R} \gamma_3 \leq \psi(B)(\gamma_2)$ therefore, $(\gamma_1 \mathscr{R} \gamma_2) \circledS (\gamma_2 \mathscr{R} \gamma_3) \leq (\gamma_1 \mathscr{R} \gamma_2) \circledS \psi(B)(\gamma_2) \leq \psi(B)(\gamma_1) = \psi(B)(\gamma_1)$ $\circledS B(\gamma_3) \leq \gamma_1 \mathscr{R} \gamma_3$ holds trivially, as $\gamma_1 \mathscr{R} \gamma_3 = \psi(B)(\gamma_3) = 1$. □

The following example shows that, if S-implication is employed with t-norm, similar to that in [09], the revealed preference relation generated by the fuzzy choice function satisfying weak rationality axioms is non-reflexiveness and hence non-regularity.

Example 5.7. *Consider the fuzzy choice function discussed in the Example 4.3. This fuzzy choice function ψ satisfies the Weak Fuzzy Choice Axiom: $\mathscr{R}(\gamma_1, \gamma_2) * \psi(A)(\gamma_2) * A(\gamma_1) \leq \psi(A)(\gamma_1), \forall \gamma_1, \gamma_2 \in \Gamma$. For this, $\mathscr{R}(\sigma, \sigma) * \psi(A)(\sigma) * A(\sigma) = (0.24).(1).(0.6) = 0.144 \leq 0.4 = \psi(A)(\sigma)$, $\mathscr{R}(\sigma, \omega) * \psi(A)(\omega) * A(\sigma) = (0.4).(1).(0.6) = 0.24 \leq 0.4 = \psi(A)(\sigma)$, $\mathscr{R}(\omega, \sigma) * \psi(A)(\sigma) * A(\omega) = (0.6).(0.4).(0.6) = 0.144 \leq 1 = \psi(A)(\omega)$ and $\mathscr{R}(\omega, \omega) * \psi(A)(\omega) * A(\omega) = (1).(1).(1) = 1 \leq 1 = \psi(A)(\omega)$. Since $\mathscr{R}(\sigma, \sigma) = 0.24 \neq 1$,the revealed preference relation $\mathscr{R}$ is not reflexive and hence not regular.*

Consequently, we conclude that the Proposition 5.1 (9) not hold in S-implication and t-norm structure. Before concluding the work we give the following table form results for S-implication-t-norms, and R-implication-t-norms structures.

Result	R-implication δ_r with t-norm $*$	S-implication δ_s with t-norm $*$
$SFCA \Rightarrow WFCA \Rightarrow SWFCA$	Hold	Hold
$SAFRP \Rightarrow WAFRP \Rightarrow SWAFRP$	Hold	Hold
Proposition 4.5 (9)	Hold	Not hold
Proposition 5.1 (9)	Hold	Not hold

6 Conclusion

Traditionally, the rationality of fuzzy choice functions (FCFs) is studied using R-implication with associated continuous t-norm [5-12]. Even though S-implication is ageneralization of classical implication and demonstrate is logically more expressive than R-implication it is not yet utilized to study the rationality. Here we attempt to employ it and obtain similar conclusions. However, we found that S-implication with t-norm is not sufficient for studying the rationality. Hence, we have introduced an operator called S-residuum operator instead of t-norm and used S-implication-S-residuum structure for studying it. We have reformulated rationality axioms of FCFs namely S-SFWARP, S-FWARP, S-FSARP, S-FSWCA, S-FWCA, S-FSCA under this structure and find their interrelations. The non-commutativity and non-associativity of S-residuum operator will provide an opportunity to reformulate the rationality axioms, so in the future work, we aim to provide a systematic characterization of the rationality axioms with this structure. We also plan to extend the analysis by engaging other classes of fuzzy implications (2),(4),(27) to investigate the rationality of given FCFs.

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