

Definition[section]

## On the Enclave Domination Number in Some Special Graph Families and Its Applications

### Research Article

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### Abstract

Domination is a classical concept in graph theory that has given rise to numerous refinements and generalizations. This paper investigates a domination variant referred to as the enclave dominating set. The exact values of the enclave domination number are determined for several families of special graphs, such as Jellyfish, Jewel, Dutch windmill, lollipop, comb, sunlet, crown, and friendship graphs, as well as for the Soifer, Franklin, Moser spindle, Chvátal, Fritsch, Herschel, and Goldner–Harary graphs. All enclave dominating vertices and their corresponding minimum enclave dominating sets are characterized. These results advance the theoretical framework of enclave domination parameter, provide new insights into graph structures, and highlight applications of enclave dominating sets.

*Keywords:* Domination Number; Enclave Dominating Vertex; Enclave Dominating Set; Enclave Domination Number

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## 1 Introduction

Domination theory constitutes a fundamental branch of graph theory with significant theoretical and practical implications, which were extensively studied in (Haynes et al., 2013), (Haynes et al., 2023), (Wallis, 1979). The origins of domination theory can be traced to 1862, when C.F.De Jaenisch examined the classical problem of determining the minimum number of queens required to dominate, or effectively control, all squares of a chessboard. The systematic development of domination as a mathematical concept within graph theory emerged much later, around the 1960s, through the pioneering contributions of Berge and Ore, who laid the foundation for its formal study (Ore, 1962). All graph terminology and notation follow (Chartrand et al., 2024).

For a finite, simple, undirected graph  $G = (V, E)$ , a dominating set  $D \subseteq V(G)$  is defined such that every vertex not in  $D$  is adjacent to at least one vertex of  $D$ . The minimum cardinality of such a dominating set is called the domination number and is denoted by  $\gamma(G)$  in (Haynes et al., 2013). This parameter plays a pivotal role in applications such as facility location, network surveillance, and resource optimization, as well as serving as an essential tool in the structural analysis of graphs. Given a vertex  $u \in V(G)$ , the open neighborhood  $N(u)$  denotes the set of vertices adjacent to  $u$ , while the closed neighborhood  $N[u] = N(u) \cup \{u\}$  includes  $u$  itself. The degree  $d(u) = |N(u)|$  quantifies the number of vertices adjacent to  $u$ . The minimum and maximum degrees among the vertices of  $G$  are denoted by  $\delta(G)$  and  $\Delta(G)$ , respectively.

Extending this framework, from (Thakkar and Savaliya, 2019), a vertex  $u \in S \subseteq V(G)$  is called an enclave vertex, if its closed neighborhood  $N[u]$  is entirely contained within  $S$ , implies that  $u$  has no neighbors in the complement  $V(G) - S$ . A vertex subset containing no enclaves is termed enclaveless. In (Santhosh Priya and Mydeen Bibi, 2025), the enclave domination number is well defined. Prior research has examined enclave domination numbers for standard graphs, specific graph constructions, including semi-total line graphs  $T_1(G)$  and semi-total point graphs  $T_2(G)$  in (Priya and Bibi, 2025). Various domination parameters have been explored extensively across different families of graphs, with numerous significant examples available in the literature (Rani and Mehra, 2024), (Bibi, 2023), (Wanguway et al., 2020), (Devi et al., 2023), (Bibi et al., 2018), (Jumalon and Cabahug, 2025). This study identifies all enclave dominating vertices with their corresponding minimum enclave dominating sets and computes the enclave domination number for specific special graphs and finite graph families.

## 2 Terminology and Notation

### 2.1 Basic Terminology

A graph is an ordered pair  $G = (V, E)$ , where  $V$  is a finite, nonempty set whose elements are called vertices (or nodes), and  $E \subseteq \{(u, v) : u, v \in V, u \neq v\}$  is a set of unordered pairs of distinct vertices, called edges. The order of a graph  $G$ , denoted by  $|V(G)|$ , is the number of vertices in the graph. The size of a graph  $G$ , denoted by  $|E(G)|$ , is the number of edges in the graph.

From (Sugumaran and Jayachandran, 2018), the largest integer less than or equal to  $x$  is the floor function of a real number  $x$ , and it is represented by the symbol  $\lfloor x \rfloor$ . If  $n$  is an integer and  $n \leq x < n + 1$ , then  $\lfloor x \rfloor = n$ . The ceiling function of a real number  $x$  is the lowest integer greater than or equal to  $x$  and it is denoted by  $\lceil x \rceil$ . Suppose that  $n - 1 \leq x < n$ , where  $n$  is an integer, then  $\lceil x \rceil = n$ .

Suppose the graph  $G$  has  $n$  vertices, namely  $u_1, u_2, \dots, u_n$ . Let  $E_{u_i}$  be a minimum enclave dominating set of  $G$  for some  $i$ , then the vertex  $u_i$  is called an enclave dominating vertex. A graph  $G$  can have one or more enclave dominating vertices with distinct enclave dominating sets.  $\gamma_\epsilon(G)$  denotes the enclave domination number, which is the minimum cardinality of an enclave dominating set in  $G$  (Santhosh Priya and Mydeen Bibi, 2025).

### 2.2 Special Graph Families

The following graph families are used throughout the paper. For each family, a brief description and standard notation are given.

For an integer  $n \geq 2$ , the path  $P_n$  is a graph of order  $n$  and size  $n - 1$  whose vertices can be labeled by  $u_1, u_2, \dots, u_n$  whose edges are  $u_i u_{i+1}$  for  $i = 1, 2, \dots, n - 1$  (Wallis, 1979).

For  $n \geq 3$ , the cycle  $C_n$  is a graph of order  $n$  and size  $n$  whose vertices can be labeled by  $u_1, u_2, \dots, u_n$  and whose edges are  $u_1 u_2, u_2 u_3, \dots, u_{n-1} u_n, u_n u_1$  (Wallis, 1979).

For  $n \geq 2$ , the complete graph  $K_n$  is a graph of order  $n$  and size  $\frac{n(n-1)}{2}$  whose vertices can be labeled by  $u_1, u_2, \dots, u_n$  and whose edges are represented as  $u_i u_j$  for all pairs of vertices where  $1 \leq i < j \leq n$  (Wallis, 1979).

The Jellyfish graph  $J_{m,n}$  is created from a 4-cycle with the vertices  $x, y, u$  and  $v$  by linking  $x$  and  $y$  with a prime edge and attaching  $m$  pendant edges to  $u$  and  $n$  pendant edges to  $v$ . The edge connecting the vertices  $x$  and  $y$  is referred to as the prime edge in Jellyfish (Rani and Mehra, 2024).

The Jewel graph  $J_n$  is derived from a four-cycle with the vertices  $x, y, u, v$  by linking  $x$  and  $y$  with a prime edge, and also by adding the edges from  $u$  and  $v$  that meet at common vertices  $v_i, 1 \leq i \leq n$ . The edge connecting the vertices  $x$  and  $y$  in a Jewel graph is defined as the prime edge (Rani and Mehra, 2024).

The Lollipop graph is represented by the symbol  $L_{m,n}$  and consists of a bridge between a complete graph  $K_m$  and a path graph  $P_n$  (Rani and Mehra, 2024).

The Dutch Windmill graph, denoted by  $D_m^n$ , is the graph obtained by taking  $n$  copies of the cycle  $C_m$  with a vertex in common. The friendship graph, denoted by  $F_n$ , is a special case of the Dutch windmill graph when  $m = 3$  (Chandrasekaran and Rajasri, 2019).

The Comb graph is the graph obtained from a path  $P_n$  by attaching a pendant edge at each vertex of the path, and is denoted by  $P_n^+$  (Dogan and Toprakkaya, 1966).

The Sunlet graph  $S_n$ , also called the  $n$ -Sunlet graph, is the graph with  $2n$  vertices obtained by attaching a pendant edge at each vertex of a cycle  $C_n$  (Jumalon and Cabahug, 2025).

A Crown graph (also known as a cocktail party graph)  $H_{n,n}$  is a graph obtained from the complete bipartite graph  $K_{n,n}$  by removing a perfect matching (Glen et al., 2018).

The Soifer graph is a planar graph on 9 vertices and 20 edges. It is an undirected graph and is known for tangling the Kempe chains in Kempe's algorithm, exemplifying the failure of Kempe's original proof of the four-colour theorem (Devi et al., 2023).

The Franklin graph is a 3-regular graph with 12 vertices and 18 edges. The Franklin graph is named after Philip Franklin. It is a 3-vertex connected and 3-edge connected perfect graph (Devi et al., 2023).

The Chavatal graph is an undirected graph with 12 vertices and 24 edges. It is triangle-free its girth is four. It is 4-regular; each vertex has exactly four neighbours (Devi et al., 2023).

The Moser spindle is the 7-node unit distance graph with 11 edges. It is sometimes called the Hajos graph. (Bibi, 2023)

The Herschel graph, the smallest non-hamiltonian polyhedral graph, is a bipartite undirected graph with 11 vertices and 18 edges (Bibi, 2023).

The Goldner-Harry graph is a simple undirected graph with 11 vertices and 27 edges. It is also a planar graph (Bibi, 2023).

The Fritsch graph is the planar graph on 9 vertices and 21 edges. It is isomorphic to the skeleton of a triangular prism, classifying it as a nonchordal graph (Bibi et al., 2018).

### 3 Definitions and Preliminary Results

For the main concepts included in this study, consider the following definitions and results.

**Definition 3.1.** (Santhosh Priya and Mydeen Bibi, 2025) A dominating set  $E_u \subset V(G)$  is called an enclave dominating set if there exists a unique enclave vertex  $u \in E_u$  such that its closed neighborhood is contained in the set  $E_u$ , i.e.,  $N[u] \subseteq E_u$ . The minimum cardinality among all enclave dominating sets of  $G$  is the **enclave domination number**, denoted by  $\gamma_e(G)$ . A vertex  $u$  is called an **enclave dominating vertex** if it is the enclave of a minimum enclave dominating set.

**Example 3.2.** For the graph in Fig. 1. the vertex  $u_4$  is the enclave dominating vertex, and the unique minimum enclave dominating set is  $E_{u_4} = \{u_4, u_1, u_5\}$ . Thus  $|E_{u_4}| = 3$  and  $\gamma_e(G) = 3$ .

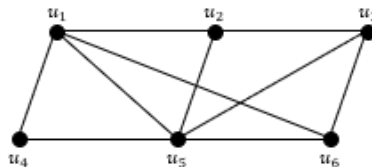


Fig. 1. Simple graph  $G$

**Theorem 3.1.** (Santhosh Priya and Mydeen Bibi, 2025) If  $G$  is a regular graph, then every vertex in  $G$  is an enclave dominating vertex. Except when  $G \cong K_n$ .

**Theorem 3.2.** (Santhosh Priya and Mydeen Bibi, 2025) In a graph  $G$ , if  $d(u) = \delta(G)$  then  $E_u$  will be the minimum enclave dominating set, and  $u$  is the enclave dominating vertex. But the converse need not be true.

**Theorem 3.3.** (Santhosh Priya and Mydeen Bibi, 2025) If  $G \cong K_n$  then  $\gamma_\epsilon(G)$  does not exist.

**Theorem 3.4.** (Frendrup et al., 2009) If  $P_n$  is a path graph with  $n$ -vertices, then  $\gamma_\epsilon(P_n) = \lceil \frac{n}{3} \rceil$ .

## 4 Enclave Domination Number of Some Graph Families

The following theorems give the characterization of enclave dominating vertex, and enclave domination number on some well-known special graph families.

**Theorem 4.1.** Let  $J_{m,n}$  be a Jellyfish graph then  $\gamma_\epsilon(J_{m,n}) = 3$ .

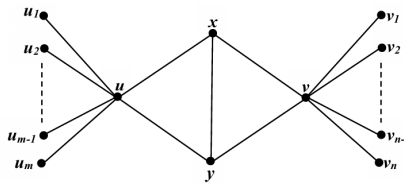


Fig. 2. Jellyfish graph  $J_{m,n}$

*Proof.* Let  $J_{m,n}$  be a Jellyfish graph with vertex set and edge set

$$V(J_{m,n}) = \{x, y, u, v, u_i, v_j : 1 \leq i \leq m, 1 \leq j \leq n\}$$

$$E(J_{m,n}) = \{ux, xv, vy, yu, xy, uu_i, vv_j : 1 \leq i \leq m, 1 \leq j \leq n\}.$$

The order of the graph  $J_{m,n}$  is  $m + n + 4$ , and the size of  $J_{m,n}$  is  $m + n + 5$ . The minimum enclave dominating set of  $J_{m,n}$  will obtained in the following cases.

**Case (i):** If  $u_i, (\forall i, 1 \leq i \leq m)$  is the enclave dominating vertex, the  $m$ -distinct enclave dominating sets generalized as  $E_{u_i} = \{u_i, u, v\}$  and  $|E_{u_i}| = 3$ .

**Case (ii):** If  $v_j, (\forall j, 1 \leq j \leq n)$  is the enclave dominating vertex, the  $n$ -distinct enclave dominating sets are generalized as  $E_{v_j} = \{v_j, u, v\}$  and  $|E_{v_j}| = 3$ .

From the above cases,  $\gamma_\epsilon(J_{m,n}) = 3$ . □

**Theorem 4.2.** Let  $J_n$  be a Jewel graph then  $\gamma_\epsilon(J_n) = 3$ .

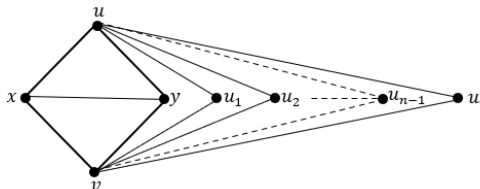


Fig. 3. Jewel graph  $J_n$

*Proof.* Let  $J_n$  be a Jewel graph, the vertex set and edge set be  
 $V(J_n) = \{x, y, u, v, u_i : 1 \leq i \leq n\}$ ,  $E(J_n) = \{ux, uy, vx, vy, xy, uu_i, vu_i : 1 \leq i \leq n\}$ .  
 The order of the graph  $J_n$  is  $n + 4$ , the size of the graph  $J_n$  is  $2n + 5$ . In this graph, each vertex  $u_i$ 's serves as an enclave dominating vertex. There exist  $n$ -distinct minimum enclave dominating sets, all having the same cardinality. For each  $i = 1, 2, \dots, n$ , the minimum enclave dominating set corresponding to  $u_i$  is  $E_{u_i} = \{u_i, u, v\}$  with  $|E_{u_i}| = 3$ . Hence  $\gamma_e(J_n) = 3$ .  $\square$

**Theorem 4.3.** For a Dutch windmill graph  $D_m^n$  with  $m \geq 5$  and  $n \geq 2$ , the enclave domination number is,  
 $\gamma_e(D_m^n) = 3 + \lceil \frac{m-5}{3} \rceil + (n-1) \lceil \frac{m-3}{3} \rceil$ .

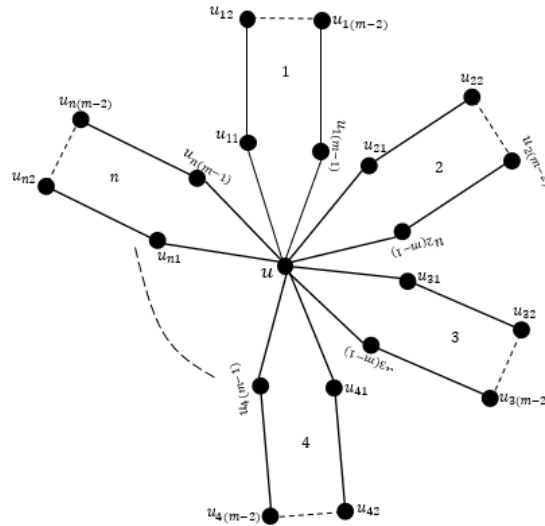


Fig. 4. Dutch windmill  $D_m^n$

*Proof.* Let  $V(D_m^n) = \{u\} \cup \{u_{ij} : 1 \leq i \leq n, 1 \leq j \leq m-1\}$ ,  $E(D_m^n) = E_1 \cup E_2$ , where  $E_1 = \{uu_{i1}, uu_{i(m-1)}\}$ , and  $E_2 = \{u_{ij}u_{i(j+1)} : 1 \leq i \leq n, 1 \leq j \leq m-2\}$ . It has  $mn$  vertices,  $m(n+1)$  edges, and maximum degree  $\Delta(D_m^n) = 2n$ . Here, the vertex  $u$  is the maximum degree vertex, that is,  $d(u) = \Delta(G)$ . For the graph  $D_m^n$ , there exist  $2n$  distinct minimum enclave dominating sets for the enclave dominating vertices  $u_{i1}, u_{i(m-1)}$ . Now choose any vertex  $u_{i1}$ , for some fixed  $i$ ,  $1 \leq i \leq n$ , adjacent to  $u$  as the enclave dominating vertex. Its closed neighborhood is

$$S = N[u_{i1}] = \{u, u_{i1}, u_{i2}\}$$

After removing  $S$  the remaining vertices are examined.

In the same  $i$ -th cycle the vertices  $u_{i3}, u_{i4}, \dots, u_{i(m-1)}$  remains. Among these vertices  $u_{i3}$  is adjacent to  $u_{i2} \in S$ , and  $u_{i(m-1)}$  is adjacent to  $u \in S$ . Thus they are already dominated. The undominated vertices in the cycle are  $u_{i4}, u_{i5}, \dots, u_{i(m-2)}$ , which form a path  $P_{m-5}$ .

For each other cycle  $j \neq i$ ,  $1 \leq j \leq n$ , the vertices  $u_{j1}, u_{j2}, \dots, u_{j(m-1)}$  remains. Here  $u_{j1}$  and  $u_{j(m-1)}$  are adjacent to  $u \in S$ , so they are dominated. The remaining vertices  $u_{j2}, u_{j3}, \dots, u_{j(m-2)}$  form a path  $P_{m-3}$  in each such cycle, and none of these vertices are adjacent to any vertex in  $S$ .

Hence the subgraph induced by the undominated vertices is,

$$P_{(m-5)} \cup \underbrace{(P_{(m-3)} \cup P_{(m-3)} \cup \dots \cup P_{(m-3)})}_{(n-1)\text{-times}}$$

By theorem 3.4, the domination number of a path  $P_k$  is  $\lceil \frac{k}{3} \rceil$ . Therefore the number of additional vertices needed to dominate the remaining graph is  $\lceil \frac{m-5}{3} \rceil + (n-1)\lceil \frac{m-3}{3} \rceil$ . Thus, the minimum enclave dominating set corresponding to the enclave dominating vertex  $u_{i1}$  has cardinality,

$$|E_{u_{i1}}| = |S| + \lceil \frac{m-5}{3} \rceil + (n-1)\lceil \frac{m-3}{3} \rceil = 3 + \lceil \frac{m-5}{3} \rceil + (n-1)\lceil \frac{m-3}{3} \rceil$$

By symmetry, choosing  $u_{i(m-1)}$  as the enclave dominating vertex yields an enclave dominating set of the same cardinality. Hence  $\gamma_\epsilon(D_m^n) = 3 + \lceil \frac{m-5}{3} \rceil + (n-1)\lceil \frac{m-3}{3} \rceil$ . □

**Theorem 4.4.** For any Lollipop graph  $L_{m,n}$  with  $m \geq 3, n \geq 1$ ,  $\gamma_\epsilon(L_{m,n}) = \begin{cases} \frac{n+6}{3}, & n \equiv 0 \pmod{3} \\ \frac{n+5}{3}, & n \equiv 1 \pmod{3} \\ \frac{n+7}{3}, & n \equiv 2 \pmod{3} \end{cases}$

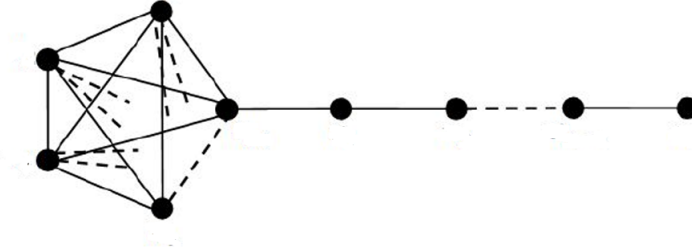


Fig. 5. Lollipop graph  $L_{m,n}$

*Proof.* Let  $L_{m,n}$  be any lollipop graph with  $(m+n)$ -vertices, such that the vertex and edge set be  $V(L_{m,n}) = \{u_1, u_2, \dots, u_m, v_1, v_2, \dots, v_n\}$ , and  $E(L_{m,n}) = E(K_m) \cup E(P_n) \cup \{u_1v_1\}$ . The enclave domination number for the graph  $L_{m,n}$  arise in the following cases.

**Case (i):** For  $n \equiv 0 \pmod{3}$

Let  $n = 3k$ , in this case the vertices  $v_n$ , and  $v_i (1 \leq i < n), i \equiv 1 \pmod{3}$  are the enclave dominating vertices and their corresponding enclave dominating sets are minimum. They are defined in the following subcases.

**Subcase (i)(a):** If  $v_n$  is the enclave dominating vertex.

Its closed neighborhood is  $N[v_n] = \{v_n, v_{n-1}\}$ . To obtain the minimum enclave domination, the vertex  $u_1$  must be included to dominate  $v_1$  and all  $u_i$ . The remaining undominated vertices on the path are  $v_2, v_3, \dots, v_{n-2}$ . However,  $v_{n-2}$  is adjacent to  $v_{n-1}$ , so it is dominated. Thus it suffices to dominate the subpath induced by  $v_2, v_3, \dots, v_{n-3}$ , which has order  $n-4 = 3k-4$ . A minimum dominating set for a path of order  $3k-4$  consists of vertices with indices congruent to 2 (mod 3). Hence a minimum enclave dominating set is

$$E_{v_n} = \{v_n, v_{n-1}, u_1\} \cup \{v_p : p \equiv 2 \pmod{3}, 2 \leq p \leq 3k-4\}$$

and  $|E_{v_n}| = 3 + (k-1) = k+2 = \frac{n+6}{3}$ .

**Subcase (i)(b):** Let the enclave dominating vertex be  $v_i$  with  $i = 3t+1$  where  $0 \leq t \leq k-1$ . Then  $N[v_i] = \{v_{i-1}, v_i, v_{i+1}\}$  where  $i-1 = 3t$ , and  $i+1 = 3t+2$ . The minimum enclave dominating set must contain the vertex  $u_1$ , which dominates  $v_1$  and all  $u_j$ . The remaining vertices on the path are dominated by including all vertices with index  $\equiv 0 \pmod{3}$  from 1 to  $i-1$  and all vertices with index  $\equiv 2 \pmod{3}$  from  $i+1$  to  $n$ . Thus a minimum enclave dominating set is

$$E_{v_i} = \{v_i, u_1\} \cup \{v_{3s} : 1 \leq s \leq t\} \cup \{v_{3s+2} : t \leq s \leq k-1\}$$

Hence  $|E_{v_i}| = 2 + t + (k - t) = k + 2 = \frac{n+6}{3}$ .

In both subcases, the minimum cardinality is  $\frac{n+6}{3}$ . Hence, for  $n \equiv 0 \pmod{3}$ ,  $\gamma_\epsilon(L_{m,n}) = \frac{n+6}{3}$ .

**Case (ii):** For  $n \equiv 1 \pmod{3}$

Let  $n = 3k + 1$ . The unique minimum enclave dominating vertex is  $v_n$ . Its closed neighborhood is  $N[v_n] = \{v_n, v_{n-1}\}$ . Include the vertex  $u_1$  to  $E_{v_n}$  to dominate  $K_m$  and  $v_1$ . The remaining path vertices are  $v_2, v_3, \dots, v_{n-2}$ . Since  $v_{n-2}$  is adjacent to  $v_{n-1}$ , it is already dominated. Thus the undominated vertices form the path  $v_2, v_3, \dots, v_{n-3}$  of order  $n - 4 = 3k - 3$ . By Theorem 3.4,  $\gamma(P_{3k-3}) = k - 1$ , achieved by vertices with indices  $\equiv 0 \pmod{3}$  from 3 to  $3k - 3$ . Hence the minimum enclave dominating set is,

$$E_{v_n} = \{v_n, v_{n-1}, u_1\} \cup \{v_p : p \equiv 0 \pmod{3}, 3 \leq p \leq 3k - 3\}$$

and

$$|E_{v_n}| = 3 + (k - 1) = k + 2 = \frac{n+5}{3}$$

Thus for  $n \equiv 1 \pmod{3}$ ,  $\gamma_\epsilon(L_{m,n}) = \frac{n+5}{3}$ .

**Case (iii):** For  $n \equiv 2 \pmod{3}$

Let  $n = 3k + 2$ . In this case, the vertices  $v_1, v_n, v_i$  ( $1 < i < n$ ),  $i \equiv 0 \pmod{3}$ , and  $v_j$  ( $1 < j < n - 2$ ),  $j \equiv 1 \pmod{3}$  are the enclave dominating vertices and their corresponding enclave dominating sets are minimum. They are defined in the following subcases.

**Subcase (iii)(a):** If the vertex  $v_1$  is the enclave dominating vertex.

Then closed neighborhood  $N[v_1] = \{v_1, v_2, u_1\}$ . The vertex  $u_1$  dominate all the vertices in  $K_m$ . To dominate the remaining path vertices  $v_4, v_5, \dots, v_n$ , the vertices with indices congruent to  $2 \pmod{3}$  from 2 to  $n$  are included. This set also includes  $v_2$  and provides the minimum enclave dominating set. Thus

$$E_{v_1} = \{v_1, u_1\} \cup \{v_p : p \equiv 2 \pmod{3}, 2 \leq p \leq n\}$$

and  $|E_{v_1}| = 2 + (k + 1) = k + 3 = \frac{n+7}{3}$ .

**Subcase (iii)(b):** If the vertex  $v_n$  is the enclave dominating vertex.

Then closed neighborhood  $N[v_n] = \{v_n, v_{n-1}\}$ . Include the vertex  $u_1$  in  $E_{v_n}$  to dominate  $K_m$  and  $v_1$ . The undominated vertices form the path  $v_2, v_3, \dots, v_{n-3}$  of order  $3k - 2$ . By Theorem 3.4,  $\gamma(P_{3k-2}) = k$ , achieved by vertices with indices  $\equiv 1 \pmod{3}$  from 1 to  $3k$ . Thus

$$E_{v_n} = \{v_n, v_{n-1}, u_1\} \cup \{v_p : p \equiv 1 \pmod{3}, 1 \leq p \leq 3k\}$$

and  $|E_{v_n}| = 3 + k = \frac{n+7}{3}$ .

**Subcase (iii)(c):** If the vertex  $v_i$  ( $1 < i < n$ ),  $i \equiv 0 \pmod{3}$  is the enclave dominating vertex.

Let  $i = 3t$ ,  $1 \leq t \leq k$ . The closed neighborhood  $N[v_i] = \{v_{i-1}, v_i, v_{i+1}\}$ . Include  $u_1$  in  $E_{v_i}$  to obtain the minimum enclave dominating set. Also add the vertices with indices  $\equiv 2 \pmod{3}$  from 1 to  $i - 1$ , and those with indices  $\equiv 1 \pmod{3}$  from  $i + 1$  to  $n$ . Hence

$$E_{v_i} = \{v_i, u_1\} \cup \{v_{3s-1} : 1 \leq s \leq t\} \cup \{v_{3s+1} : t \leq s \leq k\}$$

Hence  $|E_{v_i}| = 2 + t + (k - t + 1) = \frac{n+7}{3}$ .

**Subcase (iii)(d):** If the vertex  $v_j$  ( $1 < j \leq n - 2$ ),  $j \equiv 1 \pmod{3}$  is the enclave dominating vertex.

Let  $j = 3t + 1$ ,  $0 \leq t \leq k$ . The closed neighborhood is  $N[v_j] = \{v_{j-1}, v_j, v_{j+1}\}$ . Include  $u_1$  in  $E_{v_j}$  to get the minimum enclave dominating set. Also add the vertices with indices  $\equiv 0 \pmod{3}$  up to  $j - 1$ , and the vertices with indices  $\equiv 2 \pmod{3}$  from  $j + 1$  to  $n$ . Hence

$$E_{v_j} = \{v_j, u_1\} \cup \{v_{3s} : 1 \leq s \leq t\} \cup \{v_{3s+2} : t \leq s \leq k\}$$

Hence  $|E_{v_j}| = 2 + t + (k - t + 1) = \frac{n+7}{3}$ .

Thus, from the above subcases, for  $n \equiv 2 \pmod{3}$ ,  $\gamma_\epsilon(L_{m,n}) = \frac{n+7}{3}$ .

From all the cases, the vertices identified above are precisely the enclave dominating vertices that yield minimum enclave dominating sets; any other vertex either yields a larger enclave dominating set or contains multiple enclave vertices. Therefore, the enclave domination number of  $L_{m,n}$  is established.  $\square$

**Theorem 4.5.** For any Comb graph  $P_n^+$ ,  $n \geq 2$ ,  $\gamma_\epsilon(P_n^+) = n + 1$ .

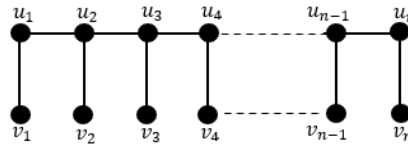


Fig. 6. Comb graph  $P_n^+$

*Proof.* Let the comb graph be denoted by  $P_n^+$ . The vertex set and edge set of  $P_n^+$  are  $V(P_n^+) = \{u_1, u_2, \dots, u_n, v_1, v_2, \dots, v_n\}$  and  $E(P_n^+) = E(P_n) \cup \{u_i v_i : 1 \leq i \leq n\}$ . The order of the graph  $P_n^+$  be  $2n$ , and the size of  $P_n^+$  be  $2n - 1$ . In the comb graph, every pendant vertex  $v_i (1 \leq i \leq n)$  is an enclave dominating vertex. For each  $i = 1, 2, \dots, n$  the corresponding minimum enclave dominating set, is  $E_{v_i} = \{u_i, v_1, v_2, \dots, v_n\}$ , with  $|E_{v_i}| = n + 1$ . Hence,  $\gamma_\epsilon(P_n^+) = n + 1$ .  $\square$

**Theorem 4.6.** For any  $n$ -Sunlet graph  $S_n$ ,  $n \geq 2$ ,  $\gamma_\epsilon(S_n) = n + 1$ .

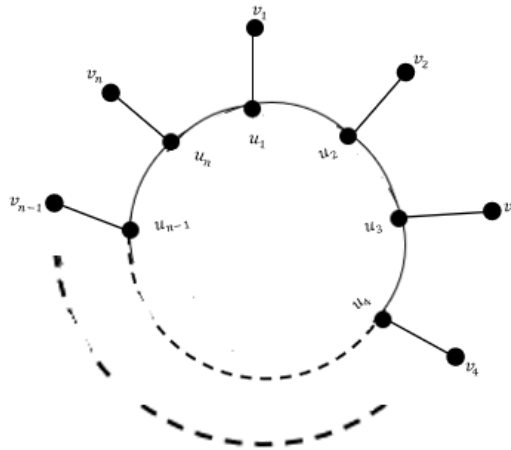


Fig. 7. Sunlet graph  $S_n$

*Proof.* Let  $S_n$  be the  $n$ -sunlet graph. The vertex set of  $S_n$  be  $V(S_n) = \{u_1, u_2, \dots, u_n, v_1, v_2, \dots, v_n\}$  and the edge set of  $S_n$  be  $E(S_n) = E(C_n) \cup \{u_i v_i : 1 \leq i \leq n\}$ . The order and size of the graph  $S_n$  are  $2n$ . Here, every pendant vertex  $v_i (1 \leq i \leq n)$  is an enclave dominating vertex and there exist  $n$ -distinct minimum enclave dominating sets. For  $i = 1, 2, \dots, n$  the minimum enclave dominating set corresponding to the vertex  $v_i$  is  $E_{v_i} = \{u_i, v_1, v_2, \dots, v_n\}$ , with  $|E_{v_i}| = n + 1$ . Thus  $\gamma_\epsilon(S_n) = n + 1$ .  $\square$

**Theorem 4.7.** For any Crown graph  $H_{n,n}$  and  $n \geq 2$ ,  $\gamma_\epsilon(H_{n,n}) = n + 1$ .

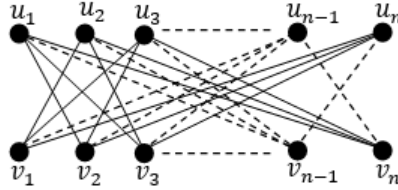


Fig. 8. Crown graph  $H_{n,n}$

*Proof.* Let  $G$  be the crown graph denoted by  $H_{n,n}$ . The vertex set and edge set of  $H_{n,n}$  be  $V(H_{n,n}) = \{u_1, u_2, \dots, u_n, v_1, v_2, \dots, v_n\}$  and  $E(H_{n,n}) = \{u_i v_j : i \neq j\}$ . The order and size of the graph  $H_{n,n}$  be  $2n$  and  $n(n-1)$  respectively. The minimum enclave dominating set of  $H_{n,n}$  will arise in the following cases.

**Case (i):** If any  $u_i$  for  $1 \leq i \leq n$  is the enclave dominating vertex then  $E_{u_i} = \{u_i, v_1, v_2, \dots, v_n\}$ . And  $|E_{u_i}| = n + 1$ .

**Case (ii):** If any  $v_j$  for  $1 \leq j \leq n$  is the enclave dominating vertex then  $E_{v_j} = \{v_j, u_1, u_2, \dots, u_n\}$ . And  $|E_{v_j}| = n + 1$ .

From the above cases,  $\gamma_\epsilon(H_{n,n}) = n + 1$ . □

**Theorem 4.8.** For any Friendship graph  $F_n$ ,  $\gamma_\epsilon(F_n)$  does not exist.

*Proof.* Let  $F_n$  be any friendship graph with  $n+1$  vertices. Suppose there exists an enclave dominating set say  $E_u$  where  $u$  be any vertex in  $F_n$ . The closed neighborhood  $N[u]$  of  $u$  consists of  $u$  and the two vertices in its unique triangle. Thus  $N[u]$  contains two enclave vertices besides  $u$ . This contradicts the definition of enclave dominating set, implies no such enclave dominating set exists. Therefore the enclave domination number of  $F_n$  does not exist. □

## 5 Enclave Domination Number of Some Special Graphs

The following theorems give the characterization of an enclave dominating vertex, and the enclave domination number on some known special graphs.

**Theorem 5.1.** If  $G$  is a Soifer graph, then  $\gamma_\epsilon(G) = 5$ .

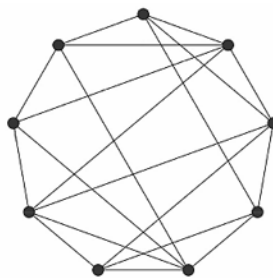


Fig. 9. Soifer graph

*Proof.* Let  $G$  be a Soifer graph with vertex set  $V(G) = \{u_1, u_2, \dots, u_9\}$  and 20 edges. Let partition the vertex set into two as  $S = \{u_1, u_2, u_3, u_4\}$ ,  $T = \{u_5, u_6, u_7, u_8, u_9\}$  where the vertices in  $S$  and  $T$  are

of degree 5 and 4 respectively. By Theorem 3.2, the vertices in the set  $T$  are the enclave dominating vertices. For  $5 \leq i \leq 9$ , there exists 5 distinct minimum enclave dominating sets as  $E_{u_i} = N[u_i]$ , with the enclave dominating vertex  $u_i$ . Moreover, every vertex in  $V - E_{u_i}$  is dominated by the vertices in  $E_{u_i}$ , and the vertex  $u_i$  is not adjacent to any other vertices in  $V - E_{u_i}$ . Since  $|E_{u_i}| = 5$ , it follows that  $\gamma_\epsilon(G) = 5$ .  $\square$

**Theorem 5.2.** *If  $G$  is a Franklin graph, then  $\gamma_\epsilon(G) = 6$ .*

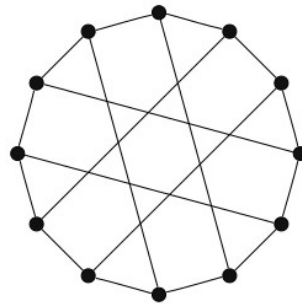


Fig. 10. Franklin graph

*Proof.* Let  $G$  be a 3-regular Franklin graph having the vertex set  $V(G) = \{u_1, u_2, \dots, u_{12}\}$  and 18 edges. By Theorem 3.1, for the Franklin graph, every vertex is an enclave dominating vertex. So, there are 12 distinct minimum enclave dominating sets with equal cardinality. The minimum enclave dominating set of  $G$  consists of any 5 consecutive vertices along with an additional vertex that is adjacent to the third vertex of these 5 vertices. Let the third vertex be any  $u_i (1 \leq i \leq 12)$  in  $G$ . Clearly,  $N[u_i]$  is the subset of the enclave dominating set  $E_{u_i}$ . And  $|E_{u_i}| = 6$ ,  $\gamma_\epsilon(G) = 6$ .  $\square$

**Theorem 5.3.** *If  $G$  is a Moser spindle graph, then  $\gamma_\epsilon(G) = 4$ .*

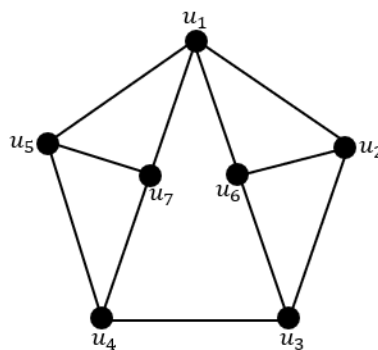


Fig. 11. Moser spindle graph

*Proof.* Let  $G$  be a Moser spindle graph with a vertex set  $V(G) = \{u_1, u_2, \dots, u_7\}$  where the vertices  $u_1, u_2, u_3, u_4, u_5$  form a 5-cycle  $u_1u_2u_3u_4u_5u_1$  and the vertex  $u_6$  adjacent to  $u_1, u_2, u_3$  and  $u_7$  adjacent

to  $u_1, u_4, u_5$ . Here, the vertices  $u_3$  and  $u_4$  are the enclave dominating vertices, and the corresponding minimum enclave dominating sets are  $E_{u_3} = \{u_3, u_2, u_4, u_6\}$  and  $E_{u_4} = \{u_4, u_3, u_5, u_7\}$ . The minimum cardinality of enclave dominating sets for Moser spindle graph is 4. Thus  $\gamma_\epsilon(G) = 4$   $\square$

**Theorem 5.4.** *If  $G$  is a Chvatal graph, then  $\gamma_\epsilon(G) = 5$ .*

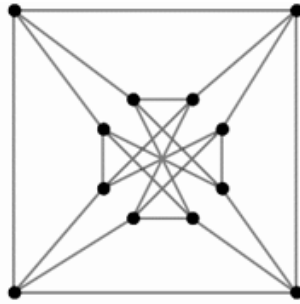


Fig. 12. Chvatal graph

*Proof.* Let  $G$  be a Chvatal graph with 12 vertices and 24 edges. Since it is a 4-regular graph, by theorem 3.1, there are 12 distinct minimum enclave dominating sets. Let  $u$  be any vertex in  $G$ , the enclave dominating set  $E_u$  is the closed neighbourhood of the vertex  $u$ . Therefore, the set  $E_u$  must have 5 vertices, and it satisfies the enclave domination conditions. Thus,  $|E_u| = 5$  and  $\gamma_\epsilon(G) = 5$ .  $\square$

**Theorem 5.5.** *If  $G$  is a Fritsch graph, then  $\gamma_\epsilon(G) = 5$ .*

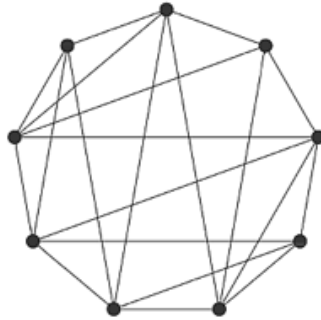


Fig. 13. Fritsch graph

*Proof.* Let  $G$  be a Fritsch graph with vertex set  $V(G) = \{u_1, u_2, \dots, u_9\}$  and 21 edges. Let the vertex set be partitioned into two as  $S = \{u_1, u_2, u_3\}$ ,  $T = \{u_4, u_5, u_6, u_7, u_8, u_9\}$  where the vertices in  $S$  and  $T$  are of degrees 4 and 5 respectively. By Theorem 3.2, the vertices in the set  $S$  are the enclave dominating vertices. For  $1 \leq i \leq 3$ , there exists 3 distinct minimum enclave dominating sets as  $E_{u_i} = N[u_i]$ , with the enclave dominating vertex  $u_i$ . Moreover, every vertex in  $V - E_{u_i}$  is dominated by the vertices in  $E_{u_i}$ , and the vertex  $u_i$  is not adjacent to any other vertices in  $V - E_{u_i}$ . Since  $|E_{u_i}| = 5$ , it follows that  $\gamma_\epsilon(G) = 5$ .  $\square$

**Theorem 5.6.** *If  $G$  is a Herschel graph, then  $\gamma_\epsilon(G) = 5$ .*

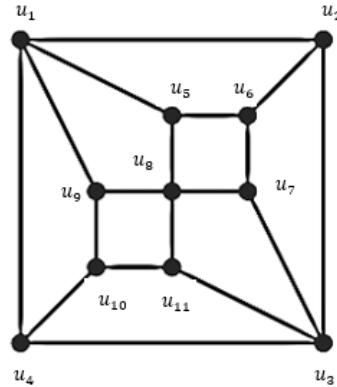


Fig. 14. Herschel graph

*Proof.* Let  $G$  be a Herschel graph with the vertex set  $\{u_1, u_2, \dots, u_{11}\}$ . The vertices  $u_2, u_5, u_6, u_7, u_9, u_{10}$  and  $u_{11}$  are the enclave dominating vertices. The corresponding minimum enclave dominating sets are as follows,  
 $E_{u_2} = \{u_2, u_1, u_3, u_6, u_{11}\}$ ,  $E_{u_5} = \{u_5, u_1, u_6, u_8, u_{11}\}$ ,  $E_{u_6} = \{u_6, u_2, u_5, u_7, u_{10}\}$ ,  
 $E_{u_7} = \{u_7, u_3, u_6, u_8, u_9\}$ ,  $E_{u_9} = \{u_9, u_1, u_2, u_8, u_{10}\}$ ,  $E_{u_{10}} = \{u_{10}, u_4, u_9, u_{11}, u_6\}$  and  
 $E_{u_{11}} = \{u_{11}, u_{10}, u_3, u_5, u_8\}$  respectively. The cardinality of all these sets is equal to 5. Hence  $\gamma_\epsilon(G) = 5$ . □

**Theorem 5.7.** If  $G$  is a Goldner-Harary graph, then  $\gamma_\epsilon(G) = 4$ .

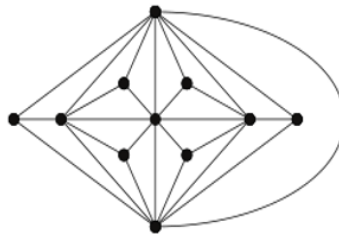


Fig. 15. Goldner-Harary graph

*Proof.* Let  $G$  be a Goldner-Harary graph with 11 vertices, and  $\delta(G) = 3$ . There are 6 vertices in  $G$  such that  $d(u) = \delta(G)$ . By theorem 3.2, there exists 6 distinct minimum enclave dominating sets. Choose a vertex  $u$  in  $G$  such that  $d(u) = \delta(G)$  then  $E_u = N[u]$  is the minimum enclave dominating set with enclave dominating set  $u$  and  $|E_u| = 4$ . It follows that  $\gamma_\epsilon(G) = 4$ . □

## 6 Applications of Enclave Dominating Sets

### 6.1 Application in Security and Surveillance

Consider a scenario where a leader organizes a rally at specific locations within a city and wants to deliver campaign speeches in some places during the rally. The security team or protection squad must protect the campaign points, their surroundings, and the entire city. This situation can be modeled using enclave dominating sets in graph theory by identifying the optimal positioning for the protection team to ensure protection throughout the rally.

Consider a city represented as a graph  $G = (V, E)$ , where vertices represent key locations and edges denote accessible routes. For the campaign, the chief security officer selects campaign sites that form an enclave dominating set  $E_u$  with the enclave dominating vertex  $u$  as the primary campaign location in the rally. The enclave dominating condition  $N[u] \subseteq E_u$  secures the campaign site  $u$  and its immediate surroundings within  $E_u$  and uniquely dominates all external areas, ensuring comprehensive coverage of the graph. This approach guarantees city-wide security for the leader's rally, minimizing threats and criminal activities throughout the city. This structure minimizes the number of locations where the security team is placed, equal to the enclave domination number  $\gamma_\epsilon(G)$ , while ensuring unambiguous monitoring. Peripheral areas are also monitored by  $E_u$ , enabling rapid response during rallies.

In a graph  $G$ , if there exist multiple enclave dominating vertices, each inducing an enclave dominating set of identical minimum cardinality, then the enclave domination number  $\gamma_\epsilon(G)$  is attained by distinct vertices. Consequently, in a rally, a leader can conduct secure campaigning across multiple locations using the same minimal contingent of optimally positioned protection squads. Enclave dominating sets model these ideal squad placements.

### 6.2 Applications in Other Domains

This versatile framework extends naturally to other domains requiring similar optimization. In each application, the enclave dominating set  $E_u \subset V(G)$  contains a unique enclave dominating vertex  $u$  whose closed neighborhood  $N[u]$  lies entirely within  $E_u$ , ensuring that  $u$  and all its immediate neighbors are part of the protected set. The minimum cardinality of such a configuration,  $\gamma_\epsilon(G)$ , represents the optimal resource allocation, while the existence of multiple enclave dominating vertices provides adaptive solutions.

In military operations, forward operating bases form an enclave dominating set  $E_u$ , with the command hub serving as the unique enclave dominating vertex  $u$  whose surrounding positions  $N[u]$  are all secured within  $E_u$ . Other bases may have adjacent hostile territory, but the dominating property of  $E_u$  ensures surveillance over the entire region.

For wireless sensor networks, data aggregation points constitute  $E_u$ , where the central aggregator serves as the unique enclave dominating vertex  $u$  with all its neighboring sensors  $N[u]$  contained in the trusted set. Other aggregation points may connect to untrusted nodes, but the set  $E_u$  as a whole monitors the entire network.

Public health officials can establish minimal quarantine zones as  $E_u$ , with the epidemic epicenter  $u$  having all directly connected communities  $N[u]$  under quarantine. Other quarantined areas may border unquarantined regions, but together they maintain surveillance over all regions.

In cybersecurity, the intrusion detection system (IDS) placements form  $E_u$ , where the primary server acts as the unique enclave dominating vertex  $u$  with all adjacent devices  $N[u]$  directly monitored. Other IDS nodes may leave some neighbors unmonitored, yet together they guarantee comprehensive network coverage.

Thus, the enclave dominating set provides a strategy for achieving comprehensive coverage with minimal resources, where multiple enclave dominating vertices offer adaptive solutions for dynamic environments.

## 7 Conclusion

In this paper, the enclave domination numbers are determined for several standard special graphs and specific graph families. The findings reveal how different graph structures influence the behaviour of the enclave domination parameter. This study advances the understanding of enclave domination in graph theory and establishes a foundation for further investigations. Practical applications of enclave dominating sets in graphs are also explored. Future research may focus on extending these results to unary and binary graph operations.

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Author(s) hereby declare that NO generative AI technologies such as Large Language Models (ChatGPT, COPILOT, etc.) and text-to-image generators have been used during the writing or editing of this manuscript.

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