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q-Leonardo Split Quaternions and Some Applications

Original Research Article

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Abstract

Quantum calculus, using in various mathematical disciplines such as combinatorics and special functions, as well as in numerous applications including fractal analysis, multifractal measures, and entropy formulations of chaotic dynamical systems, has attracted significant scholarly attention in recent years. In the present study, we present a new class of Leonardo split quaternions whose components involve quantum integers into their components. We also present the fundamental properties and identities related to the q-Leonardo split quaternion, including Binet's formula, exponential generating functions, binomial sums, as well as d'Ocagne's, Vajda's, Catalan's, and Cassini's identities. Finally, we give different polar representation using Cayley Dickson's notation applications for some q-Leonardo split quaternions.The applications can be converted into quantum integer forms under suitable conditions with similar considerations and give a deeper understanding of their algebraic and geometric interpretations, and transformations.

Keywords: q-Leonardo split quaternion; q-calculus; Leonardo number sequence.
2010 Mathematics Subject Classification: 05A15; 11B37; 11B39.

1 Introduction

Sequences of positive integers are widely recognized for their mathematical significance and continue to be an active subject of research. Among these, the Fibonacci and Lucas sequences are particularly distinguished for their profound connections across diverse scientific domains, including biology, physics, and computer science.

Building upon these classical foundations, the introduction of q-Leonardo split quaternion sequences represents a sophisticated evolution. By integrating the recursive properties of Leonardo numbers with the non-commutative algebraic structure of split quaternions through quantum calculus, this framework provides enhanced derivational clarity for analyzing complex mathematical systems.This evolution remains deeply rooted in fundamental constants, most notably the Golden Ratio.

The irrational number $\frac{1+\sqrt{5}}{2}$, valued at approximately 1.61803..., is defined as the Golden Ratio and is widely observed across various fields of mathematics and art. This relationship is characterized by

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the fact that as n increases, the ratio of successive Fibonacci numbers, $\frac{F_{n+1}}{F_n}$, consistently converges toward the Golden Ratio.

The Fibonacci number sequence is defined as

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, ...$$

The n-th Fibonacci number which denoted by F_n has as the linear reccurence relation

$$F_n = F_{n-1} + F_{n-2} \; ; \; n \geq 2,$$

with the initial conditions $F_0 = 0, \mathrm{F}_1 = 1.$

Fibonacci numbers are closely related to Lucas numbers, which were discovered in 1870 by the mathematician François Édouard Anatole Lucas. Having conducted extensive research on both sequences.

The n-th Lucas number has the recursive definition as

$$2, 1, 3, 4, 7, 11, 18, 29, 47, 76, 123, ...$$

with the same reccurence relation

$$L_n = L_{n-1} + L_{n-2} \; ; \; L_0 = 2, L_1 = 1.$$

The Binet's formulas of the F_n and L_n are as follow:

$$F_n = \frac{\varphi^n - \psi^n}{\varphi - \psi} \text{ and } L_n = \varphi^n + \psi^n,$$

where

$$\varphi = \frac{1 + \sqrt{5}}{2} \text{ and } \psi = \frac{1 - \sqrt{5}}{2}. \tag{1.1}$$

The properties, relations, results between Fibonacci and Lucas numbers can be found in Bicknell et al. (1972); Koshy (2001); Park et al. (2020); Vajda (1989); Verner and Hoggatt (1969); Halıcı (2012); Horadam (1961, 1963, 1965). In recent years, Catarino and Borges Catarino and Borges (2019) introduced a new sequence related to Fibonacci numbers known as Leonardo numbers, establishing several of their fundamental properties alongside various sum and product identities. Building on this work, Alp and Koçer Alp and Koçer (2021b) derived further identities and subsequently introduced hybrid Leonardo numbers, exploring their essential characteristics Alp and Koçer (2021a). The sequence was further expanded by Shannon Shannon (2019) through a generalization of Leonardo numbers. More recently, Kürüz et al. Kürüz et al. (2021) integrated these concepts with hybrid number theory to investigate Leonardo Pisano polynomials and hybrinomials.

Quaternions represent a four-dimensional extension of the complex number system, originally formulated by the Irish mathematician W. R. Hamilton in 1843 Hamilton (1853). They have become essential in virtual reality and gaming for modeling 3D orientations, while also finding critical use in robotics, control theory, and signal processing. Consequently, they have emerged as a versatile engineering tool whose applications continue to expand throughout various scientific disciplines Hamilton (1844); Kuipers (1999); Altmann (1986).

In 1849, Cockle Cockle (1849) introduced split quaternions as an extension of real quaternions. Unlike their classical counterparts, split quaternions do not constitute a division algebra. Their multiplication rules deviate in the vector component because they are associated with Minkowski space rather than the Euclidean 3-space of real quaternions. This fundamental distinction renders split quaternions particularly significant in spacetime-related theories, such as special relativity Kula and Yaylı (2007).

Split Leonardo quaternions are studied in Atasoy (2025) by Atasoy. Our goal in this paper is to introduce q-Leonardo split quaternion sequences using notations from quantum calculus and give some fundamental identities. Additionally, we introduce a new polar representation of q-Leonardo split quaternion sequences, which are a special type of split quaternions that have an important place in the representation of rotations in Lorentz space, is obtained. Thus, the representation of complex expressions is made easier.

2 Preliminaries

In this part, some basic terms are recollected in relation to split quaternions, Leonardo numbers and quantum calculus.

The n-th Leonardo number is given by Le_n. The following recurrence relation, Catarino and Borges (2019); Alp and Koçer (2021b,a),

$$Le_n = Le_{n-1} + Le_{n-2} + 1, \quad n \geq 2$$

introduces this sequence, and $Le_0 = Le_1 = 1$ are the initial conditions. The Leonardo number are

$$1, 1, 3, 5, 9, 15, 25, 41, 67, 109, 177, 287, 465, ...$$

The relation between the n-th Fibonacci numbers and the n-th Leonardo numbers is given by

$$Le_n = 2F_{n+1} - 1.$$

Now, we give definitions and facts from the quantum calculus necessary for understanding of this paper Babadağ (2023); Kac and Cheung (2002); Le Stum and Quirós (2015).

For positive integers n, the quantum integer n or simply the q-integer n is defined by

$$[n]_q = \frac{1-q^n}{1-q} = 1 + q + \ldots + q^{n-1}. \tag{2.1}$$

For non-negative integer numbers m and n, we have

$$[m+n]_q = [m]_q + q^m [n]_q \quad \text{and} \quad [mn]_q = [m]_q [n]_{q^m}.$$

Using (1.1), If we get

$$q = \frac{\psi}{\varphi} = \frac{1-\sqrt{5}}{1+\sqrt{5}}, \tag{2.2}$$

where q-integer forms are respectively as follows;

$$F_n(\varphi; q) = \varphi^{n-1}[n]_q = \varphi^n \frac{1-q^n}{\varphi - \varphi q} \quad , \quad L_n(\varphi; q) = \varphi^n \frac{[2n]_q}{[n]_q} = \varphi^n \frac{1-q^{2n}}{1-q^n} \tag{2.3}$$

and

$$Le_n(\varphi; q) = 2F_{n+1}(\varphi; q) - 1 = 2\varphi^{n+1} \frac{1-q^{n+1}}{\varphi - \varphi q} - 1 = 2\varphi^n [n+1]_q - 1. \tag{2.4}$$

A split quaternion Cockle (1849); Kula and Yaylı (2007); Atasoy et al. (2017) is defined as the following quadruple

$$\gamma = \gamma_0 + \gamma_1 i_1 + \gamma_2 i_2 + \gamma_3 i_3$$

with $\gamma_0, \gamma_1, \gamma_2, \gamma_3 \in \mathbb{R}$ ($\mathbb{R}$ is the set of real numbers) and split quatenionic units i_1, i_2, i_3 satisfy

$$i_1^2 = -i_2^2 = -i_3^2 = -1, \; i_1 i_2 = -i_2 i_1 = i_3, \; i_2 i_3 = -i_1. \tag{2.5}$$

Let γ and δ be split quaternions, then addition and multiplication are

$$\gamma + \delta = (S_\gamma + S_\delta) + (V_\gamma + V_\delta),$$

$$\gamma\delta = S_\gamma S_\delta + \langle V_\gamma, V_\delta \rangle + S_\gamma V_\gamma + S_\delta V_\delta + V_\gamma \times V_\delta,$$

respectively, where $\langle \, , \rangle$ and $\times$ are inner and cross products in $\mathbb{E}_1^3$ ($\mathbb{E}_1^3$ is Minkowski 3-space). The conjugate and norm of γ are defined respectively as

$$\gamma^* = \gamma_0 - \gamma_1 i_1 - \gamma_2 i_2 - \gamma_3 i_3$$

and

$$|\gamma| = \sqrt{|\gamma\gamma^*|} = \sqrt{|\gamma_0^2 + \gamma_1^2 - \gamma_2^2 - \gamma_3^2|} = \sqrt{|\mathcal{I}_\gamma|}. \tag{2.6}$$

Here,

$$\mathcal{I}_\gamma = (\gamma_0^2) + (\gamma_1^2 - \gamma_2^2 - \gamma_3^2) = \mathcal{I}_{S_\gamma} + \mathcal{I}_{V_\gamma}. \tag{2.7}$$

For any split quaternion γ with $|\gamma| \neq 0$, $\frac{\gamma}{|\gamma|}$ is a unit split quaternion. The split quaternion γ is space-like, timelike and lightlike if $\mathcal{I}_\gamma < 0$, $\mathcal{I}_\gamma > 0$ and $\mathcal{I}_\gamma = 0$ respectively Kula and Yaylı (2007); Özdemir and Ergin (2006). The multiplicative inverse of γ is given by $\gamma^{-1} = \frac{\gamma^*}{|\gamma|^2}$. It is important to note that there is no inverse for a lightlike split quaternion.

3 Identities for q-Leonardo split quaternions

In this section, we define the q-Leonardo split quaternion sequences and present their fundamental identities and properties using Binet formula.

Definition 3.1. *For positive integer n, the n-th q-Leonardo split quaternion is defined by*

$$\begin{aligned}\mathcal{L}e_n(\varphi;q) &= Le_n(\varphi;q) + Le_{n+1}(\varphi;q)i_1 + Le_{n+2}(\varphi;q)i_2 + Le_{n+3}(\varphi;q)i_3\\ &= 2\varphi^{n+1}\frac{1-q^{n+1}}{\varphi-\varphi q} + 2\varphi^{n+2}\frac{1-q^{n+2}}{\varphi-\varphi q}i_1\\ &+ 2\varphi^{n+3}\frac{1-q^{n+3}}{\varphi-\varphi q}i_2 + 2\varphi^{n+4}\frac{1-q^{n+4}}{\varphi-\varphi q}i_3 - \mathcal{A},\end{aligned}$$

by using (2.4)*, we can rewrite as*

$$\mathcal{L}e_n(\varphi;q) = 2\varphi^n[n+1]_q + 2\varphi^{n+1}[n+2]_q i_1 + 2\varphi^{n+2}[n+3]_q i_2 + 2\varphi^{n+3}[n+4]_q i_3 - \mathcal{A},$$

where $\mathcal{A} = 1 + i_1 + i_2 + i_3$.

Taking $\varphi = \frac{1+\sqrt{5}}{2}$ and $q = \frac{1-\sqrt{5}}{1+\sqrt{5}}$, the few q-Leonardo split quaternions are written as follows;

$$\begin{aligned}\mathcal{L}e_0(\varphi;q) &= 2\varphi^1\frac{1-q^1}{\varphi-\varphi q} + 2\varphi^2\frac{1-q^2}{\varphi-\varphi q}i_1 + 2\varphi^3\frac{1-q^3}{\varphi-\varphi q}i_2 + 2\varphi^4\frac{1-q^4}{\varphi-\varphi q}i_3 - \mathcal{A}\\ &= 1 + 1i_1 + 3i_2 + 5i_3,\end{aligned}$$

$$\begin{aligned}\mathcal{L}e_1(\varphi;q) &= 2\varphi^2\frac{1-q^2}{\varphi-\varphi q} + 2\varphi^3\frac{1-q^3}{\varphi-\varphi q}i_1 + 2\varphi^4\frac{1-q^4}{\varphi-\varphi q}i_2 + 2\varphi^5\frac{1-q^5}{\varphi-\varphi q}i_3 - \mathcal{A}\\ &= 1 + 3i_1 + 5i_2 + 9i_3, ...\end{aligned}$$

Theorem 3.2. *The Binet formula for the n-th q-Leonardo split quaternion is*

$$\begin{aligned}\mathcal{L}e_n(\varphi;q) &= 2\left(\frac{\varphi^{n+1}\underline{\varphi} - (\varphi q)^{n+1}\underline{\beta}}{\varphi-\varphi q}\right) - \mathcal{A}\\ &= 2\left(\varphi^n[n]_q\underline{\varphi} + (\varphi q)^n\,\underline{\psi}\right) - \mathcal{A}\end{aligned} \tag{3.1}$$

where $\mathcal{A} = 1 + i_1 + i_2 + i_3$, $\underline{\varphi} = 1 + \varphi i_1 + \varphi^2 i_2 + \varphi^3 i_3$,
$\underline{\psi} = 1 + \varphi[2]_q i_1 + \varphi^2[3]_q i_2 + \varphi^3[4]_q i_3$ and $\underline{\beta} = 1 + (\varphi q)i_1 + (\varphi q)^2 i_2 + (\varphi q)^3 i_3$.

Proof. By using (2.4), and making the appropriate calculations, we obtain

$$\begin{aligned}\mathcal{L}e_n(\varphi;q) &= Le_n(\varphi;q) + Le_{n+1}(\varphi;q)i_1 + Le_{n+2}(\varphi;q)i_2 + Le_{n+3}(\varphi;q)i_3 \\ &= 2\varphi^{n+1}\frac{1-q^{n+1}}{\varphi-\varphi q} + 2\varphi^{n+2}\frac{1-q^{n+2}}{\varphi-\varphi q}i_1 + 2\varphi^{n+3}\frac{1-q^{n+3}}{\varphi-\varphi q}i_2 \\ &+ 2\varphi^{n+4}\frac{1-q^{n+4}}{\varphi-\varphi q}i_3 - (1+i_1+i_2+i_3) \\ &= 2\frac{\varphi^{n+1}}{\varphi-\varphi q}\left(1+\varphi i_1+\varphi^2 i_2+\varphi^3 i_3\right) \\ &- 2\frac{(\varphi q)^{n+1}}{\varphi-\varphi q}\left(1+(\varphi q)i_1+(\varphi q)^2 i_2+(\varphi q)^3 i_3\right) - \mathcal{A} \\ &= 2\left(\frac{\varphi^{n+1}\underline{\varphi} - (\varphi q)^{n+1}\underline{\beta}}{\varphi-\varphi q}\right) - \mathcal{A}.\end{aligned}$$

or equivalent

$$\begin{aligned}\mathcal{L}e_n(\varphi;q) &= Le_n(\varphi;q) + Le_{n+1}(\varphi;q)i_1 + Le_{n+2}(\varphi;q)i_2 + Le_{n+3}(\varphi;q)i_3 \\ &= 2\varphi^n[n+1]_q + 2\varphi^{n+1}[n+2]_q i_1 + 2\varphi^{n+2}[n+3]_q i_2 \\ &+ 2\varphi^{n+3}[n+4]_q i_3 - (1+i_1+i_2+i_3) \\ &= 2\varphi^n\left([n]_q+q^n\right) + 2\varphi^{n+1}\left([n]_q+q^n\,[2]\right)i_1 \\ &+ 2\varphi^{n+2}\left([n]_q+q^n\,[3]\right)i_2 + 2\varphi^{n+3}\left([n]_q+q^n\,[4]\right)i_3 - \mathcal{A} \\ &= 2\varphi^n[n]_q\left(1+\varphi i_1+\varphi^2 i_2+\varphi^3 i_2\right) \\ &+ 2\varphi^n q^n\left(1+\varphi[2]_q i_1+\varphi^2[3]_q i_2+\varphi^3[4]_q i_3\right) - \mathcal{A} \\ &= 2\left(\varphi^n[n]_q\underline{\varphi} + (\varphi q)^n\,\underline{\psi}\right) - \mathcal{A}\end{aligned}$$

□

For example, let us consider the case of $n=1$ in (3.1). We will have

$$\begin{aligned}\mathcal{L}e_1(\varphi;q) =& 2\left(\frac{\varphi^2\underline{\varphi} - (\varphi q)^2\underline{\beta}}{(\varphi-\varphi q)}\right) - (1+i_1+i_2+i_3) \\ =& 2\varphi\left(\frac{1+\varphi i_1+\varphi^2 i_2+\varphi^3 i_3 - q\left(1+(\varphi q)i_1+(\varphi q)^2 i_2+(\varphi q)^3 i_3\right)}{(1-q)}\right) \\ &- \mathcal{A} \\ =& 2\left(\frac{\varphi(1-q)+\varphi^2 i_1\left(1-q^2\right)+\varphi^3 i_2\left(1-q^3\right)+\varphi^4 i_3\left(1-q^4\right)}{(1-q)}\right) - \mathcal{A},\end{aligned}$$

by using (2.1), we obtain

$$\mathcal{L}e_1(\varphi;q) = 2\left(\varphi+\varphi^2\,[2]_q\,i_1+\varphi^3\,[3]_q\,i_2+\varphi^4\,[4]_q\,i_3\right) - \mathcal{A}.$$

Theorem 3.3. *The exponential generating function for the n-th q-Leonardo split quaternion is*

$$\sum_{n=0}^{\infty}\mathcal{L}e_n(\varphi;q)\frac{x^n}{n!} = 2\frac{e^{\varphi x}\varphi\underline{\varphi} - e^{(\varphi q)x}\varphi q\underline{\psi}}{\varphi(1-q)} - \mathcal{A}e^x.$$

Proof. Using the Binet formula of the q-Leonardo split quaternion, we will have

$$\begin{aligned}
&\sum_{n=0}^{\infty}\mathcal{L}e_n(\varphi;q)\frac{x^n}{n!}\\
=&\sum_{n=0}^{\infty}\left(Le_n(\varphi;q)+Le_{n+1}(\varphi;q)i_1+Le_{n+2}(\varphi;q)i_2+Le_{n+3}(\varphi;q)i_3\right)\frac{x^n}{n!}\\
=&\sum_{n=0}^{\infty}\left(2\varphi^n\frac{1-q^{n+1}}{1-q}+2\varphi^{n+1}\frac{1-q^{n+2}}{1-q}i_1+2\varphi^{n+2}\frac{1-q^{n+3}}{1-q}i_2\right)\frac{x^n}{n!}\\
&+\sum_{n=0}^{\infty}\left(2\varphi^{n+3}\frac{1-q^{n+4}}{1-q}i_3-(1+i_1+i_2+i_3)\right)\frac{x^n}{n!}\\
=&\sum_{n=0}^{\infty}\left(2\varphi^n\frac{1-q^{n+1}}{1-q}+2\varphi^{n+1}\frac{1-q^{n+2}}{1-q}i_1+2\varphi^{n+2}\frac{1-q^{n+3}}{1-q}i_2\right)\frac{x^n}{n!}\\
&+\sum_{n=0}^{\infty}\left(2\varphi^{n+3}\frac{1-q^{n+4}}{1-q}i_3-\mathcal{A}\right)\frac{x^n}{n!}\\
=&\sum_{n=0}^{\infty}\left[2\frac{\varphi^{n+1}\underline{\varphi}-(\varphi q)^{n+1}\underline{\psi}}{\varphi(1-q)}-\mathcal{A}\right]\frac{x^n}{n!}\\
=&2\frac{\varphi\underline{\varphi}}{\varphi(1-q)}\sum_{n=0}^{\infty}\varphi^n\frac{x^n}{n!}-2\frac{\varphi q\underline{\psi}}{\varphi(1-q)}\sum_{n=0}^{\infty}(\varphi q)^n\frac{x^n}{n!}-\mathcal{A}\sum_{n=0}^{\infty}\frac{x^n}{n!}\\
=&2\frac{e^{\varphi x}\underline{\varphi}-e^{(\varphi q)x}q\underline{\psi}}{1-q}-\mathcal{A}e^x.
\end{aligned}$$

Thus, the proof is completed. □

Theorem 3.4. *Let $\mathcal{L}e_n(\alpha;q)$ be n-th q-Leonardo split quaternion. In this case, for non-negative integer numbers n and j, we can give the following relations:*

$$\begin{aligned}
&\textbf{i.}\ \sum_{k=0}^{n}\binom{n}{k}(-\varphi^2q)^{n-k}\mathcal{L}e_{2k+j}(\varphi;q)\\
&=\begin{cases}(\varphi-\varphi q)^n\left(\mathcal{L}e_{n+j}(\varphi;q)+\mathcal{A}\right)-\mathcal{A}\left(1-\varphi^2q\right)^n, & \textit{if } n \textit{ is even}\\ 2(\varphi-\varphi q)^{n-1}\mathcal{L}_{n+j+1}(\varphi;q)-\mathcal{A}\left(1-\varphi^2q\right)^n, & \textit{if } n \textit{ is odd}\end{cases}\\
&\textbf{ii.}\ \sum_{k=0}^{n}\binom{n}{k}(-1)^k(-\varphi^2q)^{n-k}\mathcal{L}e_{2k+j}(\varphi;q)\\
&=(-\varphi[2]_q)^n\mathcal{L}e_{n+j}(\varphi;q)+\left((-1)^n(\varphi[2]_q)^n+(1+\varphi^2q)^n\right)\mathcal{A}
\end{aligned}$$

where $\mathcal{L}_{n+j+1}(\varphi;q)$ is q-Lucas split quaternion.

Proof. **i.** By using the binomial coefficients for the Binet formula for q-Leonardo split quaternion in

(3.1), and doing the necessary calculations, we obtain

$$\begin{aligned}
&\sum_{k=0}^{n}\binom{n}{k}(-\varphi^2 q)^{n-k}\mathcal{L}e_{2k+j}(\varphi;q)\\
=&\sum_{k=0}^{n}\binom{n}{k}(-\varphi^2 q)^{n-k}\left(\frac{2\varphi^{2k+j+1}\underline{\varphi}-2(\varphi q)^{2k+j+1}\underline{\psi}}{\varphi(1-q)}-\mathcal{A}\right)\\
=&2\frac{\varphi^{j+1}\underline{\varphi}}{\varphi(1-q)}\sum_{k=0}^{n}\binom{n}{k}(-\varphi^2 q)^{n-k}\varphi^{2k}\\
&-2\frac{(\varphi q)^{j+1}\underline{\psi}}{\varphi(1-q)}\sum_{k=0}^{n}\binom{n}{k}(-\varphi^2 q)^{n-k}(\varphi q)^{2k}-\mathcal{A}\sum_{k=0}^{n}\binom{n}{k}(-\varphi^2 q)^{n-k}\\
=&2\frac{\varphi^{j+1}\underline{\varphi}\left(\varphi^2-\varphi^2 q\right)^n-(\varphi q)^{j+1}\underline{\psi}\left((\varphi q)^2-\varphi^2 q\right)^n}{\varphi(1-q)}-\mathcal{A}\left(1-\varphi^2 q\right)^n\\
=&2\frac{(\varphi(\varphi-\varphi q))^n\varphi^{j+1}\underline{\varphi}-(-\varphi q(\varphi-\varphi q))^n(\varphi q)^{j+1}\underline{\psi}}{\varphi(1-q)}-\mathcal{A}\left(1-\varphi^2 q\right)^n\\
=&2\frac{(\varphi-\varphi q)^n\varphi^{n+j+1}\underline{\varphi}+(-1)^{n+1}(\varphi-\varphi q)^n(\varphi q)^{n+j+1}\underline{\psi}}{\varphi(1-q)}-\mathcal{A}\left(1-\varphi^2 q\right)^n.
\end{aligned}$$

Here, if n is chosen even, we obtain

$$\begin{aligned}
&\sum_{k=0}^{n}\binom{n}{k}(-\varphi^2 q)^{n-k}\mathcal{L}e_{2k+j}(\varphi;q)\\
=&2\frac{(\varphi-\varphi q)^n\varphi^{n+j+1}\underline{\varphi}+(-1)^{n+1}(\varphi-\varphi q)^n(\varphi q)^{n+j+1}\underline{\psi}}{\varphi(1-q)}-\mathcal{A}\left(1-\varphi^2 q\right)^n\\
=&2(\varphi-\varphi q)^n\left(\frac{\varphi^{n+j+1}\underline{\varphi}-(\varphi q)^{n+j+1}\underline{\psi}}{\varphi(1-q)}\right)-\mathcal{A}\left(1-\varphi^2 q\right)^n\\
=&(\varphi-\varphi q)^n\left(2\left(\frac{\varphi^{n+j+1}\underline{\varphi}-(\varphi q)^{n+j+1}\underline{\psi}}{\varphi(1-q)}\right)-\mathcal{A}+\mathcal{A}\right)-\mathcal{A}\left(1-\varphi^2 q\right)^n\\
=&(\varphi-\varphi q)^n\left(\mathcal{L}e_{n+j}(\varphi;q)+\mathcal{A}\right)-\mathcal{A}\left(1-\varphi^2 q\right)^n.
\end{aligned}$$

On the other hand, if n is chosen odd, we have

$$\begin{aligned}
&\sum_{k=0}^{n}\binom{n}{k}(-\varphi^2 q)^{n-k}\mathcal{L}e_{2k+j}(\varphi;q)\\
=&2\frac{(\varphi(\varphi-\varphi q))^n\varphi^{j+1}\underline{\varphi}+(\varphi q(\varphi-\varphi q))^n(\varphi q)^{j+1}\underline{\psi}}{\varphi(1-q)}-\mathcal{A}\left(1-\varphi^2 q\right)^n\\
=&2(\varphi-\varphi q)^n\left(\frac{\varphi^{n+j+1}\underline{\varphi}+(\varphi q)^{n+j+1}\underline{\psi}}{\varphi(1-q)}\right)-\mathcal{A}\left(1-\varphi^2 q\right)^n\\
=&2(\varphi-\varphi q)^{n-1}\left(\varphi^{n+j+1}\underline{\varphi}+(\varphi q)^{n+j+1}\underline{\psi}\right)-\mathcal{A}\left(1-\varphi^2 q\right)^n\\
=&2(\varphi-\varphi q)^{n-1}\mathcal{L}_{n+j+1}(\varphi;q)-\mathcal{A}\left(1-\varphi^2 q\right)^n.
\end{aligned}$$

ii. If we use (3.1) and binomial coefficients, we obtain that

$$\begin{aligned}
&\sum_{k=0}^{n}\binom{n}{k}(-1)^{k}(-\varphi^{2}q)^{n-k}\mathcal{L}e_{2k+j}(\varphi;q)\\
=&\sum_{k=0}^{n}\binom{n}{k}(-1)^{k}(-\varphi^{2}q)^{n-k}\left(\frac{2\varphi^{2k+j+1}\underline{\varphi}-2(\varphi q)^{2k+j+1}\underline{\psi}}{\varphi-\varphi q}-\mathcal{A}\right)\\
=&\frac{2}{\varphi-\varphi q}\left(\sum_{k=0}^{n}\binom{n}{k}(-\varphi^{2}q)^{n-k}(-\varphi^{2})^{k}\varphi^{j+1}\underline{\varphi}\right)\\
&-\frac{2}{\varphi-\varphi q}\left(\sum_{k=0}^{n}\binom{n}{k}(-\varphi^{2}q)^{n-k}(-(\varphi q)^{2})^{k}(\varphi q)^{j+1}\underline{\psi}\right)\\
&-\sum_{i=0}^{n}\binom{n}{k}\left(-\alpha^{2}q\right)^{n-k}(-1)^{k}\mathcal{A}\\
=&2\left(\frac{(-\varphi^{2}-\varphi^{2}q)^{n}\varphi^{j+1}\underline{\varphi}-(-(\varphi q)^{2}-\varphi^{2}q)^{n}(\varphi q)^{j+1}\varphi^{j+1}\underline{\psi}}{\varphi-\varphi q}\right)\\
&-(-1-\varphi^{2}q)^{n}\mathcal{A}\\
=&2\left(\frac{\varphi^{n+j+1}\underline{\varphi}-(\varphi q)^{n+j+1}\underline{\psi}}{\varphi-\varphi q}\right)(-\varphi)^{n}(1+q)^{n}-(-1-\varphi^{2}q)^{n}\mathcal{A}\\
=&\left(2\left(\frac{\varphi^{n+j}\underline{\varphi}-(\varphi q)^{n+j}\underline{\psi}}{1-q}\right)-\mathcal{A}+\mathcal{A}\right)(-\varphi)^{n}(1+q)^{n}-(-1-\varphi^{2}q)^{n}\mathcal{A}.
\end{aligned}$$

By using (2.1), we will have

$$\begin{aligned}
&\sum_{k=0}^{n}\binom{n}{k}(-1)^{k}(-\varphi^{2}q)^{n-k}\mathcal{L}e_{2k+j}(\varphi;q)\\
&=(-\varphi[2]_{q})^{n}\mathcal{L}e_{n+j}(\varphi;q)+\left((-1)^{n}(\varphi[2]_{q})^{n}+(1+\varphi^{2}q)^{n}\right)\mathcal{A}
\end{aligned}$$

where

$$(-\varphi(1+q))^{n}=(-\varphi\frac{(1-q)(1+q)}{1-q})^{n}=(-\varphi\frac{1-q^{2}}{1-q})^{n}=(-\varphi[2]_{q})^{n}=(-1)^{n}(\varphi[2]_{q})^{n}.$$

The proof is completed. □

Theorem 3.5. *Let n be a non-negative integer and m be a natural number with $n\geq m$, d'Ocagne's identity for q-Leonardo split quaternion is given by*

$$\begin{aligned}
&\mathcal{L}e_{m+1}(\varphi;q)\mathcal{L}e_{n}(\varphi;q)-\mathcal{L}e_{m}(\varphi;q)\mathcal{L}e_{n+1}(\varphi;q)\\
&=-\frac{4\varphi^{m+n-1}}{1-q}\left(q^{m}\underline{\varphi}\,\underline{\beta}-q^{n}\underline{\beta}\,\underline{\varphi}\right)+\mathcal{A}(\mathcal{L}e_{n+1}(\varphi;q)-\mathcal{L}e_{n}(\varphi;q))\\
&-(\mathcal{L}e_{m+1}(\varphi;q)-\mathcal{L}e_{m}(\varphi;q))\mathcal{A}.
\end{aligned}$$

where

$$\underline{\varphi}\,\underline{\beta}=2+2\varphi i_{1}+2\left(\varphi q\right)^{2}i_{2}+(\varphi^{3}+(\varphi q)^{3}+\varphi-(\varphi q))i_{3}$$

and

$$\underline{\beta}\,\underline{\varphi}=2+2\left(\varphi q\right)i_{1}+2\varphi^{2}i_{2}+(\varphi^{3}+(\varphi q)^{3}-\varphi+(\varphi q))i_{3}.$$

Proof. Using the Binet formula for the n-th q-Leonardo split quaternion in Theorem (3.5), we find that

$$\begin{aligned}
&\mathcal{L}e_{m+1}(\varphi;q)\mathcal{L}e_{n}(\varphi;q)-\mathcal{L}e_{m}(\varphi;q)\mathcal{L}e_{n+1}(\varphi;q)\\
&=\left(2\left(\frac{\varphi^{m+2}\underline{\varphi}-(\varphi q)^{m+2}\underline{\beta}}{\varphi-\varphi q}\right)-\mathcal{A}\right)\left(2\left(\frac{\varphi^{n+1}\underline{\varphi}-(\varphi q)^{n+1}\underline{\beta}}{\varphi-\varphi q}\right)-\mathcal{A}\right)\\
&-\left(2\left(\frac{\varphi^{m+1}\underline{\varphi}-(\varphi q)^{m+1}\underline{\beta}}{\varphi-\varphi q}\right)-\mathcal{A}\right)\left(2\left(\frac{\varphi^{n+2}\underline{\varphi}-(\varphi q)^{n+2}\underline{\beta}}{\varphi-\varphi q}\right)-\mathcal{A}\right).
\end{aligned}$$

In this last equation, by making the necessary calculations, we get

$$\begin{aligned}
&\mathcal{L}e_{m+1}(\varphi;q)\mathcal{L}e_{n}(\varphi;q)-\mathcal{L}e_{m}(\varphi;q)\mathcal{L}e_{n+1}(\varphi;q)\\
&=\frac{4}{\varphi-(\varphi q)}\left(\varphi^{m}\left(\varphi q\right)^{n}\underline{\varphi}\,\underline{\beta}\ -\varphi^{n}\left(\varphi q\right)^{m}\underline{\beta}\,\underline{\varphi}\ \right)\\
&-\mathcal{A}\left(\frac{2\left(\varphi^{n+1}\underline{\varphi}-(\varphi q)^{n+1}\underline{\beta}\right)}{\varphi-\varphi q}-\frac{2\left(\varphi^{n+2}\underline{\varphi}-(\varphi q)^{n+2}\underline{\beta}\right)}{\varphi-\varphi q}\right)\\
&-\left(\frac{2\left(\varphi^{m+2}\underline{\varphi}-(\varphi q)^{m+2}\underline{\beta}\right)}{\varphi-\varphi q}-\frac{2\left(\varphi^{m+1}\underline{\varphi}-(\varphi q)^{m+1}\underline{\beta}\right)}{\varphi-\varphi q}\right)\mathcal{A}\\
&=\frac{4\varphi^{m+n-1}}{1-q}\left(q^{n}\underline{\beta}\,\underline{\varphi}-q^{m}\underline{\varphi}\,\underline{\beta}\right)+\mathcal{A}(\mathcal{L}e_{n+1}(\varphi;q)-\mathcal{L}e_{n}(\varphi;q))\\
&-(\mathcal{L}e_{m+1}(\varphi;q)-\mathcal{L}e_{m}(\varphi;q))\mathcal{A}
\end{aligned}$$

where $\underline{\varphi}=1+\varphi i_1+\varphi^2 i_2+\varphi^3 i_3,\quad \underline{\beta}=1+(\varphi q)i_1+(\varphi q)^2 i_2+(\varphi q)^3 i_3.$ □

Example 3.1. *Using* (2.5)*, if we take* $m=n=1$ *in Theorem* (3.5)*, we will have*

$$\begin{aligned}
&\mathcal{L}e_{2}(\varphi;q)\mathcal{L}e_{1}((\varphi;q)-\mathcal{L}e_{1}(\varphi;q)\mathcal{L}e_{2}(\varphi;q)\\
&=-\frac{4\varphi q}{1-q}\left(\underline{\varphi}\,\underline{\beta}-\underline{\beta}\,\underline{\varphi}\right)+\mathcal{A}(\mathcal{L}e_{2}(\varphi;q)-\mathcal{L}e_{1}(\varphi;q))-(\mathcal{L}e_{2}(\varphi;q)-\mathcal{L}e_{1}(\varphi;q))\mathcal{A}\\
&=-\frac{4\varphi q}{1-q}(2+2\varphi i_1+2\left(\varphi q\right)^2 i_2+(\varphi^3+\left(\varphi q\right)^3+\varphi-\varphi q)i_3)\\
&-\frac{4\varphi q}{1-q}(2+2\varphi q i_1+2\varphi^2 i_2+(\varphi^3+\left(\varphi q\right)^3-\varphi+\varphi q)i_3)\\
&+(1+i_1+i_2+i_3)[(3+5i_1+9i_2+15i_3)-(1+3i_1+5i_2+9i_3)]\\
&-[(3+5i_1+9i_2+15i_3)-(1+3i_1+5i_2+9i_3)](1+i_1+i_2+i_3)\\
&=-\frac{4\varphi q}{1-q}[2(\varphi-q\varphi)i_1-2(\varphi^2-(\varphi q)^2)i_2+2(\varphi-\varphi q)i_3]\\
&-(1+i_1+i_2+i_3)(-2-2i_1-4i_2-6i_3)\\
&-(2+2i_1+4i_2+6i_3)(1+i_1+i_2+i_3).
\end{aligned}$$

In this last equation, we take into account that $\varphi = \frac{1+\sqrt{5}}{2}$ *and* $q = \frac{1-\sqrt{5}}{1+\sqrt{5}}$ *from* (1.1) *and* (2.2)*, we obtain*

$$\begin{aligned}
&\mathcal{L}e_2(\frac{1+\sqrt{5}}{2};\frac{1-\sqrt{5}}{1+\sqrt{5}})\mathcal{L}e_1(\frac{1+\sqrt{5}}{2};\frac{1-\sqrt{5}}{1+\sqrt{5}})\\
&-\mathcal{L}e_1(\frac{1+\sqrt{5}}{2};\frac{1-\sqrt{5}}{1+\sqrt{5}})\mathcal{L}e_2(\frac{1+\sqrt{5}}{2};\frac{1-\sqrt{5}}{1+\sqrt{5}})\\
&=\frac{-8}{\sqrt{5}}[\sqrt{5}i_1-\sqrt{5}i_2+\sqrt{5}i_3]\ -(-10-2i_1-2i_2-10i_3)\\
&-(10+6i_1+10i_2+6i_3)\\
&=-8(i_1-i_2+i_3)\ -(-10-2i_1-2i_2-10i_3)-(10+6i_1+10i_2+6i_3)\\
&=-12i_1-4i_3.
\end{aligned}$$

Now, checking the result:

$$\begin{aligned}
&\mathcal{L}e_2(\frac{1+\sqrt{5}}{2};\frac{1-\sqrt{5}}{1+\sqrt{5}})\mathcal{L}e_1(\frac{1+\sqrt{5}}{2};\frac{1-\sqrt{5}}{1+\sqrt{5}})\\
&-\mathcal{L}e_1(\frac{1+\sqrt{5}}{2};\frac{1-\sqrt{5}}{1+\sqrt{5}})\mathcal{L}e_2(\frac{1+\sqrt{5}}{2};\frac{1-\sqrt{5}}{1+\sqrt{5}})\\
=&(3+5i_1+9i_2+15i_3)(1+3i_1+5i_2+9i_3)\\
&-(1+3i_1+5i_2+9i_3)(3+5i_1+9i_2+15i_3)\\
=&(168+8i_1+24i_2+40i_3)-(168+20i_1+24i_2+44i_3)\\
=&-12i_1-4i_3.
\end{aligned}$$

Theorem 3.6. *For the integers* m, n *and* k *such that* $n > m \geq k \geq 1$*, then Vajda's identity is as follows:*

$$\begin{aligned}
&\mathcal{L}e_{n+k}(\varphi;q)\mathcal{L}e_{m-k}(\varphi;q)-\mathcal{L}e_n(\varphi;q)\mathcal{L}e_m(\varphi;q)\\
&=\frac{4\varphi^{n+m}q\,[k]_q\,(\underline{\beta}\ \underline{\varphi}q^n-\underline{\varphi}\ \underline{\beta}q^{m-k})}{1-q}+(\mathcal{L}e_{n+k}(\varphi;q)-\mathcal{L}e_n(\varphi;q))\mathcal{A}\\
&+\mathcal{A}(\mathcal{L}e_{m-k}(\varphi;q)-\mathcal{L}e_m(\varphi;q)).
\end{aligned}$$

where $\underline{\varphi}=1+\varphi i_1+\varphi^2 i_2+\varphi^3 i_3$, $\underline{\beta}=1+(\varphi q)i_1+(\varphi q)^2 i_2+(\varphi q)^3 i_3$ *and* $\mathcal{A}=1+i_1+i_2+i_3$.

Proof. Using (3.1), we get

$$\begin{aligned}
&\mathcal{L}e_{n+k}(\varphi;q)\mathcal{L}e_{m-k}(\varphi;q)-\mathcal{L}e_{n}(\varphi;q)\mathcal{L}e_{m}(\varphi;q)\\
&=\left(\frac{2\varphi^{n+k+1}\underline{\varphi}-2(\varphi q)^{n+k+1}\underline{\beta}}{\varphi-\varphi q}-\mathcal{A}\right)\left(\frac{2\varphi^{m-k+1}\underline{\varphi}-2(\varphi q)^{m-k+1}\underline{\beta}}{\varphi-\varphi q}-\mathcal{A}\right)\\
&-\left(\frac{2\varphi^{n+1}\underline{\varphi}-2(\varphi q)^{n+1}\underline{\beta}}{\varphi-\varphi q}-\mathcal{A}\right)\left(\frac{2\varphi^{m+1}\underline{\varphi}-2(\varphi q)^{m+1}\underline{\beta}}{\varphi-\varphi q}-\mathcal{A}\right)\\
&=-\frac{4\varphi^{n+m+2}\underline{\varphi}\,\underline{\beta}}{(\varphi-\varphi q)^{2}}\left(q^{m-k+1}-q^{m+1}\right)-\frac{4\varphi^{n+m+2}\underline{\beta}\,\underline{\varphi}}{(\varphi-\varphi q)^{2}}\left(q^{n+k+1}-q^{n+1}\right)\\
&-\left(\frac{2\varphi^{n+k+1}\underline{\varphi}-2(\varphi q)^{n+k+1}\underline{\beta}}{\varphi-\varphi q}-\frac{2\varphi^{n+1}\underline{\varphi}-2(\varphi q)^{n+1}\underline{\beta}}{\varphi-\varphi q}\right)\mathcal{A}\\
&-\mathcal{A}\left(\frac{2\varphi^{m-k+1}\underline{\varphi}-2(\varphi q)^{m-k+1}\underline{\beta}}{\varphi-\varphi q}-\frac{2\varphi^{m+1}\underline{\varphi}-2(\varphi q)^{m+1}\underline{\beta}}{\varphi-\varphi q}\right)\\
&=\frac{4\varphi^{n+m}q(1-q^{k})(\underline{\beta}\,\underline{\varphi}q^{n}-\underline{\varphi}\,\underline{\beta}q^{m-k})}{(1-q)^{2}}-(\mathcal{L}e_{n+k}(\varphi;q)-\mathcal{L}e_{n}(\varphi;q))\mathcal{A}\\
&-\mathcal{A}(\mathcal{L}e_{m-k}(\varphi;q)-\mathcal{L}e_{m}(\varphi;q)),
\end{aligned}$$

by using (2.1), we obtain

$$\begin{aligned}
&\mathcal{L}e_{n+k}(\varphi;q)\mathcal{L}e_{m-k}(\varphi;q)-\mathcal{L}e_{n}(\varphi;q)\mathcal{L}e_{m}(\varphi;q)\\
&=\frac{4\varphi^{n+m}q\,[k]_q\,(\underline{\beta}\,\underline{\varphi}q^{n}-\underline{\varphi}\,\underline{\beta}q^{m-k})}{1-q}-(\mathcal{L}e_{n+k}(\varphi;q)-\mathcal{L}e_{n}(\varphi;q))\mathcal{A}\\
&-\mathcal{A}(\mathcal{L}e_{m-k}(\varphi;q)-\mathcal{L}e_{m}(\varphi;q)),
\end{aligned}$$

where $\underline{\varphi}=1+\varphi i_1+\varphi^2 i_2+\varphi^3 i_3$, $\underline{\beta}=1+(\varphi q)i_1+(\varphi q)^2 i_2+(\varphi q)^3 i_3$ and $\mathcal{A}=1+i_1+i_2+i_3$. The proof is completed. □

Corollary 3.7. *For $m=n$ in Theorem* (3.6)*, the Catalan's identity is as follows;*

$$\begin{aligned}
&\mathcal{L}e_{n+k}(\varphi;q)\mathcal{L}e_{n-k}(\varphi;q)-\mathcal{L}e_{n}^{2}(\varphi;q)\\
&=\frac{4\varphi^{2n}q^{n-k+1}\,[k]_q}{1-q}\left(q^{k}\underline{\beta}\,\underline{\varphi}-\underline{\varphi}\,\underline{\beta}\right)-(\mathcal{L}e_{n+k}(\varphi;q)-\mathcal{L}e_{n}(\varphi;q))\mathcal{A}\\
&-\mathcal{A}(\mathcal{L}e_{n-k}(\varphi;q)\mathcal{L}e_{n}(\varphi;q)).
\end{aligned}$$

Corollary 3.8. *For $m=n$ and $k=1$ in Corollary* (3.7)*, we get the Cassini's identity*

$$\begin{aligned}
&\mathcal{L}e_{n+1}(\varphi;q)\mathcal{L}e_{n-1}(\varphi;q)-\mathcal{L}e_{n}^{2}(\varphi;q)\\
&=\frac{4\varphi^{2n}q^{n}}{1-q}\left(q\underline{\beta}\,\underline{\varphi}-\underline{\varphi}\,\underline{\beta}\right)-(\mathcal{L}e_{n+1}(\varphi;q)-\mathcal{L}e_{n}(\varphi;q))\mathcal{A}\\
&-\mathcal{A}(\mathcal{L}e_{n-1}(\varphi;q)-\mathcal{L}e_{n}(\varphi;q)).
\end{aligned}$$

4 Applications of the q-Leonardo split quaternions

In this section we give some applications of Cayley-Dickson's form, including split quaternions with components from some special integer sequences.

4.1 Cayley-Dickson's form for split quaternions

The Cayley-Dickson's form of a split quaternion γ is $\gamma = (\gamma_0 + \gamma_1 i_1) + (\gamma_2 + \gamma_3 i_1) i_2$ which is based on two complex numbers. The classical polar representation of the given split quaternion γ can be written as follows.

i. The polar representation for spacelike split quaternion γ can be written in the form

$$\gamma = |\gamma| (\sinh\phi + \mu \cosh\phi) \tag{4.1}$$

where $\sinh\phi = \frac{\gamma_0}{|\gamma|}$, $\cosh\phi = \frac{\sqrt{-\gamma_1^2+\gamma_2^2+\gamma_3^2}}{|\gamma|}$ and $\mu = \frac{\gamma_1 i_1 + \gamma_2 i_2 + \gamma_3 i_3}{\sqrt{-\gamma_1^2+\gamma_2^2+\gamma_3^2}}$ is a spacelike unit vector.

ii. The polar representation for timelike split quaternion γ with spacelike vector part ($I_{V_\gamma} < 0$ for $V_\gamma = \gamma_1 i_1 + \gamma_2 i_2 + \gamma_3 i_3$ vector part of γ) can be written in the form

$$\gamma = |\gamma| (\cosh\phi + \mu \sinh\phi)$$

where $\sinh\phi = \frac{\sqrt{-\gamma_1^2+\gamma_2^2+\gamma_3^2}}{|\gamma|}$, $\cosh\phi = \frac{\gamma_0}{|\gamma|}$ and $\mu = \frac{\gamma_1 i_1 + \gamma_2 i_2 + \gamma_3 i_3}{\sqrt{-\gamma_1^2+\gamma_2^2+\gamma_3^2}}$ is a spacelike unit vector Kula and Yaylı (2007); Özdemir and Ergin (2006).

Split quaternions can be rewritten in the form $\gamma = z e^{w i_2}$, where z and w are complex numbers. If $P = z i_2 + w i_3 = (z + w i_1) i_2$ be an arbitrary spacelike vector, if e^P is spacelike then its exponential form is

$$e^P = \sinh|P| + \frac{z}{|P|}\cosh|P| i_2 + \frac{w}{|P|}\cosh|P| i_3 = \alpha_0 + \alpha_2 i_2 + \alpha_3 i_3 \tag{4.2}$$

and if P is timelike, then

$$e^P = \cosh|P| + \frac{z}{|P|}\sinh|P| i_2 + \frac{w}{|P|}\sinh|P| i_3 = \beta_0 + \beta_2 i_2 + \beta_3 i_3.$$

That is, it is a split quaternion which does not contain i_1.

Quantum calculus approach of Leonardo split quaternion possesses a polar form. This polar form provides an alternative representation of q-Leonardo split quaternion, offering insights into their geometric and algebraic properties beyond the traditional Cartesian representation. Indeed, the concept of constructing a polar form for these split quaternions using two complex numbers, and then applying a specific operation (multiplying the second complex number by i_2) bears resemblance to the Cayley-Dickson construction. However, there are notable distinctions between this approach and the traditional Cayley-Dickson method.

Proposition 4.1. *The norm of q-Leonardo split quaternion is given*

$$\|\mathcal{L}e_n(\varphi;q)\| = 2\sqrt{|\varphi^{2n+2}[2n+3]_q - \varphi^{2n+6}[2n+7]_q + \varphi^{n+3}[n+4]_q|}$$

where $Le_n(\varphi;q) = 2\varphi^n[n+1]_q - 1$ is n-th q Leonardo number.

Proof. Using (2), (2.4), (2.5), (2.6) and Proposition 1 in Atasoy (2025), we obtain

$$\begin{aligned}
&\|\mathcal{L}e_n(\varphi;q)\| = \sqrt{|\mathcal{I}_{\mathcal{L}e_n}(\varphi;q)|} \\
&= \sqrt{|2\left(Le_{2n+2}(\varphi;q) - Le_{2n+6}(\varphi;q) + Le_{n+3}(\varphi;q) + 1\right)|} \\
&= \sqrt{|(4\varphi^{2n+2}[2n+3]_q - 2) - (4\varphi^{2n+6}[2n+7]_q - 2) + (4\varphi^{n+3}[n+4]_q - 2) + 2|} \\
&= \sqrt{|(4\varphi^{2n+2}[2n+3]_q) - (4\varphi^{2n+6}[2n+7]_q) + (4\varphi^{n+3}[n+4]_q)|} \\
&= 2\sqrt{|\varphi^{2n+2}[2n+3]_q - \varphi^{2n+6}[2n+7]_q + \varphi^{n+3}[n+4]_q|}.
\end{aligned}$$

□

Proposition 4.2. *The vector part of q-Leonardo split quaternion $\mathcal{L}e_n(\varphi;q)$ is*

$$V_{\mathcal{L}e_n(\varphi;q)} = 2\varphi^{n+1}([n+2]_q i_1 + \varphi[n+3]_q i_2 + \varphi^{n+3}[n+4]_q i_3) - \mathcal{A} + 1$$

and

$$\begin{aligned}\mathcal{I}_{V_{\mathcal{L}e_n(\varphi;q)}} &= 4\varphi^{2n+2}([n+2]_q^2 - \varphi^2[n+3]_q^2 - \varphi^4[n+4]_q^2) \\ &- 4\varphi^{n+1}([n+2]_q - \varphi[n+3]_q - \varphi^2[n+4]_q) - 1\end{aligned}$$

where $\mathcal{A} = 1 + i_1 + i_2 + i_3$.

Proof. Using (2.4) and (2.7), we get $V_{\mathcal{L}e_n(\varphi;q)}$ and $\mathcal{I}_{V_{\mathcal{L}e_n(\varphi;q)}}$

$$\begin{aligned}V_{\mathcal{L}e_n(\varphi;q)} &= Le_{n+1}(\varphi;q)i_1 + Le_{n+2}(\varphi;q)i_2 + Le_{n+3}(\varphi;q)i_3 \\ &= \left(2\varphi^{n+1}[n+2]_q - 1\right) i_1 + \left(2\varphi^{n+2}[n+3]_q - 1\right) i_2 \\ &+ \left(2\varphi^{n+3}[n+4]_q - 1\right) i_3 \\ &= 2\varphi^{n+1}[n+2]_q i_1 + 2\varphi^{n+2}[n+3]_q i_2 + 2\varphi^{n+3}[n+4]_q i_3 \\ &- (i_1 + i_2 + i_3) \\ &= 2\varphi^{n+1}([n+2]_q i_1 + \varphi[n+3]_q i_2 + \varphi^{n+3}[n+4]_q i_3) - \mathcal{A} + 1\end{aligned}$$

and

$$\begin{aligned}\mathcal{I}_{V_{\mathcal{L}e_n(\varphi;q)}} &= Le_{n+1}^2(\varphi;q) - Le_{n+2}^2(\varphi;q) - Le_{n+3}^2(\varphi;q) \\ &= \left(2\varphi^{n+1}[n+2]_q - 1\right)^2 - \left(2\varphi^{n+2}[n+3]_q - 1\right)^2 \\ &- \left(2\varphi^{n+3}[n+4]_q - 1\right)^2 \\ &= (4\varphi^{2n+2}[n+2]_q^2 - 4\varphi^{n+1}[n+2]_q + 1) \\ &- (4\varphi^{2n+4}[n+3]_q^2 - 4\varphi^{n+2}[n+3]_q + 1) \\ &- (4\varphi^{2n+6}[n+4]_q^2 - 4\varphi^{n+3}[n+4]_q + 1) \\ &= 4\varphi^{2n+2}([n+2]_q^2 - \varphi^2[n+3]_q^2 - \varphi^4[n+4]_q^2) \\ &- 4\varphi^{n+1}([n+2]_q - \varphi[n+3]_q - \varphi^2[n+4]_q) - 1.\end{aligned}$$

□

Proposition 4.3. *Every q-Leonardo split quaternion $\mathcal{L}e_n(\varphi;q)$ is spacelike.*

Proof. For any q-Leonardo split quaternion

$$\begin{aligned}\mathcal{L}e_n(\varphi;q) &= Le_n(\varphi;q) + Le_{n+1}(\varphi;q)i_1 + Le_{n+2}(\varphi;q)i_2 + Le_{n+3}(\varphi;q)i_3 \\ &= 2\varphi^n[n+1]_q + 2\varphi^{n+1}[n+2]_q i_1 + 2\varphi^{n+2}[n+3]_q i_2 \\ &+ 2\varphi^{n+3}[n+4]_q i_3 - \mathcal{A},\end{aligned}$$

we can write that $\mathcal{I}_{\mathcal{L}e_n(\varphi;q)} < 0$, where

$$\begin{aligned}\mathcal{I}_{\mathcal{L}e_n(\varphi;q)} &= Le_n^2(\varphi;q) + Le_{n+1}^2(\varphi;q) - Le_{n+2}^2(\varphi;q) - Le_{n+3}^2(\varphi;q) \\ &= 4\varphi^{2n+2}([n+1]_q^2 - \varphi^2[n+3]_q^2 - \varphi^4[n+4]_q^2) \\ &- 4\varphi^{n+1}([n+1]_q - \varphi[n+3]_q - \varphi^2[n+4]_q).\end{aligned}$$

So, $\mathcal{L}e_n$ is spacelike. □

Theorem 4.4. *The classical polar representation of q-Leonardo split quaternion is*

$$\mathcal{L}e_n(\varphi;q) = |\mathcal{L}e_n(\varphi;q)| (\sinh\phi_n + \mu_n \cosh\phi_n)$$

where $\mu_n = \frac{V_{\mathcal{L}e_n(\varphi;q)}}{\sqrt{\left|\mathcal{I}_{V_{\mathcal{L}e_n(\varphi;q)}}\right|}}$ *is a spacelike unit vector.* $Le_n(\varphi;q)$ *is* n*-th Leonardo number,* ϕ_n *is a angle such that* $\phi_n = \tanh^{-1}\left(\frac{2\varphi^n[n+1]_q - 1}{\sqrt{\left|\mathcal{I}_{V_{\mathcal{L}e_n(\varphi;q)}}\right|}}\right)$ *and*
$\mathcal{A} = 1 + i_1 + i_2 + i_3$.

Proof. Using (4.1), for $\mathcal{L}e_n(\varphi;q)$ is a spacelike q-Leonardo split quaternion, polar representation is written as

$$\mathcal{L}e_n(\varphi;q) = |\mathcal{L}e_n(\varphi;q)| (\sinh\phi_n + \mu_n \cosh\phi_n).$$

We obtain $\sinh\phi_n = \frac{Le_n(\varphi;q)}{\|\mathcal{L}e_n(\varphi;q)\|}$, $\cosh\phi_n = \frac{\sqrt{\left|\mathcal{I}_{V_{\mathcal{L}e_n(\varphi;q)}}\right|}}{\|\mathcal{L}e_n(\varphi;q)\|}$, $\mu_n = \frac{V_{\mathcal{L}e_n(\varphi;q)}}{\sqrt{\left|\mathcal{I}_{V_{\mathcal{L}e_n(\varphi;q)}}\right|}}$. Here, by using (2.4), we obtain

$$\begin{aligned}\left|\mathcal{I}_{V_{\mathcal{L}e_n(\varphi;q)}}\right| &= Le^2_{n+3}(\varphi;q) + Le^2_{n+2}(\varphi;q) - Le^2_{n+1}(\varphi;q)\\ &= -4\varphi^{2n+2}([n+2]_q^2 - \varphi^2[n+3]_q^2 - \varphi^4[n+4]_q^2)\\ &+ 4\varphi^{n+1}([n+2]_q - \varphi[n+3]_q - \varphi^2[n+4]_q) + 1,\end{aligned}$$

$$\begin{aligned}V_{\mathcal{L}e_n(\varphi;q)} &= Le_{n+1}(\varphi;q)i_1 + Le_{n+2}(\varphi;q)i_2 + Le_{n+3}(\varphi;q)i_3\\ &= 2\varphi^{n+1}([n+2]_q i_1 + \varphi[n+3]_q i_2 + \varphi^3[n+4]_q i_3) - \mathcal{A} + 1.\end{aligned}$$

$\tanh\phi_n = \frac{2\varphi^n[n+1]_q - 1}{\sqrt{\left|\mathcal{I}_{V_{\mathcal{L}e_n(\varphi;q)}}\right|}}$. Thus, we get $\phi_n = \tanh^{-1}\left(\frac{2\varphi^n[n+1]_q - 1}{\sqrt{\left|\mathcal{I}_{V_{\mathcal{L}e_n(\varphi;q)}}\right|}}\right)$. □

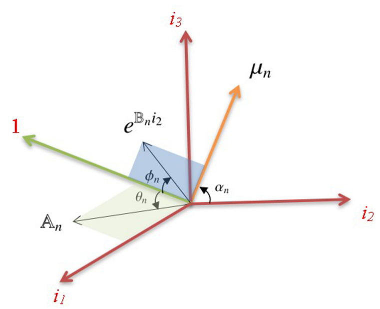
Proposition 4.5. *For every spacelike q-Leonardo split quaternion vector* $P_n(\varphi;q) = Le_n(\varphi;q)i_2 + Le_{n+1}(\varphi;q)i_3 = (Le_n(\varphi;q) + Le_{n+1}(\varphi;q)i_1)i_2$, $e^{P_n(\varphi;q)}$ *is written as follows;*

$$\begin{aligned}e^{P_n(\varphi;q)} &= \sinh|P_n(\varphi;q)| + \frac{Le_n(\varphi;q)}{|P_n(\varphi;q)|}\cosh|P_n(\varphi;q)|i_2\\ &+ \frac{Le_{n+1}(\varphi;q)}{|P_n(\varphi;q)|}\cosh|P_n(\varphi;q)|i_3\\ &= b_n + b_{n+2}i_2 + b_{n+3}i_3,\end{aligned}$$

where b_n is constant.

Proof. It is seen From (4.2). □

Figure 1: The representations of $\mathbb{A}_n$, μ_n and $e^{\mathbb{B}_n i_2}$



Theorem 4.6. *Every q-Leonardo split quaternion $\mathcal{L}e_n(\varphi;q)$ can be expressed in the form $\mathcal{L}e_n(\varphi;q) = \mathbb{A}_n e^{\mathbb{B}_n i_2}$, where $\mathbb{A}_n$ and $\mathbb{B}_n$ are complex numbers with components including q-Leonardo numbers, that is*

$$\mathbb{A}_n = \frac{2\varphi^n[n+1]_q - 1 + \left(2\varphi^{n+1}[n+2]_q - 1\right) i_1}{\sqrt{\mathcal{X}_n}} \left\|\mathcal{L}e_n(\varphi;q)\right\|$$

and

$$\mathbb{B}_n = \frac{\mathcal{Y}_n + \mathcal{Z}_n i_1}{\sqrt{\mathcal{Y}_n^2 + \mathcal{Z}_n^2}} \tanh^{-1}\left(\frac{\mathcal{X}_n}{\sqrt{\mathcal{Y}_n^2 + \mathcal{Z}_n^2}}\right)$$

where

$$\begin{aligned}
\mathcal{X}_n &= 4\varphi^{2n+2}[2n+3]_q - 4\varphi^{n+2}[n+3]_q + 2\\
\mathcal{Y}_n &= 4\varphi^{2n+4}[2n+5]_q - 2\varphi^{n+2}[n+3]_q - 2\varphi^{n+4}[n+5]_q + 2\\
\mathcal{Z}_n &= 4\varphi^{2n+3}[n+1]_q[n+4]_q - 2\varphi^{n}[n+1]_q - 2\varphi^{n+3}[n+4]_q\\
&\quad - 4\varphi^{2n+3}[n+2]_q[n+3]_q + 2\varphi^{n+1}[n+2]_q + 2\varphi^{n+2}[n+3]_q + 2.
\end{aligned}$$

Proof. If $\mathbb{A}_n = a_n + a_{n+1}i_1$ and $e^{\mathbb{B}_n i_2} = b_n + b_{n+2}i_2 + b_{n+3}i_3$, we can write that

$$\begin{aligned}
\mathcal{L}e_n(\varphi;q) = \mathbb{A}_n e^{\mathbb{B}_n i_2} &= a_n b_n + a_{n+1}b_n i_1 + (a_n b_{n+2} - a_{n+1}b_{n+3})i_2\\
&\quad + (a_n b_{n+3} + a_{n+1}b_{n+2})i_3.
\end{aligned}$$

Taking $b_n \neq 0$, we obtain a complex number

$$\omega_n = a_n b_n + a_{n+1}b_n i_1 = Le_n(\varphi;q) + Le_{n+1}(\varphi;q)i_1,$$

where $\mathbb{A}_n$ and $\mathbb{B}_n$ are complex numbers with components including $q-$Leonardo numbers. We can write that $\mathbb{A}_n = \frac{\omega_n}{|\omega_n|}\left\|\mathcal{L}e_n(\varphi;q)\right\|$. So, $\left\|\mathbb{A}_n\right\| = \left\|\mathcal{L}e_n(\varphi;q)\right\|$. By using (2.4), we get

$$\begin{aligned}
\mathbb{A}_n &= \frac{Le_n(\varphi;q) + Le_{n+1}(\varphi;q)i_1}{\sqrt{Le_n^2(\varphi;q) + Le_{n+1}^2(\varphi;q)}} \left\|\mathcal{L}e_n(\varphi;q)\right\|\\
&= \frac{Le_n(\varphi;q) + Le_{n+1}(\varphi;q)i_1}{\sqrt{2(Le_{2n+2}(\varphi;q) - Le_{n+2}(\varphi;q) + 1)}} \left\|\mathcal{L}e_n(\varphi;q)\right\|\\
&= \frac{2\varphi^n[n+1]_q - 1 + \left(2\varphi^{n+1}[n+2]_q - 1\right) i_1}{\sqrt{2(2\varphi^{2n+2}[2n+3]_q - 1 - (2\varphi^{n+2}[n+3]_q - 1) + 1)}} \left\|\mathcal{L}e_n(\varphi;q)\right\|\\
&= \frac{2\varphi^n[n+1]_q - 1 + \left(2\varphi^{n+1}[n+2]_q - 1\right) i_1}{\sqrt{(4\varphi^{2n+2}[2n+3]_q - 2 - (4\varphi^{n+2}[n+3]_q - 2) + 2)}} \left\|\mathcal{L}e_n(\varphi;q)\right\|\\
&= \frac{2\varphi^n[n+1]_q - 1 + \left(2\varphi^{n+1}[n+2]_q - 1\right) i_1}{\sqrt{4\varphi^{2n+2}[2n+3]_q - 4\varphi^{n+2}[n+3]_q + 2}} \left\|\mathcal{L}e_n(\varphi;q)\right\|\\
&= \frac{2\varphi^n[n+1]_q - 1 + \left(2\varphi^{n+1}[n+2]_q - 1\right) i_1}{\sqrt{\mathcal{X}_n}} \left\|\mathcal{L}e_n(\varphi;q)\right\|
\end{aligned}$$

where $\mathbb{A}_n \in Sp\{1, i_1\}$ and θ_n is argument of $\mathbb{A}_n$. Therefore,

$$\begin{aligned}
\mathbb{A}_n^{-1} &= \frac{\mathbb{A}_n^*}{\|\mathbb{A}_n\|^2} \\
&= \frac{Le_n(\varphi;q) - Le_{n+1}(\varphi;q)i_1}{\|\mathcal{L}e_n(\varphi;q)\| \sqrt{Le_n^2(\varphi;q) + Le_{n+1}^2(\varphi;q)}} \\
&= \frac{Le_n(\varphi;q) - Le_{n+1}(\varphi;q)i_1}{\|\mathcal{L}e_n(\varphi;q)\| \sqrt{2(Le_{2n+2}(\varphi;q) - Le_{n+2}(\varphi;q) + 1)}} \\
&= \frac{2\varphi^n[n+1]_q - 1 - \left(2\varphi^{n+1}[n+1]_q - 1\right) i_1}{\|\mathcal{L}e_n(\varphi;q)\| \sqrt{\mathcal{X}_n}}
\end{aligned}$$

where $\mathbb{A}_n^{-1}$ and $\mathbb{A}_n^*$ are the inverse and the conjugate of $\mathbb{A}_n$, respectively. So, we get

$$\begin{aligned}
e^{\mathbb{B}_n i_2} &= \mathbb{A}_n^{-1} \mathcal{L}e_n(\varphi;q) \\
&= \frac{(Le_n^2(\varphi;q) + Le_{n+1}^2(\varphi;q))}{\|\mathcal{L}e_n(\varphi;q)\| \sqrt{Le_n^2(\varphi;q) + Le_{n+1}^2(\varphi;q)}} \\
&+ \frac{(Le_n(\varphi;q)Le_{n+2}(\varphi;q) + Le_{n+1}(\varphi;q)Le_{n+3}(\varphi;q))i_2}{\|\mathcal{L}e_n(\varphi;q)\| \sqrt{Le_n^2(\varphi;q) + Le_{n+1}^2(\varphi;q)}} \\
&+ \frac{(Le_n(\varphi;q)Le_{n+3}(\varphi;q) - Le_{n+1}(\varphi;q)Le_{n+2}(\varphi;q))i_3}{\|\mathcal{L}e_n(\varphi;q)\| \sqrt{Le_n^2(\varphi;q) + Le_{n+1}^2(\varphi;q)}}
\end{aligned}$$

and then

$$\begin{aligned}
e^{\mathbb{B}_n i_2} &= \frac{2(Le_{2n+2}(\varphi;q) - Le_{n+2}(\varphi;q) + 1)}{\|\mathcal{L}e_n(\varphi;q)\| \sqrt{2(Le_{2n+2}(\varphi;q) - Le_{n+2}(\varphi;q) + 1)}} \\
&+ \frac{(2Le_{2n+4}(\varphi;q) - Le_{n+2}(\varphi;q) - Le_{n+4}(\varphi;q) + 2)i_2}{\|\mathcal{L}e_n(\varphi;q)\| \sqrt{2(Le_{2n+2}(\varphi;q) - Le_{n+2}(\varphi;q) + 1)}} \\
&+ \frac{(Le_n(\varphi;q)Le_{n+3}(\varphi;q) - Le_{n+1}(\varphi;q)Le_{n+2}(\varphi;q))i_3}{\|\mathcal{L}e_n(\varphi;q)\| \sqrt{2(Le_{2n+2}(\varphi;q) - Le_{n+2}(\varphi;q) + 1)}} \\
&= \frac{\mathcal{X}_n + \mathcal{Y}_n i_2 + \mathcal{Z}_n i_3}{\|\mathcal{L}e_n(\varphi;q)\| \sqrt{4\varphi^{2n+2}[2n+3]_q - 4\varphi^{n+2}[n+3]_q + 2}}
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{X}_n &= 4\varphi^{2n+2}[2n+3]_q - 4\varphi^{n+2}[n+3]_q + 2 \\
\mathcal{Y}_n &= 2\left(2\varphi^{2n+4}[2n+5]_q - 1\right) - \left(2\varphi^{n+2}[n+3]_q - 1\right) - \left(2\varphi^{n+4}[n+5]_q - 1\right) + 2 \\
&= 4\varphi^{2n+4}[2n+5]_q - 2\varphi^{n+2}[n+3]_q - 2\varphi^{n+4}[n+5]_q + 2 \\
\mathcal{Z}_n &= (2\varphi^n[n+1]_q - 1)\left(2\varphi^{n+3}[n+4]_q - 1\right) \\
&- \left(2\varphi^{n+1}[n+2]_q - 1\right)(2\varphi^{n+2}[n+3]_q - 1) \\
&= 4\varphi^{2n+3}[n+1]_q[n+4]_q - 2\varphi^n[n+1]_q \\
&- 2\varphi^{n+3}[n+4]_q - 4\varphi^{2n+3}[n+2]_q[n+3]_q \\
&+ 2\varphi^{n+1}[n+2]_q + 2\varphi^{n+2}[n+3]_q + 2.
\end{aligned}$$

In this case, we have $\left|e^{\mathbb{B}_n i_2}\right| = 1$, and $\mathcal{L}e_n(\varphi;q) = \mathbb{A}_n e^{\mathbb{B}_n i_2}$ and $\mathbb{B}_n$ is unimportant. Since $e^{\mathbb{B}_n i_2}$ is a unit spacelike q-Leonardo split quaternion, its classical polar form is written

$$e^{\mathbb{B}_n i_2} = \sinh \phi_n + \mu_n \cosh \phi_n,$$

where, ϕ_n is a Lorentzian angle. Therefore, it can be shown

$$\sinh\phi_n = \frac{\sqrt{\mathcal{X}_n}}{\|\mathcal{L}e_n\|}, \quad \cosh\phi_n = \frac{\sqrt{\mathcal{Y}_n^2+\mathcal{Z}_n^2}}{\sqrt{\mathcal{X}_n}\,\|\mathcal{L}e_n\|}, \quad \mu_n = \frac{\mathcal{Y}_n i_2 + \mathcal{Z}_n i_3}{\sqrt{\mathcal{Y}_n^2+\mathcal{Z}_n^2}}$$

where ϕ_n is argument of $\mu_n \in Sp\{i_2, i_3\}$ (see Figure 1). Then, $\tanh\phi_n = \frac{\mathcal{X}_n}{\sqrt{\mathcal{Y}_n^2+\mathcal{Z}_n^2}}$ and

$$\phi_n = \tanh^{-1}\left(\frac{\mathcal{X}_n}{\sqrt{\mathcal{Y}_n^2+\mathcal{Z}_n^2}}\right)$$

and we obtain

$$\mathbb{B}_n = \frac{\mathcal{Y}_n + \mathcal{Z}_n i_1}{\sqrt{\mathcal{Y}_n^2+\mathcal{Z}_n^2}} \tanh^{-1}\left(\frac{\mathcal{X}_n}{\sqrt{\mathcal{Y}_n^2+\mathcal{Z}_n^2}}\right).$$

Because, it is $\mathbb{B}_n i_2 = \mu_n \phi_n$. □

Example 4.1. *For $n = 1$ in Theorem (4.6), and by making the necessary calculations, we obtain*

$$\begin{aligned}
\mathbb{A}_1 &= \frac{2\varphi^1[2]_q - 1 + \left(2\varphi^2[3]_q - 1\right) i_1}{\sqrt{\mathcal{X}_1}} \|\mathcal{L}e_1(\varphi;q)\| \\
&= 2\frac{(2-1)+(4-1)\,i_1}{\sqrt{10}} \sqrt{\varphi^4[5]_q - \varphi^8[9]_q + \varphi^4[5]_q} \\
&= \frac{2+6i_1}{\sqrt{10}} \sqrt{|5-34+5|} = \frac{2+6i_1}{\sqrt{10}}\sqrt{24} = \frac{2+6i_1}{\sqrt{5}}\sqrt{12} = \sqrt{\frac{48}{5}} + \sqrt{\frac{432}{5}} i_1
\end{aligned}$$

and

$$\begin{aligned}
\mathbb{B}_1 &= \frac{\mathcal{Y}_1 + \mathcal{Z}_1 i_1}{\sqrt{\mathcal{Y}_1^2+\mathcal{Z}_1^2}} \tanh^{-1} \frac{\mathcal{X}_1}{\sqrt{\mathcal{Y}_1^2+\mathcal{Z}_1^2}} \\
&= \frac{32-6i_1}{\sqrt{1040}} \tanh^{-1}\frac{10}{\sqrt{1060}} = \frac{16-3i_1}{\sqrt{265}} \tanh^{-1}\sqrt{\frac{5}{53}} \\
&= \frac{16}{\sqrt{265}} \tanh^{-1}\sqrt{\frac{5}{53}} - i_1 \frac{3}{\sqrt{265}} \tanh^{-1}\sqrt{\frac{5}{53}}
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{X}_1 &= 4\varphi^4[5]_q - 4\varphi^3[4]_q + 2 = 20 - 12 + 2 = 10 \\
\mathcal{Y}_1 &= 4\varphi^6[7]_q - 2\varphi^3[4]_q - 2\varphi^5[6]_q + 2 = 52 - 6 - 16 + 2 = 32 \\
\mathcal{Z}_1 &= 4\varphi^5[2]_q[5]_q - 2\varphi^1[2]_q - 2\varphi^4[5]_q - 4\varphi^5[3]_q[4]_q + 2\varphi^2[3]_q + 2\varphi^3[4]_q \\
&= 20 - 2 - 10 - 24 + 4 + 6 = -6.
\end{aligned}$$

5 Conclusions

In this study, q-Leonardo split quaternion sequences have been introduced by integrating the robust framework of quantum calculus with the algebraic structure of split quaternions. We have established several fundamental identities and provided numerical examples that validate the theoretical consistency of these sequences. By employing quantum calculus notation, we have achieved enhanced derivational clarity. This research offers a new perspective on the structural evolution of q-Leonardo split quaternions, providing a deeper understanding of their geometric interpretations and algebraic transformations.

Furthermore, the results obtained here suggest several promising directions for further studies. Future research could explore the extension of these sequences into octonion algebras or investigate their potential implications in digital signal processing and cryptographic key generation.

Disclaimer (Artificial Intelligence)

Author(s) hereby declare that NO generative AI technologies such as Large Language Models (ChatGPT, COPILOT, etc) and text-to-image generators have been used during writing or editing of this manuscript.

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